Chapter 2

Fractions and Problem Solving

2.1 Solving Problems and Explaining Solutions

2.2 Defining and Reasoning About Fractions

1. Anna says that the dark blocks pictured in Figure 2.1 can't represent $\frac{1}{4}$ because there are 6 dark blocks and 6 is more than 1 but $\frac{1}{4}$ is supposed to be less than 1. What should you clarify in order to interpret the dark blocks as $\frac{1}{4}$?

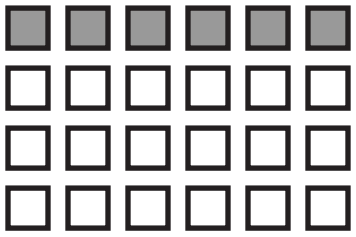

Figure 2.1: Representing the Fraction $\frac{1}{4}$

2. When Martin was asked to say what the 5 in the fraction $\frac{2}{5}$ means, Martin said that the 5 is the whole. Explain why it is not completely correct to say that "5 is the whole." What is a better way to say what the 5 in the fraction $\frac{2}{5}$ means?

3. If Harry needs $\frac{4}{5}$ of a liter of dragon snot to make a full batch of potion but he only has $\frac{3}{5}$ of a liter of dragon snot, then what fraction of a batch of potion can Harry make (assuming he has enough of the other ingredients)?

   (a) Make a math drawing to help you solve the problem and explain your solution. Use our definition of fraction in your explanation and attend to the whole (unit amount) that each fraction is *of*.

   (b) Describe the different wholes that occur in part (a). Discuss how one amount can be described with two different fractions depending on what the whole is taken to be.

4. The rectangle of Xs below is $\frac{4}{3}$ of another rectangle of Xs. Show the original rectangle and explain how to determine it. Use our definition of fraction in your explanation.

X X X X X X X X
X X X X X X X X
X X X X X X X X
X X X X X X X X
X X X X X X X X
X X X X X X X X

5. The rectangle of Xs below is $\frac{3}{5}$ of another rectangle of Xs. Show the original rectangle and explain how to determine it. Use our definition of fraction in your explanation.

   X X X X X X X X X X
   X X X X X X X X X X
   X X X X X X X X X X
   X X X X X X X X X X
   X X X X X X X X X X
   X X X X X X X X X X

6. The rectangle of Xs below is $\frac{3}{5}$ of another rectangle of Xs. Show $\frac{2}{5}$ of the original rectangle and explain how to determine it. Use our definition of fraction in your explanation.

   X X X X X X X X X X
   X X X X X X X X X X
   X X X X X X X X X X
   X X X X X X X X X X
   X X X X X X X X X X
   X X X X X X X X X X

7. If $\frac{3}{4}$ of a cup of a food gives you your daily value of potassium, then what fraction of your daily value of potassium is in 1 cup of the food?

   Make a math drawing to help you solve the problem and explain your solution. Use our definition of fraction in your explanation and attend to the whole (unit amount) that each fraction is *of*.

8. See Figure 2.2.

   (a) How many of the short strip does it take to make the long strip exactly? Use our definition of fraction to explain your answer.

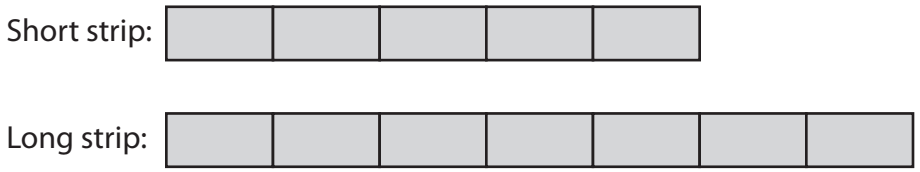

Figure 2.2: A short strip and a long strip.

(b) How many of the long strip does it take to make the short strip exactly? Use our definition of fraction to explain your answer.

9. (a) (See Figure 2.3.) The strip below represents $\frac{8}{5}$ of a unit amount. Using our definition of fractions, explain how to draw another strip to represent the unit amount.

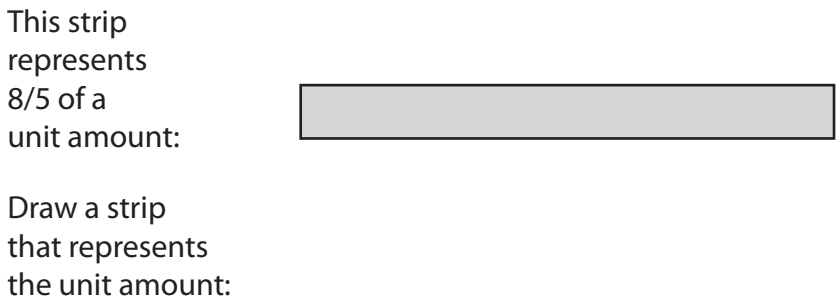

Figure 2.3: Fraction strips.

(b) In part (a), how many of the gray $\frac{8}{5}$ strip does it take to make the strip you drew exactly? Explain.

10. (a) (See Figure 2.4.) The strip below represents $\frac{3}{8}$ of a unit amount. Using our definition of fractions, explain how to draw another strip to represent the unit amount.

(b) In part (a), how many of the gray $\frac{3}{8}$ strip does it take to make the strip you drew exactly? Explain.

11. (a) (See Figure 2.5.) The strip below represents $\frac{4}{37}$ of a unit amount. Using our definition of fractions, explain how to draw another strip to represent $\frac{15}{37}$ of the unit amount.

This strip represents 3/8 of a unit amount:

Draw a strip that represents the unit amount:

Figure 2.4: Fraction strips.

This strip represents 4/37 of a unit amount:

Draw a strip that represents 15/37 of the unit amount:

Figure 2.5: Fraction strips.

(b) In part (a), how many of the gray $\frac{4}{37}$ strip does it take to make the strip you drew exactly? Explain.

12. (a) (See Figure 2.6.) The strip below represents $\frac{4}{65}$ of a unit amount. Using our definition of fractions, explain how to draw another strip to represent $\frac{11}{65}$ of the unit amount.

    (b) In part (a), how many of the gray $\frac{4}{65}$ strip does it take to make the strip you drew? Explain.

13. The strip in Figure 2.7 shows a canal that is $\frac{16}{100}$ of a mile long. Using our definition of fractions, explain how to draw another strip to show a canal that is $\frac{7}{100}$ of a mile long.

14. The diagram in Figure 2.8 shows a map of Mr. McGregor's garden, which consists of two plots of different areas. Each plot is divided into 3 parts of equal

This strip represents 4/65 of a unit amount:

Draw a strip that represents 11/65 of the unit amount:

Figure 2.6: Fraction strips.

Figure 2.7: A canal

area. Mr. McGregor planted peas in the two parts that are shaded. What fraction of Mr. McGregor's (entire) garden is planted with peas? Explain why your answer is correct by using our definition of fraction without further subdividing the plots.

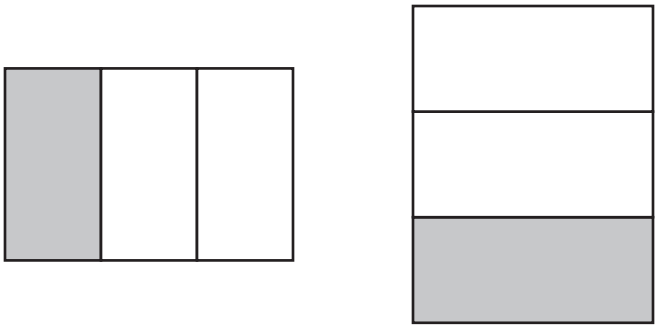

Figure 2.8: Mr. McGregor's Garden

15. Plot $\frac{3}{4}$ on the number line in Figure 2.9 and explain why this location fits with our definition of fraction.

16. Plot $\frac{5}{4}$ on the number line in Figure 2.9 and explain why this location fits with our definition of fraction.

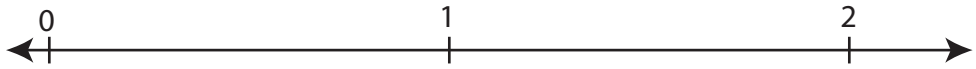

Figure 2.9: A Number Line

17. According to our text, what does $\frac{5}{3}$ mean?

18. Explain what $\frac{7}{4}$ means according to our definition. Discuss how to make sense of it.

19. Make drawings and discuss what $\frac{7}{5}$ means according to the definition of fraction we are using.

20. Students often have trouble interpreting improper fractions such as $\frac{7}{4}$ and they sometimes think of a fraction as a pair of numbers rather than as a single number. Use our definition of fraction to interpret $\frac{7}{4}$. Explain how to understand $\frac{7}{4}$ as a number just like any other number, such as 3.

21. Students often have trouble interpreting improper fractions such as $\frac{4}{3}$ and they sometimes think of a fraction as a pair of numbers rather than as a single number. Discuss how to interpret $\frac{4}{3}$ according to our definition of fractions. As part of your discussion:

    - explain how to see $\frac{4}{3}$ as a single number (not a pair of numbers);
    - explain how to plot $\frac{4}{3}$ on the number line in Figure 2.10.

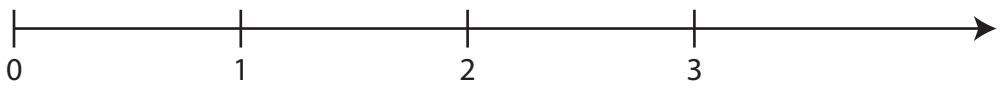

Figure 2.10: Plot the fraction $\frac{4}{3}$.

22. (a) Give two different fractions that you can legitimately use to describe the shaded region in Figure 2.11. For each fraction, explain why you can use that fraction to describe the shaded region.

    (b) Write an unambiguous question about the shaded region in Figure 2.11 that can be answered by naming a fraction. Explain why your question is not ambiguous.

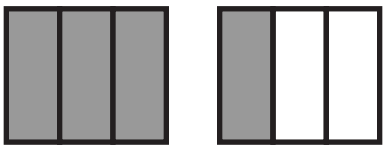

Figure 2.11: What Fraction is Shaded?

23. You showed Johnny the picture in Figure 2.12 to help Johnny understand the meaning of the fraction $\frac{5}{3}$. But Johnny doesn't get it. He says the picture shows $\frac{5}{6}$, not $\frac{5}{3}$.

Figure 2.12: A Picture for $\frac{5}{3}$

   (a) Is Johnny right that the picture can be interpreted as showing $\frac{5}{6}$? Explain briefly.

   (b) What must you clarify in order to interpret the picture as showing $\frac{5}{3}$?

   (c) What is another way to show $\frac{5}{3}$ to Johnny, other than with parts of objects?

24. Many students have difficulty understanding improper fractions, such as $\frac{7}{4}$. State what $\frac{7}{4}$ means according to our definition. Then describe a way that might help a student understand this meaning.

25. Discuss why it can be confusing to show an improper fraction such as $\frac{7}{3}$ with pieces of pie or the like. What is another way to show the fraction $\frac{7}{3}$?

2.3 Reasoning About Equivalent Fractions

1. Using the example

$$\frac{2}{3} = \frac{2 \cdot 4}{3 \cdot 4}$$

and a math drawing, explain why multiplying the numerator and denominator of a fraction by the same number results in the same number (equivalent fraction). Give a "general" explanation, in the sense that the explanation would work the same way if other numbers had been used. In your explanation, attend carefully to the number and the size of the parts.

2. Using the example
$$\frac{2}{5} = \frac{2 \cdot 3}{5 \cdot 3}$$
and a math drawing, explain why multiplying the numerator and denominator of a fraction by the same number results in the same number (equivalent fraction). In your explanation, discuss the following:

   - what happens to the number of parts and the size of the parts;
   - *how your math drawing* shows that the numerator and denominator are each multiplied by 3;
   - *how your math drawing* shows why those two fractions are equal.

3. Using a math drawing, explain why multiplying both the numerator and denominator of
$$\frac{2}{3}$$
by 4 produces the same number (an equivalent fraction). Discuss how to see multiplication by 4 in both the numerator and denominator in terms of your math drawing. Attend carefully to points that might be difficult for students.

4. Without using multiplication by 1, explain why multiplying the numerator and denominator of a fraction by the same number produces an equivalent fraction. Use the example
$$\frac{2}{3} = \frac{2 \cdot 4}{3 \cdot 4}$$
to illustrate.

5. Using the example
$$\frac{2}{3} = \frac{2 \cdot 4}{3 \cdot 4}$$
explain in *two different ways* why multiplying the numerator and denominator by the same number produces an equivalent fraction.

6. Using the fractions $\frac{1}{3}$ and $\frac{3}{4}$, describe how to give two fractions common denominators. In terms of a math drawing, what are you doing when you give fractions common denominators?

7. Using a math drawing, explain why dividing both the numerator and denominator of

$$\frac{6}{9}$$

by 3 produces the same number (an equivalent fraction). Discuss how to see division by 3 in both the numerator and denominator in terms of your math drawing. Attend carefully to points that might be difficult for students.

8. Simplify

$$\frac{4}{6}$$

and use a math drawing to show the process.

9. Plot $\frac{5}{6}$, $\frac{5}{4}$, and $\frac{4}{3}$ on the number line in Figure 2.13 in such a way that each number falls on a tick mark. Lengthen the tick marks of whole numbers.

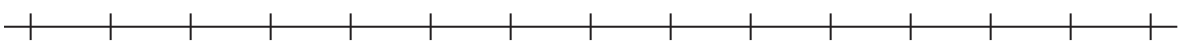

Figure 2.13: A Number Line

10. (See Figure 2.13.) Plot $\frac{5}{6}$, 0.7, and $\frac{4}{5}$ on the number line in such a way that each number falls on a tick mark. Lengthen the tick marks of whole numbers (if any).

11. Place equally spaced tick marks on the number line in Figure 2.14 so that you can plot both $\frac{1}{7}$ and $\frac{1}{5}$ at tick marks. Explain your reasoning.

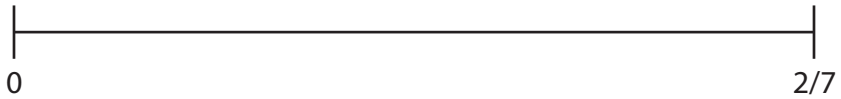

Figure 2.14: An interval

12. (See Figure 2.15.) Explain how to place equally spaced tick marks on the number line below so that you can plot both $\frac{1}{5}$ and $\frac{1}{6}$ on the number line. Explain your reasoning.

13. (See Figure 2.16.) Place equally spaced tick marks on the number line below so that you can plot both $\frac{1}{3}$ and $\frac{1}{5}$ at tick marks. Explain your reasoning.

Figure 2.15: Plot $\frac{1}{5}$ and $\frac{1}{6}$.

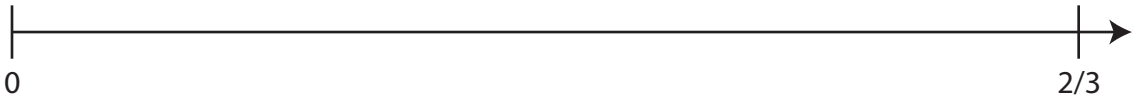

Figure 2.16: Plot $\frac{1}{3}$ and $\frac{1}{5}$ at tick marks.

14. (See Figure 2.17.) So far, Torrey has run $\frac{3}{4}$ of a mile, but that is only $\frac{2}{5}$ of her total running distance.

    Explain how to use equivalent fractions to place equally spaced tick marks on the number line below so that you can (i) plot Torrey's total running distance at a tick mark and (ii) determine Torrey's total running distance in miles.

Figure 2.17: Torrey's run

15. Recipe problem: A recipe requires $\frac{4}{5}$ of a liter of juice. You have $\frac{2}{3}$ of a liter of juice. What fraction of the recipe can you make?

    Explain how to reason with a math drawing and equivalent fractions to solve the recipe problem. Discuss how equivalent fractions help you solve the problem.

16. Swimming problem: You swam $\frac{2}{5}$ of a kilometer but that was only $\frac{3}{8}$ of the total distance you wanted to swim in your workout. How many kilometers did you want to swim in your full workout?

    Explain how to reason with a math drawing and equivalent fractions to solve the recipe problem. Discuss how equivalent fractions help you solve the problem.

17. The line segment below is $\frac{2}{3}$ units long. Describe how to remove or add a portion to this line segment to create a line segment that is $\frac{3}{5}$ units long. Explain how

you know your new segment will be the correct length.

$\frac{2}{3}$ unit

18. Susie says that when you do the same thing to the top and bottom of a fraction you get an equivalent fraction. Is Susie right, or is it possible to do the same thing to the top and the bottom of a fraction and *not* get an equivalent fraction?

19. Ken ordered $\frac{4}{5}$ of a ton of sand. Ken wants to receive $\frac{1}{3}$ of his order now (and the rest of his order later). What fraction of a ton of sand should Ken receive now?

    Make a math drawing to help you solve the problem. Explain how your drawing helps you to solve the problem. In your explanation, attend carefully to the unit amount (whole) that each fraction is *of*.

20. Frank is making a recipe that calls for $\frac{3}{4}$ of a cup of ketchup. Frank only has $\frac{1}{3}$ of a cup of ketchup. Assuming that Frank has enough of the other ingredients, what fraction of the recipe can Frank make?

    Make a math drawing to help you solve the problem. Explain how your drawing helps you to solve the problem. In your explanation, attend carefully to the unit amount (whole) that each fraction is *of*.

21. *Ketchup problem*: You are making a recipe that calls for $\frac{3}{4}$ of a cup of ketchup. You only have $\frac{1}{3}$ of a cup of ketchup. Assuming that you have enough of the other ingredients, what fraction of the recipe can you make?

    Explain how to solve the ketchup problem with the aid of a math drawing. In your explanation, attend carefully to the unit amount that each fraction is *of*.

22. The strips in Figure 2.18 show the relative amounts of cans that Ben and Charles collected for the can-a-thon. (Note that each rectangle represents some fixed number of cans, but this number may be greater than 1.)

    (a) Write a sentence in which you use a fraction to describe how the amount of cans that Ben collected compares with the amount of can that Charles collected. Explain briefly.

    (b) Write a sentence in which you use a fraction to describe how the amount of cans that Charles collected compares with the amount of can that Ben collected. Explain briefly.

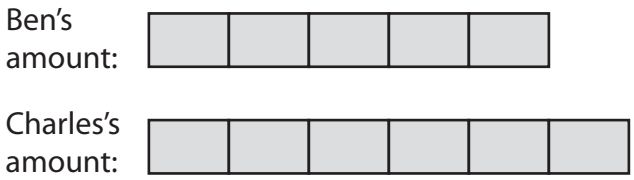

Figure 2.18: The Amounts of Cans that Ben and Charles Collected

2.4 Reasoning to Compare Fractions

1. Describe three general methods for determining which of two fractions is greater. Illustrate the three methods with the fractions $\frac{3}{5}$ and $\frac{5}{8}$.

2. Using the fractions $\frac{2}{3}$ and $\frac{3}{5}$ to illustrate, explain clearly and in detail why we can determine which of two fractions is greater by giving the two fractions common denominators. What is the rationale behind this method? In terms of pictures, what are we doing when we give the fractions common denominators?

3. Show how to use the cross-multiplying method to determine which of $\frac{5}{8}$ and $\frac{7}{12}$ is greater.

4. Using the fractions $\frac{2}{3}$ and $\frac{3}{5}$ to illustrate, explain clearly and in detail why we can determine which of two fractions is greater by using the cross-multiplying method. What is the rationale behind this method? What are we really doing when we cross-multiply in order to compare fractions?

5. Give two different methods for solving the following problem. Find two different fractions in between $\frac{3}{4}$ and $\frac{4}{5}$ whose numerators and denominators are all whole numbers.

6. Conrad says that $\frac{3}{8} > \frac{2}{7}$ because $3 > 2$ and $8 > 7$. Regardless of whether or not Conrad's conclusion is correct, discuss whether or not Conrad's reasoning is valid.

7. Ron says that $\frac{5}{8} > \frac{3}{7}$ because $5 > 3$ and $8 > 7$.

   (a) Discuss whether or not Ron's reasoning is valid (whether or not his conclusion is correct).

   (b) Describe another way to reason (legitimately) to compare $\frac{5}{8}$ and $\frac{3}{7}$ without converting to decimals, giving the fractions common denominators, or cross-multiplying.

8. Minju says that fractions that use bigger numbers are greater than fractions that use smaller numbers. Make up two problems for Minju to help her reconsider her ideas. For each problem, explain how to solve it, and explain why you chose that problem for Minju.

9. Use reasoning other than converting to decimals, using common denominators or cross-multiplying to determine which of $\frac{19}{94}$ and $\frac{19}{107}$ is greater. Explain your reasoning clearly and in detail.

10. Use reasoning other than converting to decimals, using common denominators or common numerators, or cross-multiplying to determine which of $\frac{4}{19}$ and $\frac{5}{17}$ is greater. Explain your reasoning clearly and in detail.

11. Use reasoning other than converting to decimals, using common denominators or common numerators, or cross-multiplying to determine which of $\frac{13}{17}$ and $\frac{14}{15}$ is greater. Explain your reasoning clearly and in detail.

12. Use reasoning other than converting to decimals, using common denominators or common numerators, or cross-multiplying to determine which of $\frac{38}{39}$ and $\frac{45}{46}$ is greater. Explain your reasoning clearly and in detail.

13. Use reasoning other than converting to decimals, using common denominators or common numerators, or cross-multiplying to determine which of $\frac{37}{39}$ and $\frac{43}{45}$ is greater. Explain your reasoning clearly and in detail.

14. Use reasoning other than converting to decimals, using common denominators or common numerators, or cross-multiplying to determine which of $\frac{21}{43}$ and $\frac{41}{81}$ is greater. Explain your reasoning clearly and in detail.

15. Use reasoning other than converting to decimals, using common denominators or common numerators, or cross-multiplying to determine which of $\frac{23}{44}$ and $\frac{25}{48}$ is greater. Explain your reasoning clearly and in detail.

16. Use reasoning *other than* converting to decimals, using common denominators or common numerators, or cross-multiplying to determine which of $\frac{13}{24}$ and $\frac{17}{32}$ is greater. Explain your reasoning clearly and in detail.

17. Use reasoning other than converting to decimals, using common denominators or common numerators, or cross-multiplying to determine which of $\frac{24}{23}$ and $\frac{26}{25}$ is greater. Explain your reasoning clearly and in detail.

18. Use reasoning other than converting to decimals, using common denominators or common numerators, or cross-multiplying to determine which of $\frac{7}{15}$ and $\frac{11}{19}$ is greater. Explain your reasoning clearly and in detail.

2.5 Reasoning About Percent

1. If your full daily value of potassium is 3600 milligrams, then how many milligrams is 45% of your daily value of potassium? Show how to solve the problem with the aid of a math drawing or a percent table. Explain your reasoning.

2. If the normal rainfall for August is 2.5 inches, but only 1.75 inches of rain fell in August, then what percent of the normal rainfall fell in August? Show how to solve the problem with the aid of a math drawing or a percent table. Explain your reasoning.

3. If the full capacity of a tank is 25 liters and the tank is filled with only 15 liters, then what percent full is the tank? Show how to solve the problem with the aid of a math drawing or a percent table. Explain your reasoning.

4. If \$85,000 is 40% of the budget, then what is the full budget? Show how to solve the problem with the aid of a math drawing or a percent table. Explain your reasoning.

5. If $\frac{3}{4}$ of a cup of food gives you 40% of the daily value of calcium, then how many cups of the food will give you 100% of the daily value of calcium? Explain how to solve this problem with a math drawing *and* a percent table. Relate the information in the table and the drawing.

6. If $\frac{3}{4}$ of a cup of juice gives you 100% of your daily value of vitamin C, then what percent of your daily value of vitamin C will you get in 1 full cup of juice? Show how to solve the problem with the aid of a math drawing or a percent table. Explain your reasoning.

7. If $\frac{2}{5}$ of a liter of juice gives you 100% of your daily value of vitamin C, then what percent of your daily value of vitamin C will you get in $\frac{1}{2}$ liter of juice? Show how to solve the problem with the aid of a math drawing or a percent table. Explain your reasoning.

8. If $\frac{2}{5}$ of a liter of juice gives you 100% of your daily value of vitamin C, then how much of your daily value of vitamin C will you get in $\frac{1}{2}$ of a liter of juice?

Explain how to solve this problem with a math drawing *and* a percent table. Relate the information in the table and the drawing.

9. If $\frac{1}{3}$ of a pound of cheese gives you 100% of your daily value of calcium, then what percent of your daily value of calcium do you get in $\frac{3}{4}$ of a pound of cheese? Explain how to solve this problem with the aid of a percent table *and* a math drawing.

10. If running $\frac{5}{8}$ of a mile is 100% of your workout, then what percent of your workout is running $\frac{1}{2}$ mile? Show how to solve the problem with the aid of a math drawing or a percent table. Explain your reasoning.

11. If Company A sells 30% as many cars as Company B, then what are Company B's car sales, when they are calculated as a percentage of Company A's sales? Use a math drawing or percent table to help you solve the problem. Use your drawing or table to help you explain your answer.

12. Explaining your reasoning clearly and use math drawings or tables to help you answer the following:

    a. What percent of $\frac{1}{5}$ is $\frac{2}{5}$?

    b. What percent of $\frac{2}{5}$ is $\frac{1}{5}$?

13. Solve using equivalent fractions without cross-multiplying: A shirt had cost \$15, but now it is on sale at a \$6 discount. What percent is the discount?

14. Solve using equivalent fractions without cross-multiplying: A company's profits were 12% of its revenues. If the company's profits were \$360,000, what were its revenues?

15. Solve using equivalent fractions without cross-multiplying: 12% of a company's 75 employees walk to work every day. How many employees walk to work every day?

16. There were 70 members in a club. Of the 70 members, 60% were girls and the rest were boys. After some more girls joined the club (and no boys joined or left), the club was 75% girls. How many girls joined the club?