

CHAPTER 17 SECOND-ORDER DIFFERENTIAL EQUATIONS

17.1 SECOND-ORDER LINEAR EQUATIONS

1. $y'' - y' - 12y = 0 \Rightarrow r^2 - r - 12 = 0 \Rightarrow (r - 4)(r + 3) = 0 \Rightarrow r = 4 \text{ or } r = -3 \Rightarrow y = c_1 e^{4x} + c_2 e^{-3x}$
2. $3y'' - y' = 0 \Rightarrow 3r^2 - r = 0 \Rightarrow r(3r - 1) = 0 \Rightarrow r = 0 \text{ or } r = \frac{1}{3} \Rightarrow y = c_1 e^{0 \cdot x} + c_2 e^{\frac{1}{3}x} \Rightarrow y = c_1 + c_2 e^{\frac{1}{3}x}$
3. $y'' + 3y' - 4y = 0 \Rightarrow r^2 + 3r - 4 = 0 \Rightarrow (r + 4)(r - 1) = 0 \Rightarrow r = -4 \text{ or } r = 1 \Rightarrow y = c_1 e^{-4x} + c_2 e^x$
4. $y'' - 9y = 0 \Rightarrow r^2 - 9 = 0 \Rightarrow (r - 3)(r + 3) = 0 \Rightarrow r = 3 \text{ or } r = -3 \Rightarrow y = c_1 e^{3x} + c_2 e^{-3x}$
5. $y'' - 4y = 0 \Rightarrow r^2 - 4 = 0 \Rightarrow (r - 2)(r + 2) = 0 \Rightarrow r = 2 \text{ or } r = -2 \Rightarrow y = c_1 e^{2x} + c_2 e^{-2x}$
6. $y'' - 64y = 0 \Rightarrow r^2 - 64 = 0 \Rightarrow (r - 8)(r + 8) = 0 \Rightarrow r = 8 \text{ or } r = -8 \Rightarrow y = c_1 e^{8x} + c_2 e^{-8x}$
7. $2y'' - y' - 3y = 0 \Rightarrow 2r^2 - r - 3 = 0 \Rightarrow (2r - 3)(r + 1) = 0 \Rightarrow r = \frac{3}{2} \text{ or } r = -1 \Rightarrow y = c_1 e^{\frac{3}{2}x} + c_2 e^{-x}$
8. $9y'' - y = 0 \Rightarrow 9r^2 - 1 = 0 \Rightarrow (3r - 1)(3r + 1) = 0 \Rightarrow r = \frac{1}{3} \text{ or } r = -\frac{1}{3} \Rightarrow y = c_1 e^{\frac{1}{3}x} + c_2 e^{-\frac{1}{3}x}$
9. $8y'' - 10y' - 3y = 0 \Rightarrow 8r^2 - 10r - 3 = 0 \Rightarrow (4r + 1)(2r - 3) = 0 \Rightarrow r = -\frac{1}{4} \text{ or } r = \frac{3}{2} \Rightarrow y = c_1 e^{-\frac{1}{4}x} + c_2 e^{\frac{3}{2}x}$
10. $3y'' - 20y' + 12y = 0 \Rightarrow 3r^2 - 20r + 12 = 0 \Rightarrow (3r - 2)(r - 6) = 0 \Rightarrow r = \frac{2}{3} \text{ or } r = 6 \Rightarrow y = c_1 e^{\frac{2}{3}x} + c_2 e^{6x}$
11. $y'' + 9y = 0 \Rightarrow r^2 + 9 = 0 \Rightarrow r = 0 \pm 3i \Rightarrow y = e^{0 \cdot x}(c_1 \cos 3x + c_2 \sin 3x) \Rightarrow y = c_1 \cos 3x + c_2 \sin 3x$
12. $y'' + 4y' + 5y = 0 \Rightarrow r^2 + 4r + 5 = 0 \Rightarrow r = \frac{-4 \pm \sqrt{4^2 - 4(1)(5)}}{2(1)} = -2 \pm i \Rightarrow y = e^{-2x}(c_1 \cos x + c_2 \sin x)$
13. $y'' + 25y = 0 \Rightarrow r^2 + 25 = 0 \Rightarrow r = 0 \pm 5i \Rightarrow y = e^{0 \cdot x}(c_1 \cos 5x + c_2 \sin 5x) \Rightarrow y = c_1 \cos 5x + c_2 \sin 5x$
14. $y'' + y = 0 \Rightarrow r^2 + 1 = 0 \Rightarrow r = 0 \pm i \Rightarrow y = e^{0 \cdot x}(c_1 \cos x + c_2 \sin x) \Rightarrow y = c_1 \cos x + c_2 \sin x$
15. $y'' - 2y' + 5y = 0 \Rightarrow r^2 - 2r + 5 = 0 \Rightarrow r = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} = 1 \pm 2i \Rightarrow y = e^x(c_1 \cos 2x + c_2 \sin 2x)$
16. $y'' + 16y = 0 \Rightarrow r^2 + 16 = 0 \Rightarrow r = 0 \pm 4i \Rightarrow y = e^{0 \cdot x}(c_1 \cos 4x + c_2 \sin 4x) \Rightarrow y = c_1 \cos 4x + c_2 \sin 4x$
17. $y'' + 2y' + 4y = 0 \Rightarrow r^2 + 2r + 4 = 0 \Rightarrow r = \frac{-2 \pm \sqrt{2^2 - 4(1)(4)}}{2(1)} = -1 \pm \sqrt{3}i \Rightarrow y = e^{-x}(c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x)$
18. $y'' - 2y' + 3y = 0 \Rightarrow r^2 - 2r + 3 = 0 \Rightarrow r = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(3)}}{2(1)} = 1 \pm \sqrt{2}i \Rightarrow y = e^x(c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x)$
19. $y'' + 4y' + 9y = 0 \Rightarrow r^2 + 4r + 9 = 0 \Rightarrow r = \frac{-4 \pm \sqrt{4^2 - 4(1)(9)}}{2(1)} = -2 \pm \sqrt{5}i \Rightarrow y = e^{-2x}(c_1 \cos \sqrt{5}x + c_2 \sin \sqrt{5}x)$

20. $4y'' - 4y' + 13y = 0 \Rightarrow 4r^2 - 4r + 13 = 0 \Rightarrow r = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(4)(13)}}{2(4)} = \frac{1}{2} \pm \sqrt{3}i$
 $\Rightarrow y = e^{\frac{1}{2}x} (c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x)$
21. $y'' = 0 \Rightarrow r^2 = 0 \Rightarrow r = 0$, repeated twice $\Rightarrow y = c_1 e^{0 \cdot x} + c_2 x e^{0 \cdot x} \Rightarrow y = c_1 + c_2 x$
22. $y'' + 8y' + 16y = 0 \Rightarrow r^2 + 8r + 16 = 0 \Rightarrow (r + 4)^2 = 0 \Rightarrow r = -4$, repeated twice $\Rightarrow y = c_1 e^{-4x} + c_2 x e^{-4x}$
23. $\frac{d^2 y}{dx^2} + 4\frac{dy}{dx} + 4y = 0 \Rightarrow r^2 + 4r + 4 = 0 \Rightarrow (r + 2)^2 = 0 \Rightarrow r = -2$, repeated twice $\Rightarrow y = c_1 e^{-2x} + c_2 x e^{-2x}$
24. $\frac{d^2 y}{dx^2} - 6\frac{dy}{dx} + 9y = 0 \Rightarrow r^2 - 6r + 9 = 0 \Rightarrow (r - 3)^2 = 0 \Rightarrow r = 3$, repeated twice $\Rightarrow y = c_1 e^{3x} + c_2 x e^{3x}$
25. $\frac{d^2 y}{dx^2} + 6\frac{dy}{dx} + 9y = 0 \Rightarrow r^2 + 6r + 9 = 0 \Rightarrow (r + 3)^2 = 0 \Rightarrow r = -3$, repeated twice $\Rightarrow y = c_1 e^{-3x} + c_2 x e^{-3x}$
26. $4\frac{d^2 y}{dx^2} - 12\frac{dy}{dx} + 9y = 0 \Rightarrow 4r^2 - 12r + 9 = 0 \Rightarrow (2r - 3)^2 = 0 \Rightarrow r = \frac{3}{2}$, repeated twice $\Rightarrow y = c_1 e^{\frac{3}{2}x} + c_2 x e^{\frac{3}{2}x}$
27. $4\frac{d^2 y}{dx^2} + 4\frac{dy}{dx} + y = 0 \Rightarrow 4r^2 + 4r + 1 = 0 \Rightarrow (2r + 1)^2 = 0 \Rightarrow r = -\frac{1}{2}$, repeated twice $\Rightarrow y = c_1 e^{-\frac{1}{2}x} + c_2 x e^{-\frac{1}{2}x}$
28. $4\frac{d^2 y}{dx^2} - 4\frac{dy}{dx} + y = 0 \Rightarrow 4r^2 - 4r + 1 = 0 \Rightarrow (2r - 1)^2 = 0 \Rightarrow r = \frac{1}{2}$, repeated twice $\Rightarrow y = c_1 e^{\frac{1}{2}x} + c_2 x e^{\frac{1}{2}x}$
29. $9\frac{d^2 y}{dx^2} + 6\frac{dy}{dx} + y = 0 \Rightarrow 9r^2 + 6r + 1 = 0 \Rightarrow (3r + 1)^2 = 0 \Rightarrow r = -\frac{1}{3}$, repeated twice $\Rightarrow y = c_1 e^{-\frac{1}{3}x} + c_2 x e^{-\frac{1}{3}x}$
30. $9\frac{d^2 y}{dx^2} - 12\frac{dy}{dx} + 4y = 0 \Rightarrow 9r^2 - 12r + 4 = 0 \Rightarrow (3r - 2)^2 = 0 \Rightarrow r = \frac{2}{3}$, repeated twice $\Rightarrow y = c_1 e^{\frac{2}{3}x} + c_2 x e^{\frac{2}{3}x}$
31. $y'' + 6y' + 5y = 0, y(0) = 0, y'(0) = 3 \Rightarrow r^2 + 6r + 5 = 0 \Rightarrow (r + 5)(r + 1) = 0 \Rightarrow r = -5$ or $r = -1$
 $\Rightarrow y = c_1 e^{-5x} + c_2 e^{-x} \Rightarrow y' = -5c_1 e^{-5x} - c_2 e^{-x}; y(0) = 0 \Rightarrow c_1 + c_2 = 0$, and $y'(0) = 3 \Rightarrow -5c_1 - c_2 = 3$
 $\Rightarrow c_1 = -\frac{3}{4}$ and $c_2 = \frac{3}{4} \Rightarrow y = -\frac{3}{4}e^{-5x} + \frac{3}{4}e^{-x}$
32. $y'' + 16y = 0, y(0) = 2, y'(0) = -2 \Rightarrow r^2 + 16 = 0 \Rightarrow r = 0 \pm 4i \Rightarrow y = c_1 \cos 4x + c_2 \sin 4x$
 $\Rightarrow y' = -4c_1 \sin 4x + 4c_2 \cos 4x; y(0) = 2 \Rightarrow c_1 = 2$, and $y'(0) = -2 \Rightarrow 4c_2 = -2 \Rightarrow c_1 = 2$ and $c_2 = -\frac{1}{2}$
 $\Rightarrow y = 2 \cos 4x - \frac{1}{2} \sin 4x$
33. $y'' + 12y = 0, y(0) = 0, y'(0) = 1 \Rightarrow r^2 + 12 = 0 \Rightarrow r = 0 \pm 2\sqrt{3}i \Rightarrow y = c_1 \cos 2\sqrt{3}x + c_2 \sin 2\sqrt{3}x$
 $\Rightarrow y' = -2\sqrt{3}c_1 \sin 2\sqrt{3}x + 2\sqrt{3}c_2 \cos 2\sqrt{3}x; y(0) = 0 \Rightarrow c_1 = 0$, and $y'(0) = 1 \Rightarrow 2\sqrt{3}c_2 = 1$
 $\Rightarrow c_1 = 0$ and $c_2 = \frac{1}{2\sqrt{3}} \Rightarrow y = \frac{1}{2\sqrt{3}} \sin 2\sqrt{3}x$
34. $12y'' + 5y' - 2y = 0, y(0) = 1, y'(0) = -1 \Rightarrow 12r^2 + 5r - 2 = 0 \Rightarrow (4r - 1)(3r + 2) = 0 \Rightarrow r = \frac{1}{4}$ or $r = -\frac{2}{3}$
 $\Rightarrow y = c_1 e^{(1/4)x} + c_2 e^{-(2/3)x} \Rightarrow y' = \frac{1}{4}c_1 e^{(1/4)x} - \frac{2}{3}c_2 e^{-(2/3)x}; y(0) = 1 \Rightarrow c_1 + c_2 = 1$, and $y'(0) = -1$
 $\Rightarrow \frac{1}{4}c_1 - \frac{2}{3}c_2 = -1 \Rightarrow c_1 = -\frac{4}{11}$ and $c_2 = \frac{15}{11} \Rightarrow y = -\frac{4}{11}e^{(1/4)x} + \frac{15}{11}e^{-(2/3)x}$
35. $y'' + 8y = 0, y(0) = -1, y'(0) = 2 \Rightarrow r^2 + 8 = 0 \Rightarrow r = 0 \pm 2\sqrt{2}i \Rightarrow y = c_1 \cos 2\sqrt{2}x + c_2 \sin 2\sqrt{2}x$
 $\Rightarrow y' = -2\sqrt{2}c_1 \sin 2\sqrt{2}x + 2\sqrt{2}c_2 \cos 2\sqrt{2}x; y(0) = -1 \Rightarrow c_1 = -1$, and $y'(0) = 2 \Rightarrow 2\sqrt{2}c_2 = 2$
 $\Rightarrow c_1 = -1$ and $c_2 = \frac{1}{\sqrt{2}} \Rightarrow y = -\cos 2\sqrt{2}x + \frac{1}{\sqrt{2}} \sin 2\sqrt{2}x$

36. $y'' + 4y' + 4y = 0, y(0) = 0, y'(0) = 1 \Rightarrow r^2 + 4r + 4 = 0 \Rightarrow (r + 2)^2 = 0 \Rightarrow r = -2$ repeated twice
 $\Rightarrow y = c_1 e^{-2x} + c_2 x e^{-2x} \Rightarrow y' = -2c_1 e^{-2x} + c_2 e^{-2x} - 2c_2 x e^{-2x}; y(0) = 0 \Rightarrow c_1 = 0$, and
 $y'(0) = 1 \Rightarrow -2c_1 + c_2 = 1 \Rightarrow c_1 = 0$ and $c_2 = 1 \Rightarrow y = x e^{-2x}$
37. $y'' - 4y' + 4y = 0, y(0) = 1, y'(0) = 0 \Rightarrow r^2 - 4r + 4 = 0 \Rightarrow (r - 2)^2 = 0 \Rightarrow r = 2$ repeated twice
 $\Rightarrow y = c_1 e^{2x} + c_2 x e^{2x} \Rightarrow y' = 2c_1 e^{2x} + c_2 e^{2x} + 2c_2 x e^{2x}; y(0) = 1 \Rightarrow c_1 = 1$, and $y'(0) = 0 \Rightarrow 2c_1 + c_2 = 0$
 $\Rightarrow c_1 = 1$ and $c_2 = -2 \Rightarrow y = e^{2x} - 2x e^{2x}$
38. $4y'' - 4y' + y = 0, y(0) = 4, y'(0) = 4 \Rightarrow 4r^2 - 4r + 1 = 0 \Rightarrow (2r - 1)^2 = 0 \Rightarrow r = \frac{1}{2}$ repeated twice
 $\Rightarrow y = c_1 e^{\frac{1}{2}x} + c_2 x e^{\frac{1}{2}x} \Rightarrow y' = \frac{1}{2}c_1 e^{\frac{1}{2}x} + c_2 e^{\frac{1}{2}x} + \frac{1}{2}c_2 x e^{\frac{1}{2}x}; y(0) = 4 \Rightarrow c_1 = 4$,
and $y'(0) = 4 \Rightarrow \frac{1}{2}c_1 + c_2 = 4 \Rightarrow c_1 = 4$ and $c_2 = 2 \Rightarrow y = 4e^{\frac{1}{2}x} + 2x e^{\frac{1}{2}x}$
39. $4\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 9y = 0, y(0) = 2, \frac{dy}{dx}(0) = 1 \Rightarrow 4r^2 + 12r + 9 = 0 \Rightarrow (2r + 3)^2 = 0 \Rightarrow r = -\frac{3}{2}$ repeated twice
 $\Rightarrow y = c_1 e^{-\frac{3}{2}x} + c_2 x e^{-\frac{3}{2}x} \Rightarrow \frac{dy}{dx} = -\frac{3}{2}c_1 e^{-\frac{3}{2}x} + c_2 e^{-\frac{3}{2}x} - \frac{3}{2}c_2 x e^{-\frac{3}{2}x}; y(0) = 2 \Rightarrow c_1 = 2$, and $\frac{dy}{dx}(0) = 1$
 $\Rightarrow -\frac{3}{2}c_1 + c_2 = 1 \Rightarrow c_1 = 2$ and $c_2 = 4 \Rightarrow y = 2e^{-\frac{3}{2}x} + 4x e^{-\frac{3}{2}x}$
40. $9\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 4y = 0, y(0) = -1, \frac{dy}{dx}(0) = 1 \Rightarrow 9r^2 - 12r + 4 = 0 \Rightarrow (3r - 2)^2 = 0 \Rightarrow r = \frac{2}{3}$ repeated twice
 $\Rightarrow y = c_1 e^{\frac{2}{3}x} + c_2 x e^{\frac{2}{3}x} \Rightarrow \frac{dy}{dx} = \frac{2}{3}c_1 e^{\frac{2}{3}x} + c_2 e^{\frac{2}{3}x} + \frac{2}{3}c_2 x e^{\frac{2}{3}x}; y(0) = -1 \Rightarrow c_1 = -1$, and $\frac{dy}{dx}(0) = 1$
 $\Rightarrow \frac{2}{3}c_1 + c_2 = 1 \Rightarrow c_1 = -1$ and $c_2 = \frac{5}{3} \Rightarrow y = -e^{\frac{2}{3}x} + \frac{5}{3}x e^{\frac{2}{3}x}$
41. $y'' - 2y' - 3y = 0 \Rightarrow r^2 - 2r - 3 = 0 \Rightarrow (r - 3)(r + 1) = 0 \Rightarrow r = 3$ or $r = -1 \Rightarrow y = c_1 e^{3x} + c_2 e^{-x}$
42. $6y'' - y' - y = 0 \Rightarrow 6r^2 - r - 1 = 0 \Rightarrow (2r - 1)(3r + 1) = 0 \Rightarrow r = \frac{1}{2}$ or $r = -\frac{1}{3} \Rightarrow y = c_1 e^{\frac{1}{2}x} + c_2 e^{-\frac{1}{3}x}$
43. $4y'' + 4y' + y = 0 \Rightarrow 4r^2 + 4r + 1 = 0 \Rightarrow (2r + 1)^2 = 0 \Rightarrow r = -\frac{1}{2}$ repeated twice $\Rightarrow y = c_1 e^{-\frac{1}{2}x} + c_2 x e^{-\frac{1}{2}x}$
44. $9y'' + 12y' + 4y = 0 \Rightarrow 9r^2 + 12r + 4 = 0 \Rightarrow (3r + 2)^2 = 0 \Rightarrow r = -\frac{2}{3}$ repeated twice $\Rightarrow y = c_1 e^{-\frac{2}{3}x} + c_2 x e^{-\frac{2}{3}x}$
45. $4y'' + 20y = 0 \Rightarrow 4r^2 + 20 = 0 \Rightarrow r = 0 \pm 5i \Rightarrow y = e^{0x} (c_1 \cos \sqrt{5}x + c_2 \sin \sqrt{5}x) \Rightarrow y = c_1 \cos \sqrt{5}x + c_2 \sin \sqrt{5}x$
46. $y'' + 2y' + 2y = 0 \Rightarrow r^2 + 2r + 2 = 0 \Rightarrow r = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(2)}}{2(1)} = -1 \pm i \Rightarrow y = e^{-x}(c_1 \cos x + c_2 \sin x)$
47. $25y'' + 10y' + y = 0 \Rightarrow 25r^2 + 10r + 1 = 0 \Rightarrow (5r + 1)^2 = 0 \Rightarrow r = -\frac{1}{5}$ repeated twice $\Rightarrow y = c_1 e^{-\frac{1}{5}x} + c_2 x e^{-\frac{1}{5}x}$
48. $6y'' + 13y' - 5y = 0 \Rightarrow 6r^2 + 13r - 5 = 0 \Rightarrow (3r - 1)(2r + 5) = 0 \Rightarrow r = \frac{1}{3}$ or $r = -\frac{5}{2} \Rightarrow y = c_1 e^{\frac{1}{3}x} + c_2 e^{-\frac{5}{2}x}$
49. $4y'' + 4y' + 5y = 0 \Rightarrow 4r^2 + 4r + 5 = 0 \Rightarrow r = \frac{-4 \pm \sqrt{(4)^2 - 4(4)(5)}}{2(4)} = -\frac{1}{2} \pm i \Rightarrow y = e^{-\frac{1}{2}x}(c_1 \cos x + c_2 \sin x)$
50. $y'' + 4y' + 6y = 0 \Rightarrow r^2 + 4r + 6 = 0 \Rightarrow r = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(6)}}{2(1)} = -2 \pm \sqrt{2}i \Rightarrow y = e^{-2x}(c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x)$
51. $16y'' - 24y' + 9y = 0 \Rightarrow 16r^2 - 24r + 9 = 0 \Rightarrow (4r - 3)^2 = 0 \Rightarrow r = \frac{3}{4}$ repeated twice $\Rightarrow y = c_1 e^{\frac{3}{4}x} + c_2 x e^{\frac{3}{4}x}$

$$52. 6y'' - 5y' - 6y = 0 \Rightarrow 6r^2 - 5r - 6 = 0 \Rightarrow (2r - 3)(3r + 2) = 0 \Rightarrow r = \frac{3}{2} \text{ or } r = -\frac{2}{3} \Rightarrow y = c_1 e^{\frac{3}{2}x} + c_2 e^{-\frac{2}{3}x}$$

$$53. 9y'' + 24y' + 16y = 0 \Rightarrow 9r^2 + 24r + 16 = 0 \Rightarrow (3r + 4)^2 = 0 \Rightarrow r = -\frac{4}{3} \text{ repeated twice} \Rightarrow y = c_1 e^{-\frac{4}{3}x} + c_2 x e^{-\frac{4}{3}x}$$

$$54. 4y'' + 16y' + 52y = 0 \Rightarrow 4r^2 + 16r + 52 = 0 \Rightarrow r = \frac{-16 \pm \sqrt{(16)^2 - 4(4)(52)}}{2(4)} = -2 \pm 3i \Rightarrow y = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x)$$

$$55. 6y'' - 5y' - 4y = 0 \Rightarrow 6r^2 - 5r - 4 = 0 \Rightarrow (3r - 4)(2r + 1) = 0 \Rightarrow r = \frac{4}{3} \text{ or } r = -\frac{1}{2} \Rightarrow y = c_1 e^{\frac{4}{3}x} + c_2 e^{-\frac{1}{2}x}$$

$$56. y'' - 2y' + 2y = 0, y(0) = 0, y'(0) = 2 \Rightarrow r^2 - 2r + 2 = 0 \Rightarrow r = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} = 1 \pm i$$

$$\Rightarrow y = e^x(c_1 \cos x + c_2 \sin x) \Rightarrow y' = e^x(c_1 \cos x + c_2 \sin x) + e^x(-c_1 \sin x + c_2 \cos x); y(0) = 0 \Rightarrow c_1 = 0, \text{ and}$$

$$y'(0) = 2 \Rightarrow c_1 + c_2 = 2 \Rightarrow c_1 = 0 \text{ and } c_2 = 2 \Rightarrow y = 2e^x \sin x$$

$$57. y'' + 2y' + y = 0, y(0) = 1, y'(0) = 1 \Rightarrow r^2 + 2r + 1 = 0 \Rightarrow (r + 1)^2 = 0 \Rightarrow r = -1, \text{ repeated twice}$$

$$\Rightarrow y = c_1 e^{-x} + c_2 x e^{-x} \Rightarrow y' = -c_1 e^{-x} - c_2 x e^{-x} + c_2 e^{-x}; y(0) = 1 \Rightarrow c_1 = 1, \text{ and } y'(0) = 1 \Rightarrow -c_1 + c_2 = 1$$

$$\Rightarrow c_1 = 1 \text{ and } c_2 = 2 \Rightarrow y = e^{-x} + 2x e^{-x}$$

$$58. 4y'' - 4y' + y = 0, y(0) = -1, y'(0) = 2 \Rightarrow 4r^2 - 4r + 1 = 0 \Rightarrow (2r - 1)^2 = 0 \Rightarrow r = \frac{1}{2}, \text{ repeated twice}$$

$$\Rightarrow y = c_1 e^{\frac{1}{2}x} + c_2 x e^{\frac{1}{2}x} \Rightarrow y' = \frac{1}{2}c_1 e^{\frac{1}{2}x} + \frac{1}{2}c_2 x e^{\frac{1}{2}x} + c_2 e^{\frac{1}{2}x}; y(0) = -1 \Rightarrow c_1 = -1, \text{ and } y'(0) = 2$$

$$\Rightarrow \frac{1}{2}c_1 + c_2 = 2 \Rightarrow c_1 = -1 \text{ and } c_2 = \frac{5}{2} \Rightarrow y = -e^{\frac{1}{2}x} + \frac{5}{2}x e^{\frac{1}{2}x}$$

$$59. 3y'' + y' - 14y = 0, y(0) = 2, y'(0) = -1 \Rightarrow 3r^2 + r - 14 = 0 \Rightarrow (3r + 7)(r - 2) = 0 \Rightarrow r = -\frac{7}{3} \text{ or } r = 2$$

$$\Rightarrow y = c_1 e^{-\frac{7}{3}x} + c_2 e^{2x} \Rightarrow y' = -\frac{7}{3}c_1 e^{-\frac{7}{3}x} + 2c_2 e^{2x}; y(0) = 2 \Rightarrow c_1 + c_2 = 2, \text{ and } y'(0) = -1 \Rightarrow -\frac{7}{3}c_1 + 2c_2 = -1$$

$$\Rightarrow c_1 = \frac{15}{13} \text{ and } c_2 = \frac{11}{13} \Rightarrow y = \frac{15}{13}e^{-(7/3)x} + \frac{11}{13}e^{2x}$$

$$60. 4y'' + 4y' + 5y = 0, y(\pi) = 1, y'(\pi) = 0 \Rightarrow 4r^2 + 4r + 5 = 0 \Rightarrow r = \frac{-4 \pm \sqrt{(4)^2 - 4(4)(5)}}{2(4)} = -\frac{1}{2} \pm i$$

$$\Rightarrow y = e^{-\frac{1}{2}x}(c_1 \cos x + c_2 \sin x) \Rightarrow y' = -\frac{1}{2}e^{-\frac{1}{2}x}(c_1 \cos x + c_2 \sin x) + e^{-\frac{1}{2}x}(-c_1 \sin x + c_2 \cos x); y(\pi) = 1$$

$$\Rightarrow -e^{-\frac{\pi}{2}}c_1 = 1, \text{ and } y'(\pi) = 0 \Rightarrow -\frac{1}{2}e^{-\frac{\pi}{2}}c_1 + e^{-\frac{\pi}{2}}c_2 = 0 \Rightarrow c_1 = -e^{\frac{\pi}{2}} \text{ and } c_2 = -\frac{1}{2}e^{\frac{\pi}{2}}$$

$$\Rightarrow y = e^{-\frac{1}{2}x}\left(-e^{\frac{\pi}{2}}\cos x - \frac{1}{2}e^{\frac{\pi}{2}}\sin x\right)$$

61. Let r_1 and r_2 be real roots with $r_1 \neq r_2$. If $e^{r_1 x}$ and $e^{r_2 x}$ are linearly independent, then $e^{r_1 x}$ is not a constant multiple of $e^{r_2 x}$ (and vice versa). Assume that $e^{r_1 x}$ is a constant multiple of $e^{r_2 x}$, then for some nonzero constant c , $e^{r_1 x} = c e^{r_2 x} \Rightarrow \frac{e^{r_1 x}}{e^{r_2 x}} = c$
 $\Rightarrow e^{r_1 x - r_2 x} = c \Rightarrow e^{(r_1 - r_2)x} = c$. Since $r_1 \neq r_2$, c is not a constant, which is a contradiction. Thus $e^{r_1 x}$ and $e^{r_2 x}$ are linearly independent.

62. Let r be the only repeated real root. If e^{rx} and $x e^{rx}$ are linearly independent, then e^{rx} is not a constant multiple of $x e^{rx}$ (and vice versa). Assume that $x e^{rx}$ is a constant multiple of e^{rx} , then for some nonzero constant c , $x e^{rx} = c e^{rx}$
 $\Rightarrow x e^{rx} - c e^{rx} = 0 \Rightarrow e^{rx}(x - c) = 0 \Rightarrow e^{rx} = 0$ or $x - c = 0$. Since $e^{rx} \neq 0 \Rightarrow x = c$, thus c is not a constant, which is a contradiction. Thus e^{rx} and $x e^{rx}$ are linearly independent.

63. Let $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$ be complex roots. If $e^{\alpha x} \cos \beta x$ and $e^{\alpha x} \sin \beta x$ are linearly independent, then $e^{\alpha x} \cos \beta x$ is not a constant multiple of $e^{\alpha x} \sin \beta x$ (and vice versa). Assume that $e^{\alpha x} \cos \beta x$ is a constant multiple of $e^{\alpha x} \sin \beta x$, then for some nonzero constant c , $e^{\alpha x} \cos \beta x = c e^{\alpha x} \sin \beta x \Rightarrow e^{\alpha x} \cos \beta x - c e^{\alpha x} \sin \beta x = 0 \Rightarrow e^{\alpha x} (\cos \beta x - c \sin \beta x) = 0 \Rightarrow e^{\alpha x} = 0$ or $\cos \beta x - c \sin \beta x = 0$. Since $e^{\alpha x} \neq 0 \Rightarrow c = \cot \beta x$, thus c is not a constant, which is a contradiction. Thus $e^{\alpha x} \cos \beta x$ and $e^{\alpha x} \sin \beta x$ are linearly independent.

64. Let y_1 and y_2 be linearly independent solutions of $P(x)y'' + Q(x)y' + R(x)y = 0$. Let $y_3 = y_1 + y_2 \Rightarrow y_3' = y_1' + y_2'$
 $\Rightarrow y_3'' = y_1'' + y_2''$. Then $P(x)y_3'' + Q(x)y_3' + R(x)y_3 = P(x)(y_1'' + y_2'') + Q(x)(y_1' + y_2') + R(x)(y_1 + y_2)$
 $= P(x)y_1'' + P(x)y_2'' + Q(x)y_1' + Q(x)y_2' + R(x)y_1 + R(x)y_2$
 $= P(x)y_1'' + Q(x)y_1' + R(x)y_1 + P(x)y_2'' + Q(x)y_2' + R(x)y_2 = 0 + 0 = 0$. Let $y_4 = y_1 - y_2 \Rightarrow y_4' = y_1' - y_2'$
 $\Rightarrow y_4'' = y_1'' - y_2''$. Then $P(x)y_4'' + Q(x)y_4' + R(x)y_4 = P(x)(y_1'' - y_2'') + Q(x)(y_1' - y_2') + R(x)(y_1 - y_2)$
 $= P(x)y_1'' - P(x)y_2'' + Q(x)y_1' - Q(x)y_2' + R(x)y_1 - R(x)y_2$
 $= P(x)y_1'' + Q(x)y_1' + R(x)y_1 - [P(x)y_2'' + Q(x)y_2' + R(x)y_2] = 0 - 0 = 0$. Thus y_3 and y_4 are both solutions.
 Suppose that y_3 is a constant multiple of y_4 , then there is a nonzero constant c such that $y_3 = c y_4$.
 $\Rightarrow y_1 + y_2 = c(y_1 - y_2)$. If we solve this equation for y_1 we obtain $y_1 = -\frac{1+c}{1-c}y_2 \Rightarrow y_1$ is a constant multiple of y_2 , which is a contradiction since y_1 and y_2 are linearly independent $\Rightarrow y_1 + y_2$ and $y_1 - y_2$ are linearly independent.

65. (a) $y'' + 4y = 0$, $y(0) = 0$, $y(\pi) = 1 \Rightarrow r^2 + 4 = 0 \Rightarrow r = 0 \pm 2i \Rightarrow y = e^{0 \cdot x}(c_1 \cos 2x + c_2 \sin 2x)$
 $\Rightarrow y = c_1 \cos 2x + c_2 \sin 2x$; $y(0) = 0 \Rightarrow c_1 = 0$, and $y(\pi) = 1 \Rightarrow c_2 = 1 \Rightarrow$ no solution
 (b) $y'' + 4y = 0$, $y(0) = 0$, $y(\pi) = 0 \Rightarrow r^2 + 4 = 0 \Rightarrow r = 0 \pm 2i \Rightarrow y = e^{0 \cdot x}(c_1 \cos 2x + c_2 \sin 2x)$
 $\Rightarrow y = c_1 \cos 2x + c_2 \sin 2x$; $y(0) = 0 \Rightarrow c_1 = 0$, and $y(\pi) = 0 \Rightarrow c_2 = 0 \Rightarrow c_1 = 0$, c_2 can be any real number
 $\Rightarrow y = c_2 \sin 2x$

66. Let a , b , and c be positive constants, then $r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$. There are three cases to consider.

Case I: Two distinct real solutions $\Rightarrow b^2 - 4ac > 0$. Since a , b , and c are positive $\Rightarrow 4ac > 0$ and $b > 0$
 $\Rightarrow -4ac < 0$ and $-b < 0 \Rightarrow 0 < b^2 - 4ac < b^2 \Rightarrow 0 < \sqrt{b^2 - 4ac} < b \Rightarrow -b + \sqrt{b^2 - 4ac} < 0$
 and $-b - \sqrt{b^2 - 4ac} < 0$. Since $2a > 0 \Rightarrow \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} < 0$. Thus $r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} < 0$ and
 $r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} < 0$. The general solution to the differential equation is $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$.
 $\lim_{x \rightarrow \infty} (c_1 e^{r_1 x} + c_2 e^{r_2 x}) = 0 + 0 = 0$.

Case II: One repeated real solution $\Rightarrow b^2 - 4ac = 0$. Since a , b , and c are positive $\Rightarrow 2a > 0$ and $-b < 0$
 $\Rightarrow r = -\frac{b}{2a} < 0$. The general solution to the differential equation is $y = c_1 e^{rx} + c_2 x e^{rx}$. Since
 $r < 0 \Rightarrow -r > 0$. $\lim_{x \rightarrow \infty} (c_1 e^{rx} + c_2 x e^{rx}) = \lim_{x \rightarrow \infty} (c_1 e^{rx} + \frac{c_2 x}{e^{-rx}}) = \lim_{x \rightarrow \infty} (c_1 e^{rx}) + \lim_{x \rightarrow \infty} (\frac{c_2 x}{e^{-rx}})$
 $= 0 + \lim_{x \rightarrow \infty} (\frac{c_2}{-r e^{-rx}}) = 0$, using L'Hopital's rule to evaluate the limit of the second expression.

Case III: Two complex, nonreal solutions $\Rightarrow b^2 - 4ac < 0 \Rightarrow 4ac - b^2 > 0$. Since a , b , and c are positive $\Rightarrow 2a > 0$ and
 $-b < 0 \Rightarrow -\frac{b}{2a} < 0$. The roots are $r_1 = -\frac{b}{2a} + \frac{\sqrt{4ac - b^2}}{2a}i$ and $r_2 = -\frac{b}{2a} - \frac{\sqrt{4ac - b^2}}{2a}i$, thus $\alpha = -\frac{b}{2a} < 0$ and
 $\beta = \frac{\sqrt{4ac - b^2}}{2a} > 0$. The general solution to the differential equation is $y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$. Since
 $-1 \leq \sin \beta x \leq 1$ and $-1 \leq \cos \beta x \leq 1 \Rightarrow -|c_2| \leq c_2 \sin \beta x \leq |c_2|$ and $-|c_1| \leq c_1 \cos \beta x \leq |c_1|$
 $\Rightarrow -|c_1| - |c_2| \leq c_1 \cos \beta x + c_2 \sin \beta x \leq |c_1| + |c_2|$
 $\Rightarrow e^{\alpha x}(-|c_1| - |c_2|) \leq e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) \leq e^{\alpha x}(|c_1| + |c_2|)$
 $\lim_{x \rightarrow \infty} [e^{\alpha x}(-|c_1| - |c_2|)] = (-|c_1| - |c_2|) \lim_{x \rightarrow \infty} e^{\alpha x} = 0$ and $\lim_{x \rightarrow \infty} [e^{\alpha x}(|c_1| + |c_2|)] = (|c_1| + |c_2|) \lim_{x \rightarrow \infty} e^{\alpha x} = 0$
 thus by the Sandwich theorem, $\lim_{x \rightarrow \infty} [e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)] = 0$.

17.2 NONHOMOGENEOUS LINEAR EQUATIONS

1. $y'' - 3y' - 10y = -3 \Rightarrow r^2 - 3r - 10 = 0 \Rightarrow (r - 5)(r + 2) = 0 \Rightarrow r = 5 \text{ or } r = -2 \Rightarrow y_c = c_1 e^{5x} + c_2 e^{-2x}; y_p = A$
 $\Rightarrow y'_p = 0 \Rightarrow y''_p = 0 \Rightarrow 0 - 3(0) - 10A = -3 \Rightarrow A = \frac{3}{10} \Rightarrow y = c_1 e^{5x} + c_2 e^{-2x} + \frac{3}{10}$
2. $y'' - 3y' - 10y = 2x - 3 \Rightarrow r^2 - 3r - 10 = 0 \Rightarrow (r - 5)(r + 2) = 0 \Rightarrow r = 5 \text{ or } r = -2 \Rightarrow y_c = c_1 e^{5x} + c_2 e^{-2x};$
 $y_p = Ax + B \Rightarrow y'_p = A \Rightarrow y''_p = 0 \Rightarrow 0 - 3A - 10(Ax + B) = 2x - 3 \Rightarrow -10Ax + (-3A - 10B) = 2x - 3$
 $\Rightarrow -10A = 2, -3A - 10B = -3 \Rightarrow A = -\frac{1}{5}, B = \frac{9}{25} \Rightarrow y = c_1 e^{5x} + c_2 e^{-2x} - \frac{1}{5}x + \frac{9}{25}$
3. $y'' - y' = \sin x \Rightarrow r^2 - r = 0 \Rightarrow r(r - 1) = 0 \Rightarrow r = 0 \text{ or } r = 1 \Rightarrow y_c = c_1 e^{0 \cdot x} + c_2 e^x = c_1 + c_2 e^x;$
 $y_p = A \sin x + B \cos x \Rightarrow y'_p = A \cos x - B \sin x \Rightarrow y''_p = -A \sin x - B \cos x$
 $\Rightarrow -A \sin x - B \cos x - (A \cos x - B \sin x) = \sin x \Rightarrow (-A + B) \sin x + (-A - B) \cos x = \sin x$
 $\Rightarrow -A + B = 1, -A - B = 0 \Rightarrow A = -\frac{1}{2}, B = \frac{1}{2} \Rightarrow y = c_1 + c_2 e^x - \frac{1}{2} \sin x + \frac{1}{2} \cos x$
4. $y'' + 2y' + y = x^2 \Rightarrow r^2 + 2r + 1 = 0 \Rightarrow (r + 1)^2 = 0 \Rightarrow r = -1 \text{ repeated twice} \Rightarrow y_c = c_1 e^{-x} + c_2 x e^{-x};$
 $y_p = Ax^2 + Bx + C \Rightarrow y'_p = 2Ax + B \Rightarrow y''_p = 2A \Rightarrow (Ax^2 + Bx + C) + 2(2Ax + B) + 2A = x^2$
 $\Rightarrow Ax^2 + (4A + B)x + (2A + 2B + C) = x^2 \Rightarrow A = 1, 4A + B = 0, 2A + 2B + C = 0 \Rightarrow A = 1, B = -4,$
 $C = 6 \Rightarrow y = c_1 e^{-x} + c_2 x e^{-x} + x^2 - 4x + 6$
5. $y'' + y = \cos 3x \Rightarrow r^2 + 1 = 0 \Rightarrow r = 0 \pm i \Rightarrow y_c = e^{0 \cdot x}(c_1 \cos x + c_2 \sin x) = c_1 \cos x + c_2 \sin x;$
 $y_p = A \sin 3x + B \cos 3x \Rightarrow y'_p = 3A \cos 3x - 3B \sin 3x \Rightarrow y''_p = -9A \sin 3x - 9B \cos 3x$
 $\Rightarrow -9A \sin 3x - 9B \cos 3x + (A \sin 3x + B \cos 3x) = \cos 3x \Rightarrow -8A \sin x - 8B \cos x = \cos 3x$
 $\Rightarrow -8A = 0, -8B = 1 \Rightarrow A = 0, B = -\frac{1}{8} \Rightarrow y = c_1 \cos x + c_2 \sin x - \frac{1}{8} \cos 3x$
6. $y'' + y = e^{2x} \Rightarrow r^2 + 1 = 0 \Rightarrow r = 0 \pm i \Rightarrow y_c = e^{0 \cdot x}(c_1 \cos x + c_2 \sin x) = c_1 \cos x + c_2 \sin x;$
 $y_p = A e^{2x} \Rightarrow y'_p = 2A e^{2x} \Rightarrow y''_p = 4A e^{2x} \Rightarrow 4A e^{2x} + A e^{2x} = e^{2x} \Rightarrow 5A e^{2x} = e^{2x} \Rightarrow 5A = 1$
 $\Rightarrow A = \frac{1}{5} \Rightarrow y = c_1 \cos x + c_2 \sin x + \frac{1}{5} e^{2x}$
7. $y'' - y' - 2y = 20 \cos x \Rightarrow r^2 - r - 2 = 0 \Rightarrow (r - 2)(r + 1) = 0 \Rightarrow r = 2 \text{ or } r = -1 \Rightarrow y_c = c_1 e^{2x} + c_2 e^{-x};$
 $y_p = A \sin x + B \cos x \Rightarrow y'_p = A \cos x - B \sin x \Rightarrow y''_p = -A \sin x - B \cos x$
 $\Rightarrow -A \sin x - B \cos x - (A \cos x - B \sin x) - 2(A \sin x + B \cos x) = 20 \cos x$
 $\Rightarrow (-3A + B) \sin x + (-A - 3B) \cos x = 20 \cos x \Rightarrow -3A + B = 0, -A - 3B = 20 \Rightarrow A = -2, B = -6$
 $\Rightarrow y = c_1 e^{2x} + c_2 e^{-x} - 2 \sin x - 6 \cos x$
8. $y'' + y = 2x + 3e^x \Rightarrow r^2 + 1 = 0 \Rightarrow r = 0 \pm i \Rightarrow y_c = c_1 \cos x + c_2 \sin x; y_p = Ax + B + Ce^x \Rightarrow y'_p = A + Ce^x$
 $\Rightarrow y''_p = Ce^x \Rightarrow Ce^x + (Ax + B + Ce^x) = 2x + 3e^x \Rightarrow Ax + B + 2Ce^x = 2x + 3e^x \Rightarrow A = 2, B = 0, 2C = 3$
 $\Rightarrow A = 2, B = 0, C = \frac{3}{2} \Rightarrow y = c_1 \cos x + c_2 \sin x + 2x + \frac{3}{2} e^x$
9. $y'' - y = e^x + x^2 \Rightarrow r^2 - 1 = 0 \Rightarrow (r - 1)(r + 1) = 0 \Rightarrow r = 1 \text{ or } r = -1 \Rightarrow y_c = c_1 e^x + c_2 e^{-x};$
 $y_p = Ax e^x + Bx^2 + Cx + D \Rightarrow y'_p = Ae^x + Ax e^x + 2Bx + C \Rightarrow y''_p = 2Ae^x + Ax e^x + 2B$
 $\Rightarrow (2Ae^x + Ax e^x + 2B) - (Ax e^x + Bx^2 + Cx + D) = e^x + x^2 \Rightarrow 2Ae^x - Bx^2 - Cx + (2B - D) = e^x + x^2$
 $\Rightarrow 2A = 1, -B = 1, -C = 0, 2B - D = 0 \Rightarrow A = \frac{1}{2}, B = -1, C = 0, D = -2$
 $\Rightarrow y = c_1 e^x + c_2 e^{-x} + \frac{1}{2} x e^x - x^2 - 2$

10. $y'' + 2y' + y = 6\sin 2x \Rightarrow r^2 + 2r + 1 = 0 \Rightarrow (r + 1)^2 = 0 \Rightarrow r = -1$, repeated twice $\Rightarrow y_c = c_1e^{-x} + c_2xe^{-x}$;
 $y_p = A \sin 2x + B \cos 2x \Rightarrow y_p' = 2A \cos 2x - 2B \sin 2x \Rightarrow y_p'' = -4A \sin 2x - 4B \cos 2x$
 $\Rightarrow -4A \sin 2x - 4B \cos 2x + 2(2A \cos 2x - 2B \sin 2x) + (A \sin 2x + B \cos 2x) = 6\sin 2x$
 $\Rightarrow (-3A - 4B)\sin 2x + (4A - 3B)\cos 2x = 6\sin 2x \Rightarrow -3A - 4B = 6, 4A - 3B = 0 \Rightarrow A = -\frac{24}{25}, B = -\frac{18}{25}$
 $\Rightarrow y = c_1e^{-x} + c_2xe^{-x} - \frac{24}{25}\sin 2x - \frac{18}{25}\cos 2x$
11. $y'' - y' - 6y = e^{-x} - 7\cos x \Rightarrow r^2 - r - 6 = 0 \Rightarrow (r - 3)(r + 2) = 0 \Rightarrow r = 3$ or $r = -2 \Rightarrow y_c = c_1e^{3x} + c_2e^{-2x}$;
 $y_p = Ae^{-x} + B \sin x + C \cos x \Rightarrow y_p' = -Ae^{-x} + B \cos x - C \sin x \Rightarrow y_p'' = Ae^{-x} - B \sin x - C \cos x$
 $\Rightarrow Ae^{-x} - B \sin x - C \cos x - (-Ae^{-x} + B \cos x - C \sin x) - 6(Ae^{-x} + B \sin x + C \cos x) = e^{-x} - 7\cos x$
 $\Rightarrow -4Ae^{-x} + (-7B + C)\sin x + (-B - 7C)\cos x = e^{-x} - 7\cos x \Rightarrow -4A = 1, -7B + C = 0, -B - 7C = -7$
 $\Rightarrow A = -\frac{1}{4}, B = \frac{7}{50}, C = \frac{49}{50} \Rightarrow y = c_1e^{3x} + c_2e^{-2x} - \frac{1}{4}e^{-x} + \frac{7}{50}\sin x + \frac{49}{50}\cos x$
12. $y'' + 3y' + 2y = e^{-x} + e^{-2x} - x \Rightarrow r^2 + 3r + 2 = 0 \Rightarrow (r + 1)(r + 2) = 0 \Rightarrow r = -1$ or $r = -2 \Rightarrow y_c = c_1e^{-x} + c_2e^{-2x}$;
 $y_p = Ax e^{-x} + Bx e^{-2x} + Cx + D \Rightarrow y_p' = Ae^{-x} - Ax e^{-x} + B e^{-2x} - 2Bx e^{-2x} + C$
 $\Rightarrow y_p'' = -2Ae^{-x} + Ax e^{-x} - 4B e^{-2x} + 4Bx e^{-2x} \Rightarrow -2Ae^{-x} + Ax e^{-x} - 4B e^{-2x} + 4Bx e^{-2x}$
 $+ 3(Ae^{-x} - Ax e^{-x} + B e^{-2x} - 2Bx e^{-2x} + C) + 2(Ax e^{-x} + Bx e^{-2x} + Cx + D) = e^{-x} + e^{-2x} - x$
 $\Rightarrow Ax e^{-x} - B e^{-2x} + 2Cx + (3C + 2D) = e^{-x} + e^{-2x} - x \Rightarrow A = 1, -B = 1, 2C = -1, 3C + 2D = 0$
 $\Rightarrow A = 1, B = -1, C = -\frac{1}{2}, D = \frac{1}{4} \Rightarrow y = c_1e^{-x} + c_2e^{-2x} + x e^{-x} - x e^{-2x} - \frac{1}{2}x + \frac{1}{4}$
13. $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} = 15x^2 \Rightarrow r^2 + 5r = 0 \Rightarrow r(r + 5) = 0 \Rightarrow r = 0$ or $r = -5 \Rightarrow y_c = c_1e^{0 \cdot x} + c_2e^{-5x} = c_1 + c_2e^{-5x}$;
 $y_p = Ax^3 + Bx^2 + Cx \Rightarrow y_p' = 3Ax^2 + 2Bx + C \Rightarrow y_p'' = 6Ax + 2B \Rightarrow 6Ax + 2B + 5(3Ax^2 + 2Bx + C) = 15x^2$
 $\Rightarrow 15Ax^2 + (6A + 10B)x + (2B + 5C) = 15x^2 \Rightarrow 15A = 15, 6A + 10B = 0, 2B + 5C = 0 \Rightarrow A = 1, B = -\frac{3}{5}, C = \frac{6}{25}$
 $\Rightarrow y = c_1 + c_2e^{-5x} + x^3 - \frac{3}{5}x^2 + \frac{6}{25}x$
14. $\frac{d^2y}{dx^2} - \frac{dy}{dx} = -8x + 3 \Rightarrow r^2 - r = 0 \Rightarrow r(r - 1) = 0 \Rightarrow r = 0$ or $r = 1 \Rightarrow y_c = c_1e^{0 \cdot x} + c_2e^{1 \cdot x} = c_1 + c_2e^x$;
 $y_p = Ax^2 + Bx \Rightarrow y_p' = 2Ax + B \Rightarrow y_p'' = 2Ax \Rightarrow 2A - (2Ax + B) = -8x + 3 \Rightarrow -2Ax + (2A - B) = -8x + 3$
 $\Rightarrow -2A = -8, 2A - B = 3 \Rightarrow A = 4, B = 5 \Rightarrow y = c_1 + c_2e^x + 4x^2 + 5x$
15. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} = e^{3x} - 12x \Rightarrow r^2 - 3r = 0 \Rightarrow r(r - 3) = 0 \Rightarrow r = 0$ or $r = 3 \Rightarrow y_c = c_1e^{0 \cdot x} + c_2e^{3x} = c_1 + c_2e^{3x}$;
 $y_p = Ax e^{3x} + Bx^2 + Cx \Rightarrow y_p' = Ae^{3x} + 3Axe^{3x} + 2Bx + C \Rightarrow y_p'' = 6Ae^{3x} + 9Axe^{3x} + 2B$
 $\Rightarrow 6Ae^{3x} + 9Axe^{3x} + 2B - 3(Ae^{3x} + 3Axe^{3x} + 2Bx + C) = e^{3x} - 12x \Rightarrow 3Ae^{3x} - 6Bx + (2B - 3C) = e^{3x} - 12x$
 $\Rightarrow 3A = 1, -6B = 12, 2B - 3C = 0 \Rightarrow A = \frac{1}{3}, B = 2, C = \frac{4}{3} \Rightarrow y = c_1 + c_2e^{3x} + \frac{1}{3}x e^{3x} + 2x^2 + \frac{4}{3}x$
16. $\frac{d^2y}{dx^2} + 7\frac{dy}{dx} = 42x^2 + 5x + 1 \Rightarrow r^2 + 7r = 0 \Rightarrow r(r + 7) = 0 \Rightarrow r = 0$ or $r = -7 \Rightarrow y_c = c_1e^{0 \cdot x} + c_2e^{-7x} = c_1 + c_2e^{-7x}$;
 $y_p = Ax^3 + Bx^2 + Cx \Rightarrow y_p' = 3Ax^2 + 2Bx + C \Rightarrow y_p'' = 6Ax + 2B$
 $\Rightarrow 6Ax + 2B + 7(3Ax^2 + 2Bx + C) = 42x^2 + 5x + 1 \Rightarrow 21Ax^2 + (6A + 14B)x + (2B + 7C) = 42x^2 + 5x + 1$
 $\Rightarrow 21A = 42, 6A + 14B = 5, 2B + 7C = 1 \Rightarrow A = 2, B = -\frac{1}{2}, C = \frac{2}{7} \Rightarrow y = c_1 + c_2e^{-7x} + 2x^3 - \frac{1}{2}x^2 + \frac{2}{7}x$
17. $y'' + y' = x, \Rightarrow r^2 + r = 0 \Rightarrow r(r + 1) = 0 \Rightarrow r = 0$ or $r = -1 \Rightarrow y_c = c_1e^{0 \cdot x} + c_2e^{-x} = c_1 + c_2e^{-x} \Rightarrow y_1 = 1, y_2 = e^{-x}$
 $\Rightarrow v_1' = \begin{vmatrix} 0 & e^{-x} \\ x & -e^{-x} \end{vmatrix} = \frac{-xe^{-x}}{-e^{-x}} = x$ and $v_2' = \begin{vmatrix} 1 & 0 \\ 0 & x \end{vmatrix} = \frac{x}{-e^{-x}} = -xe^x \Rightarrow v_1 = \int x dx = \frac{1}{2}x^2$ and
 $v_2 = \int -xe^x dx = -xe^x + e^x \Rightarrow y_p = \frac{1}{2}x^2(1) + (-xe^x + e^x)e^{-x} = \frac{1}{2}x^2 - x + 1 \Rightarrow y = c_1 + c_2e^{-x} + \frac{1}{2}x^2 - x$

$$18. y'' + y = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2} \Rightarrow r^2 + 1 = 0 \Rightarrow r = 0 \pm i \Rightarrow y_c = c_1 \cos x + c_2 \sin x \Rightarrow y_1 = \cos x, y_2 = \sin x$$

$$\Rightarrow v_1' = \frac{\begin{vmatrix} 0 & \sin x \\ \tan x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{-\sin x \tan x}{1} = -\sin x \cdot \frac{\sin x}{\cos x} = -\frac{\sin^2 x}{\cos x} \text{ and } v_2' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \tan x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{\cos x \tan x}{1} = \cos x \cdot \frac{\sin x}{\cos x} = \sin x$$

$$\Rightarrow v_1 = -\int \frac{\sin^2 x}{\cos x} dx = \int \frac{\cos^2 x - 1}{\cos x} dx = \int \left(\frac{\cos^2 x}{\cos x} - \frac{1}{\cos x} \right) dx = \int (\cos x - \sec x) dx = \sin x - \ln|\sec x + \tan x| \text{ and}$$

$$v_2 = \int \sin x dx = -\cos x \Rightarrow y_p = (\sin x - \ln|\sec x + \tan x|)(\cos x) + (-\cos x)(\sin x) = -(\cos x) \ln|\sec x + \tan x|$$

$$\Rightarrow y = c_1 \cos x + c_2 \sin x - (\cos x) \ln|\sec x + \tan x|$$

$$19. y'' + y = \sin x \Rightarrow r^2 + 1 = 0 \Rightarrow r = 0 \pm i \Rightarrow y_c = c_1 \cos x + c_2 \sin x \Rightarrow y_1 = \cos x, y_2 = \sin x$$

$$\Rightarrow v_1' = \frac{\begin{vmatrix} 0 & \sin x \\ \sin x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{-\sin^2 x}{1} = -\sin^2 x \text{ and } v_2' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \sin x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{\sin x \cos x}{1} = \sin x \cos x$$

$$\Rightarrow v_1 = \int -\sin^2 x dx = \int \frac{\cos 2x - 1}{2} dx = \frac{1}{4} \sin 2x - \frac{1}{2} x, \text{ and } v_2 = \int \sin x \cos x dx = \frac{1}{2} \sin^2 x$$

$$\Rightarrow y_p = \left(\frac{1}{4} \sin 2x - \frac{1}{2} x \right) (\cos x) + \left(\frac{1}{2} \sin^2 x \right) (\sin x) = \frac{1}{2} \sin x \cos^2 x - \frac{1}{2} x \cos x + \frac{1}{2} \sin x - \frac{1}{2} \sin x \cos^2 x$$

$$= -\frac{1}{2} x \cos x + \frac{1}{2} \sin x \Rightarrow y = c_1 \cos x + c_2 \sin x - \frac{1}{2} x \cos x$$

$$20. y'' + 2y' + y = e^x \Rightarrow r^2 + 2r + 1 = 0 \Rightarrow (r + 1)^2 = 0 \Rightarrow r = -1, \text{ repeated twice} \Rightarrow y_c = c_1 e^{-x} + c_2 x e^{-x}$$

$$\Rightarrow y_1 = e^{-x}, y_2 = x e^{-x}$$

$$\Rightarrow v_1' = \frac{\begin{vmatrix} 0 & x e^{-x} \\ e^x & e^{-x} - x e^{-x} \end{vmatrix}}{\begin{vmatrix} e^{-x} & x e^{-x} \\ -e^{-x} & e^{-x} - x e^{-x} \end{vmatrix}} = \frac{-x}{e^{-2x}} = -x e^{2x} \text{ and } v_2' = \frac{\begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & e^x \end{vmatrix}}{\begin{vmatrix} e^{-x} & x e^{-x} \\ -e^{-x} & e^{-x} - x e^{-x} \end{vmatrix}} = \frac{1}{e^{-2x}} = e^{2x} \Rightarrow v_1 = -\int x e^{2x} dx = -\frac{1}{2} x e^{2x} + \frac{1}{4} e^{2x}$$

$$\text{and } v_2 = \int e^{2x} dx = \frac{1}{2} e^{2x} \Rightarrow y_p = \left(-\frac{1}{2} x e^{2x} + \frac{1}{4} e^{2x} \right) (e^{-x}) + \left(\frac{1}{2} e^{2x} \right) (x e^{-x}) = \frac{1}{4} e^x \Rightarrow y = c_1 e^{-x} + c_2 x e^{-x} + \frac{1}{4} e^x$$

$$21. y'' + 2y' + y = e^{-x} \Rightarrow r^2 + 2r + 1 = 0 \Rightarrow (r + 1)^2 = 0 \Rightarrow r = -1, \text{ repeated twice} \Rightarrow y_c = c_1 e^{-x} + c_2 x e^{-x}$$

$$\Rightarrow y_1 = e^{-x}, y_2 = x e^{-x}$$

$$\Rightarrow v_1' = \frac{\begin{vmatrix} 0 & x e^{-x} \\ e^{-x} & e^{-x} - x e^{-x} \end{vmatrix}}{\begin{vmatrix} e^{-x} & x e^{-x} \\ -e^{-x} & e^{-x} - x e^{-x} \end{vmatrix}} = \frac{-x e^{-2x}}{e^{-2x}} = -x \text{ and } v_2' = \frac{\begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & e^{-x} \end{vmatrix}}{\begin{vmatrix} e^{-x} & x e^{-x} \\ -e^{-x} & e^{-x} - x e^{-x} \end{vmatrix}} = \frac{e^{-2x}}{e^{-2x}} = 1 \Rightarrow v_1 = -\int x dx = -\frac{1}{2} x^2$$

$$\text{and } v_2 = \int 1 dx = x \Rightarrow y_p = \left(-\frac{1}{2} x^2 \right) (e^{-x}) + (x)(x e^{-x}) = \frac{1}{2} x^2 e^{-x} \Rightarrow y = c_1 e^{-x} + c_2 x e^{-x} + \frac{1}{2} x^2 e^{-x}$$

$$22. y'' - y = x \Rightarrow r^2 - 1 = 0 \Rightarrow (r - 1)(r + 1) = 0 \Rightarrow r = 1 \text{ or } r = -1 \Rightarrow y_c = c_1 e^x + c_2 e^{-x} \Rightarrow y_1 = e^x, y_2 = e^{-x}$$

$$\Rightarrow v_1' = \frac{\begin{vmatrix} 0 & e^{-x} \\ x & -e^{-x} \end{vmatrix}}{\begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}} = \frac{-x e^{-x}}{-2} = \frac{1}{2} x e^{-x} \text{ and } v_2' = \frac{\begin{vmatrix} e^x & 0 \\ e^x & x \end{vmatrix}}{\begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}} = \frac{x e^x}{-2} = -\frac{1}{2} x e^x \Rightarrow v_1 = \frac{1}{2} \int x e^{-x} dx = -\frac{1}{2} x e^{-x} - \frac{1}{2} e^{-x}$$

$$\text{and } v_2 = -\frac{1}{2} \int x e^x dx = -\frac{1}{2} x e^x + \frac{1}{2} e^x \Rightarrow y_p = \left(-\frac{1}{2} x e^{-x} - \frac{1}{2} e^{-x} \right) e^x + \left(-\frac{1}{2} x e^x + \frac{1}{2} e^x \right) e^{-x} = -x$$

$$\Rightarrow y = c_1 e^x + c_2 e^{-x} - x$$

$$23. y'' - y = e^x \Rightarrow r^2 - 1 = 0 \Rightarrow (r - 1)(r + 1) = 0 \Rightarrow r = 1 \text{ or } r = -1 \Rightarrow y_c = c_1 e^x + c_2 e^{-x} \Rightarrow y_1 = e^x, y_2 = e^{-x}$$

$$\Rightarrow v_1' = \frac{\begin{vmatrix} 0 & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}}{\begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}} = \frac{-1}{-2} = \frac{1}{2} \text{ and } v_2' = \frac{\begin{vmatrix} e^x & 0 \\ e^x & e^{-x} \end{vmatrix}}{\begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}} = \frac{e^{2x}}{-2} = -\frac{1}{2} e^{2x} \Rightarrow v_1 = \int \frac{1}{2} dx = \frac{1}{2} x \text{ and } v_2 = -\frac{1}{2} \int e^{2x} dx = -\frac{1}{4} e^{2x}$$

$$\Rightarrow y_p = \left(\frac{1}{2} x \right) e^x + \left(-\frac{1}{4} e^{2x} \right) e^{-x} = \frac{1}{2} x e^x - \frac{1}{4} e^x \Rightarrow y = c_1 e^x + c_2 e^{-x} + \frac{1}{2} x e^x$$

$$24. y'' - y = \sin x \Rightarrow r^2 - 1 = 0 \Rightarrow (r-1)(r+1) = 0 \Rightarrow r = 1 \text{ or } r = -1 \Rightarrow y_c = c_1 e^x + c_2 e^{-x} \Rightarrow y_1 = e^x, y_2 = e^{-x}$$

$$\Rightarrow v'_1 = \frac{\begin{vmatrix} 0 & e^{-x} \\ \sin x & -e^{-x} \end{vmatrix}}{\begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}} = \frac{-e^{-x} \sin x}{-2} = \frac{1}{2} e^{-x} \sin x \text{ and } v'_2 = \frac{\begin{vmatrix} e^x & 0 \\ e^x & \sin x \end{vmatrix}}{\begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}} = \frac{e^x \sin x}{-2} = -\frac{1}{2} e^x \sin x$$

$$\Rightarrow v_1 = \frac{1}{2} \int e^{-x} \sin x \, dx = -\frac{1}{4} e^{-x} \cos x - \frac{1}{4} e^{-x} \sin x \text{ and } v_2 = -\frac{1}{2} \int e^x \sin x \, dx = \frac{1}{4} e^x \cos x - \frac{1}{4} e^x \sin x$$

$$\Rightarrow y_p = \left(-\frac{1}{4} e^{-x} \cos x - \frac{1}{4} e^{-x} \sin x\right) e^x + \left(\frac{1}{4} e^x \cos x - \frac{1}{4} e^x \sin x\right) e^{-x} = -\frac{1}{2} \sin x \Rightarrow y = c_1 e^x + c_2 e^{-x} - \frac{1}{2} \sin x$$

$$25. y'' + 4y' + 5y = 10 \Rightarrow r^2 + 4r + 5 = 0 \Rightarrow r = \frac{-4 \pm \sqrt{4^2 - 4(1)(5)}}{2(1)} = -2 \pm i \Rightarrow y = e^{-2x}(c_1 \cos x + c_2 \sin x)$$

$$\Rightarrow y_1 = e^{-2x} \cos x, y_2 = e^{-2x} \sin x \Rightarrow v'_1 = \frac{\begin{vmatrix} 0 & e^{-2x} \sin x \\ 10 & e^{-2x} \cos x - 2e^{-2x} \sin x \end{vmatrix}}{\begin{vmatrix} e^{-2x} \cos x & e^{-2x} \sin x \\ -e^{-2x} \sin x - 2e^{-2x} \cos x & e^{-2x} \cos x - 2e^{-2x} \sin x \end{vmatrix}} = \frac{-10e^{-2x} \sin x}{e^{-4x}} = -10e^{2x} \sin x$$

$$\text{and } v'_2 = \frac{\begin{vmatrix} e^{-2x} \cos x & 0 \\ -e^{-2x} \sin x - 2e^{-2x} \cos x & 10 \end{vmatrix}}{\begin{vmatrix} e^{-2x} \cos x & e^{-2x} \sin x \\ -e^{-2x} \sin x - 2e^{-2x} \cos x & e^{-2x} \cos x - 2e^{-2x} \sin x \end{vmatrix}} = \frac{10e^{-2x} \cos x}{e^{-4x}} = 10e^{2x} \cos x$$

$$\Rightarrow v_1 = -\int 10e^{2x} \sin x \, dx = 2e^{2x} \cos x - 4e^{2x} \sin x \text{ and } v_2 = \int 10e^{2x} \cos x \, dx = 2e^{2x} \sin x + 4e^{2x} \cos x$$

$$\Rightarrow y_p = (2e^{2x} \cos x - 4e^{2x} \sin x)(e^{-2x} \cos x) + (2e^{2x} \sin x + 4e^{2x} \cos x)(e^{-2x} \sin x) = 2 \Rightarrow y = e^{-2x}(c_1 \cos x + c_2 \sin x) + 2$$

$$26. y'' - y' = 2^x \Rightarrow r^2 - r = 0 \Rightarrow r(r-1) = 0 \Rightarrow r = 0 \text{ or } r = 1 \Rightarrow y_c = c_1 e^{0 \cdot x} + c_2 e^x = c_1 + c_2 e^x \Rightarrow y_1 = 1, y_2 = e^x$$

$$\Rightarrow v'_1 = \frac{\begin{vmatrix} 0 & e^x \\ 2^x & e^x \end{vmatrix}}{\begin{vmatrix} 1 & e^x \\ 0 & e^x \end{vmatrix}} = \frac{-2^x e^x}{e^x} = -2^x \text{ and } v'_2 = \frac{\begin{vmatrix} 1 & 0 \\ 0 & 2^x \end{vmatrix}}{\begin{vmatrix} 1 & e^x \\ 0 & e^x \end{vmatrix}} = \frac{2^x}{e^x} = \left(\frac{2}{e}\right)^x \Rightarrow v_1 = -\int 2^x \, dx = \frac{1}{\ln 2} 2^x \text{ and}$$

$$v_2 = \int \left(\frac{2}{e}\right)^x \, dx = \frac{1}{\ln 2 - 1} \left(\frac{2}{e}\right)^x \Rightarrow y_p = \left(\frac{1}{\ln 2} 2^x\right)(1) + \left(\frac{1}{\ln 2 - 1} \left(\frac{2}{e}\right)^x\right) e^x = \left(\frac{1}{\ln 2} + \frac{1}{\ln 2 - 1}\right) 2^x = \frac{\ln 2 - 1}{\ln 2(\ln 2 - 1)} 2^x$$

$$\Rightarrow y = c_1 + c_2 e^x + \frac{\ln 2 - 1}{\ln 2(\ln 2 - 1)} 2^x$$

$$27. \frac{d^2 y}{dx^2} + y = \sec x, -\frac{\pi}{2} < x < \frac{\pi}{2} \Rightarrow r^2 + 1 = 0 \Rightarrow r = 0 \pm i \Rightarrow y_c = c_1 \cos x + c_2 \sin x \Rightarrow y_1 = \cos x, y_2 = \sin x$$

$$\Rightarrow v'_1 = \frac{\begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{-\sin x \sec x}{1} = -\tan x \text{ and } v'_2 = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{1}{1} = \sin x \Rightarrow v_1 = \int -\tan x \, dx$$

$$= -\int \frac{\sin x}{\cos x} \, dx = \ln|\cos x| \text{ and } v_2 = \int 1 \, dx = x \Rightarrow y_p = (\ln|\cos x|)(\cos x) + (x)(\sin x) = \cos x \ln|\cos x| + x \sin x$$

$$\Rightarrow y = c_1 \cos x + c_2 \sin x + \cos x \ln|\cos x| + x \sin x$$

$$28. \frac{d^2 y}{dx^2} - \frac{dy}{dx} = e^x \cos x \Rightarrow r^2 - r = 0 \Rightarrow r(r-1) = 0 \Rightarrow r = 0 \text{ or } r = 1 \Rightarrow y_c = c_1 e^{0 \cdot x} + c_2 e^x = c_1 + c_2 e^x$$

$$\Rightarrow y_1 = 1, y_2 = e^x \Rightarrow v'_1 = \frac{\begin{vmatrix} 0 & e^x \\ e^x \cos x & e^x \end{vmatrix}}{\begin{vmatrix} 1 & e^x \\ 0 & e^x \end{vmatrix}} = \frac{-e^{2x} \cos x}{e^x} = -e^x \cos x \text{ and } v'_2 = \frac{\begin{vmatrix} 1 & 0 \\ 0 & e^x \cos x \end{vmatrix}}{\begin{vmatrix} 1 & e^x \\ 0 & e^x \end{vmatrix}} = \frac{e^x \cos x}{e^x} = \cos x$$

$$\Rightarrow v_1 = -\int e^x \cos x \, dx = -\frac{1}{2} e^x \sin x - \frac{1}{2} e^x \cos x \text{ and } v_2 = \int \cos x \, dx = \sin x$$

$$\Rightarrow y_p = \left(-\frac{1}{2} e^x \sin x - \frac{1}{2} e^x \cos x\right)(1) + (\sin x)(e^x) = \frac{1}{2} e^x \sin x - \frac{1}{2} e^x \cos x \Rightarrow y = c_1 + c_2 e^x + \frac{1}{2} e^x \sin x - \frac{1}{2} e^x \cos x$$

$$29. y'' - 5y' = x e^{5x}, y_p = Ax^2 e^{5x} + Bx e^{5x} \Rightarrow y'_p = 5Ax^2 e^{5x} + 2Ax e^{5x} + 5Bx e^{5x} + B e^{5x}$$

$$\Rightarrow y''_p = 25Ax^2 e^{5x} + 20Ax e^{5x} + 2A e^{5x} + 25Bx e^{5x} + 10B e^{5x}$$

$$\Rightarrow (25Ax^2 e^{5x} + 20Ax e^{5x} + 2A e^{5x} + 25Bx e^{5x} + 10B e^{5x}) - 5(5Ax^2 e^{5x} + 2Ax e^{5x} + 5Bx e^{5x} + B e^{5x}) = x e^{5x}$$

$$\Rightarrow 10Ax e^{5x} + (2A + 5B) e^{5x} = x e^{5x} \Rightarrow 10A = 1, 2A + 5B = 0 \Rightarrow A = \frac{1}{10}, B = -\frac{1}{25} \Rightarrow y_p = \frac{1}{10} x^2 e^{5x} - \frac{1}{25} x e^{5x};$$

$$r^2 - 5r = 0 \Rightarrow r(r-5) = 0 \Rightarrow r = 0 \text{ or } r = 5 \Rightarrow y_c = c_1 e^{0 \cdot x} + c_2 e^{5x} = c_1 + c_2 e^{5x}$$

$$\Rightarrow y = c_1 + c_2 e^{5x} + \frac{1}{10} x^2 e^{5x} - \frac{1}{25} x e^{5x}$$

30. $y'' - y' = \cos x + \sin x$, $y_p = A \cos x + B \sin x \Rightarrow y'_p = -A \sin x + B \cos x \Rightarrow y''_p = -A \cos x - B \sin x$
 $\Rightarrow -A \cos x - B \sin x - (-A \sin x + B \cos x) = \cos x + \sin x \Rightarrow (A - B) \sin x + (-A - B) \cos x = \cos x + \sin x$
 $\Rightarrow A - B = 1, -A - B = 1 \Rightarrow A = 0, B = -1 \Rightarrow y_p = -\sin x; r^2 - r = 0 \Rightarrow r(r - 1) = 0 \Rightarrow r = 0 \text{ or } r = 1$
 $\Rightarrow y_c = c_1 e^{0 \cdot x} + c_2 e^x = c_1 + c_2 e^x \Rightarrow y = c_1 + c_2 e^x - \sin x$
31. $y'' + y = 2 \cos x + \sin x$, $y_p = Ax \sin x + Bx \cos x \Rightarrow y'_p = Ax \cos x + A \sin x - Bx \sin x + B \cos x$
 $\Rightarrow y''_p = -Ax \sin x + 2A \cos x - Bx \cos x - 2B \sin x$
 $\Rightarrow (-Ax \sin x + 2A \cos x - Bx \cos x - 2B \sin x) + (Ax \sin x + Bx \cos x) = \cos x + \sin x$
 $\Rightarrow -2B \sin x + 2A \cos x = 2 \cos x + \sin x \Rightarrow -2B = 1, 2A = 2 \Rightarrow A = 1, B = -\frac{1}{2} \Rightarrow y_p = x \sin x - \frac{1}{2} x \cos x;$
 $r^2 + 1 = 0 \Rightarrow r = 0 \pm i \Rightarrow y_c = c_1 \cos x + c_2 \sin x \Rightarrow y = c_1 \cos x + c_2 \sin x + x \sin x - \frac{1}{2} x \cos x$
32. $y'' + y' - 2y = x e^x$, $y_p = Ax^2 e^x + Bx e^x \Rightarrow y'_p = Ax^2 e^x + 2Ax e^x + Bx e^x + B e^x$
 $\Rightarrow y''_p = Ax^2 e^x + 4Ax e^x + 2Ae^x + Bx e^x + 2B e^x$
 $\Rightarrow (Ax^2 e^x + 4Ax e^x + 2Ae^x + Bx e^x + 2B e^x) + (Ax^2 e^x + 2Ax e^x + Bx e^x + B e^x) - 2(Ax^2 e^x + Bx e^x) = x e^x$
 $\Rightarrow 6Ax e^x + (2A + 3B) e^x = x e^x \Rightarrow 6A = 1, 2A + 3B = 0 \Rightarrow A = \frac{1}{6}, B = -\frac{1}{9} \Rightarrow y_p = \frac{1}{6} x^2 e^x - \frac{1}{9} x e^x;$
 $r^2 + r - 2 = 0 \Rightarrow (r + 2)(r - 1) = 0 \Rightarrow r = -2 \text{ or } r = 1 \Rightarrow y_c = c_1 e^{-2x} + c_2 e^x \Rightarrow y = c_1 e^{-2x} + c_2 e^x + \frac{1}{6} x^2 e^x - \frac{1}{9} x e^x$
33. $\frac{d^2 y}{dx^2} - \frac{dy}{dx} = e^x + e^{-x} \Rightarrow r^2 - r = 0 \Rightarrow r(r - 1) = 0 \Rightarrow r = 0 \text{ or } r = 1 \Rightarrow y_c = c_1 e^{0 \cdot x} + c_2 e^x = c_1 + c_2 e^x$
(a) $y_1 = 1, y_2 = e^x \Rightarrow v'_1 = \frac{\begin{vmatrix} 0 & e^x \\ e^x + e^{-x} & e^x \end{vmatrix}}{\begin{vmatrix} 1 & e^x \\ 0 & e^x \end{vmatrix}} = \frac{-e^{2x} - 1}{e^x} = -e^x - e^{-x} \text{ and } v'_2 = \frac{\begin{vmatrix} 1 & 0 \\ 0 & e^x + e^{-x} \end{vmatrix}}{\begin{vmatrix} 1 & e^x \\ 0 & e^x \end{vmatrix}} = \frac{e^x + e^{-x}}{e^x} = 1 + e^{-2x}$
 $\Rightarrow v_1 = \int (-e^x - e^{-x}) dx = -e^x - e^{-x} \text{ and } v_2 = \int (1 + e^{-2x}) dx = x - \frac{1}{2} e^{-2x}$
 $\Rightarrow y_p = (-e^x - e^{-x})(1) + (x - \frac{1}{2} e^{-2x})(e^x) = -e^x - e^{-x} + x e^x - \frac{1}{2} e^{-x} \Rightarrow y = c_1 + c_2 e^x + x e^x - \frac{1}{2} e^{-x}$
(b) $y_p = Ax e^x + B e^{-x} \Rightarrow y'_p = Ax e^x + A e^x - B e^{-x} \Rightarrow y''_p = Ax e^x + 2A e^x + B e^{-x}$
 $\Rightarrow (Ax e^x + 2A e^x + B e^{-x}) - (Ax e^x + A e^x - B e^{-x}) = e^x + e^{-x} \Rightarrow A e^x + 2B e^{-x} = e^x + e^{-x}$
 $\Rightarrow A = 1, 2B = 1 \Rightarrow A = 1, B = \frac{1}{2} \Rightarrow y = c_1 + c_2 e^x + x e^x + \frac{1}{2} e^{-x}$
34. $\frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 4y = 2e^{2x} \Rightarrow r^2 - 4r + 4 = 0 \Rightarrow (r - 2)^2 = 0 \Rightarrow r = 2, \text{ repeated twice} \Rightarrow y_c = c_1 e^{2x} + c_2 x e^{2x}$
(a) $y_1 = e^{2x}, y_2 = x e^{2x} \Rightarrow v'_1 = \frac{\begin{vmatrix} 0 & x e^{2x} \\ 2e^{2x} & e^{2x} + 2x e^{2x} \end{vmatrix}}{\begin{vmatrix} e^{2x} & x e^{2x} \\ 2e^{2x} & e^{2x} + 2x e^{2x} \end{vmatrix}} = \frac{-2x e^{4x}}{e^{4x}} = -2x \text{ and } v'_2 = \frac{\begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & 2e^{2x} \end{vmatrix}}{\begin{vmatrix} e^{2x} & x e^{2x} \\ 2e^{2x} & e^{2x} + 2x e^{2x} \end{vmatrix}} = \frac{2e^{4x}}{e^{4x}} = 2$
 $\Rightarrow v_1 = \int (-2x) dx = -x^2 \text{ and } v_2 = \int 2 dx = 2x \Rightarrow y_p = (-x^2)(e^{2x}) + (2x)(x e^{2x}) = x^2 e^{2x}$
 $\Rightarrow y = c_1 e^{2x} + c_2 x e^{2x} + x^2 e^{2x}$
(b) $y_p = Ax^2 e^{2x} \Rightarrow y'_p = 2Ax e^{2x} + 2Ax^2 e^{2x} \Rightarrow y''_p = 2A e^{2x} + 8Ax e^{2x} + 4Ax^2 e^{2x}$
 $\Rightarrow (2A e^{2x} + 8Ax e^{2x} + 4Ax^2 e^{2x}) - 4(2Ax e^{2x} + 2Ax^2 e^{2x}) + 4Ax^2 e^{2x} = 2e^{2x} \Rightarrow 2A e^{2x} = 2e^{2x}$
 $\Rightarrow 2A = 2 \Rightarrow A = 1 \Rightarrow y = c_1 e^{2x} + c_2 x e^{2x} + x^2 e^{2x}$
35. $\frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} - 5y = e^x + 4 \Rightarrow r^2 - 4r - 5 = 0 \Rightarrow (r - 5)(r + 1) = 0 \Rightarrow r = 5 \text{ or } r = -1 \Rightarrow y_c = c_1 e^{5x} + c_2 e^{-x}$
(a) $y_1 = e^{5x}, y_2 = e^{-x} \Rightarrow v'_1 = \frac{\begin{vmatrix} 0 & e^{-x} \\ e^x + 4 & -e^{-x} \end{vmatrix}}{\begin{vmatrix} e^{5x} & e^{-x} \\ 5e^{5x} & -e^{-x} \end{vmatrix}} = \frac{-1 - 4e^{-x}}{-6e^{4x}} = \frac{1}{6} e^{-4x} + \frac{2}{3} e^{-5x} \text{ and } v'_2 = \frac{\begin{vmatrix} e^{5x} & 0 \\ 5e^{5x} & e^x + 4 \end{vmatrix}}{\begin{vmatrix} e^{5x} & e^{-x} \\ 5e^{5x} & -e^{-x} \end{vmatrix}} = \frac{e^{6x} + 4e^{5x}}{-6e^{4x}} = -\frac{1}{6} e^{2x} - \frac{2}{3} e^x$
 $\Rightarrow v_1 = \int (\frac{1}{6} e^{-4x} + \frac{2}{3} e^{-5x}) dx = -\frac{1}{24} e^{-4x} - \frac{2}{15} e^{-5x} \text{ and } v_2 = \int (-\frac{1}{6} e^{2x} - \frac{2}{3} e^x) dx = -\frac{1}{12} e^{2x} - \frac{2}{3} e^x$
 $\Rightarrow y_p = (-\frac{1}{24} e^{-4x} - \frac{2}{15} e^{-5x})(e^{5x}) + (-\frac{1}{12} e^{2x} - \frac{2}{3} e^x)(e^{-x}) = -\frac{1}{8} e^x - \frac{4}{5} \Rightarrow y = c_1 e^{5x} + c_2 e^{-x} - \frac{1}{8} e^x - \frac{4}{5}$
(b) $y_p = Ae^x + B \Rightarrow y'_p = Ae^x \Rightarrow y''_p = Ae^x \Rightarrow Ae^x - 4Ae^x - 5(Ae^x + B) = e^x + 4 \Rightarrow -8Ae^x - 5B = e^x + 4$
 $\Rightarrow -8A = 1, -5B = 4 \Rightarrow A = -\frac{1}{8}, B = -\frac{4}{5} \Rightarrow y = c_1 e^{5x} + c_2 e^{-x} - \frac{1}{8} e^x - \frac{4}{5}$

$$36. \frac{d^2y}{dx^2} - 9\frac{dy}{dx} = 9e^{9x} \Rightarrow r^2 - 9r = 0 \Rightarrow r(r-9) = 0 \Rightarrow r = 0 \text{ or } r = 9 \Rightarrow y_c = c_1e^{0 \cdot x} + c_2e^{9x} = c_1 + c_2e^{9x}$$

$$(a) \ y_1 = 1, y_2 = e^{9x} \Rightarrow v_1' = \frac{\begin{vmatrix} 0 & e^{9x} \\ 9e^{9x} & 9e^{9x} \end{vmatrix}}{\begin{vmatrix} 1 & e^{9x} \\ 0 & 9e^{9x} \end{vmatrix}} = \frac{-9e^{18x}}{9e^{9x}} = -e^{9x} \text{ and } v_2' = \frac{\begin{vmatrix} 1 & 0 \\ 0 & 9e^{9x} \end{vmatrix}}{\begin{vmatrix} 1 & e^{9x} \\ 0 & 9e^{9x} \end{vmatrix}} = \frac{9e^{9x}}{9e^{9x}} = 1$$

$$\Rightarrow v_1 = \int (-e^{9x}) dx = -\frac{1}{9}e^{9x} \text{ and } v_2 = \int (1) dx = x \Rightarrow y_p = \left(-\frac{1}{9}e^{9x}\right)(1) + (x)(e^{9x}) = -\frac{1}{9}e^{9x} + xe^{9x}$$

$$\Rightarrow y = c_1 + c_2e^{9x} + xe^{9x}$$

$$(b) \ y_p = Ax e^{9x} \Rightarrow y_p' = 9Ax e^{9x} + A e^{9x} \Rightarrow y_p'' = 81Ax e^{9x} + 18A e^{9x}$$

$$\Rightarrow (81Ax e^{9x} + 18A e^{9x}) - 9(9Ax e^{9x} + A e^{9x}) = 9e^{9x} \Rightarrow 9A e^{9x} = 9e^{9x} \Rightarrow 9A = 9 \Rightarrow A = 1$$

$$\Rightarrow y = c_1 + c_2e^{9x} + xe^{9x}$$

$$37. \ y'' + y = \cot x, 0 < x < \pi \Rightarrow r^2 + 1 = 0 \Rightarrow r = 0 \pm i \Rightarrow y_c = c_1 \cos x + c_2 \sin x \Rightarrow y_1 = \cos x, y_2 = \sin x$$

$$\Rightarrow v_1' = \frac{\begin{vmatrix} 0 & \sin x \\ \cot x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{-\sin x \cot x}{1} = -\sin x \frac{\cos x}{\sin x} = -\cos x \text{ and } v_2' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \cot x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{\cos x \cot x}{1} = \cos x \cdot \frac{\cos x}{\sin x} = \frac{\cos^2 x}{\sin x}$$

$$\Rightarrow v_1 = -\int \cos x dx = -\sin x \text{ and } v_2 = \int \frac{\cos^2 x}{\sin x} dx = \int \frac{1 - \sin^2 x}{\sin x} dx = \int \left(\frac{1}{\sin x} - \frac{\sin^2 x}{\sin x}\right) dx = \int (\csc x - \sin x) dx$$

$$= \ln|\csc x + \cot x| - \sin x \Rightarrow y_p = (\sin x)(\cos x) + (\ln|\csc x + \cot x| - \sin x)(\sin x) = -(\sin x) \ln|\csc x + \cot x|$$

$$\Rightarrow y = c_1 \cos x + c_2 \sin x - (\sin x) \ln|\csc x + \cot x|$$

$$38. \ y'' + y = \csc x, 0 < x < \pi \Rightarrow r^2 + 1 = 0 \Rightarrow r = 0 \pm i \Rightarrow y_c = c_1 \cos x + c_2 \sin x \Rightarrow y_1 = \cos x, y_2 = \sin x$$

$$\Rightarrow v_1' = \frac{\begin{vmatrix} 0 & \sin x \\ \csc x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{-\sin x \csc x}{1} = -\sin x \frac{1}{\sin x} = -1 \text{ and } v_2' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \csc x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{\cos x \csc x}{1} = \cos x \cdot \frac{1}{\sin x} = \frac{\cos x}{\sin x}$$

$$\Rightarrow v_1 = \int (-1) dx = -x \text{ and } v_2 = \int \frac{\cos x}{\sin x} dx = \ln|\sin x| \Rightarrow y_p = (-x)(\cos x) + (\ln|\sin x|)(\sin x)$$

$$= -x \cos x + (\sin x) \ln|\sin x| \Rightarrow y = c_1 \cos x + c_2 \sin x - x \cos x + (\sin x) \ln|\sin x|$$

$$39. \ y'' - 8y' = e^{8x} \Rightarrow r^2 - 8r = 0 \Rightarrow r(r-8) = 0 \Rightarrow r = 0 \text{ or } r = 8 \Rightarrow y_c = c_1e^{0 \cdot x} + c_2e^{8x} = c_1 + c_2e^{8x};$$

$$y_p = Ax e^{8x} \Rightarrow y_p' = Ae^{8x} + 8Ax e^{8x} \Rightarrow y_p'' = 16Ae^{8x} + 64Ax e^{8x} \Rightarrow (16Ae^{8x} + 64Ax e^{8x}) - 8(Ae^{8x} + 8Ax e^{8x}) = e^{8x}$$

$$\Rightarrow 8Ae^{8x} = e^{8x} \Rightarrow 8A = 1 \Rightarrow A = \frac{1}{8} \Rightarrow y = c_1 + c_2e^{8x} + \frac{1}{8}x e^{8x}$$

$$40. \ y'' + 4y = \sin x \Rightarrow r^2 + 4 = 0 \Rightarrow r = 0 \pm 2i \Rightarrow y_c = e^{0 \cdot x}(c_1 \cos 2x + c_2 \sin 2x) = c_1 \cos 2x + c_2 \sin 2x;$$

$$y_p = A \sin x + B \cos x \Rightarrow y_p' = A \cos x - B \sin x \Rightarrow y_p'' = -A \sin x - B \cos x$$

$$\Rightarrow (-A \sin x - B \cos x) + 4(A \sin x + B \cos x) = \sin x \Rightarrow 3A \sin x + 3B \cos x = \sin x \Rightarrow 3A = 1, 3B = 0$$

$$\Rightarrow A = \frac{1}{3}, B = 0 \Rightarrow y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{3} \sin x$$

$$41. \ y'' - y' = x^3 \Rightarrow r^2 - r = 0 \Rightarrow r(r-1) = 0 \Rightarrow r = 0 \text{ or } r = 1 \Rightarrow y_c = c_1e^{0 \cdot x} + c_2e^x = c_1 + c_2e^x;$$

$$y_p = Ax^4 + Bx^3 + Cx^2 + Dx \Rightarrow y_p' = 4Ax^3 + 3Bx^2 + 2Cx + D \Rightarrow y_p'' = 12Ax^2 + 6Bx + 2C$$

$$\Rightarrow (12Ax^2 + 6Bx + 2C) - (4Ax^3 + 3Bx^2 + 2Cx + D) = x^3$$

$$\Rightarrow -4Ax^3 + (12A - 3B)x^2 + (6B - 2C)x + (2C - D) = x^3 \Rightarrow -4A = 1, 12A - 3B = 0, 6B - 2C = 0, 2C - D = 0$$

$$\Rightarrow A = -\frac{1}{4}, B = -1, C = -3, D = -6 \Rightarrow y = c_1 + c_2e^x - \frac{1}{4}x^4 - x^3 - 3x^2 - 6x$$

$$42. \ y'' + 4y' + 5y = x + 2 \Rightarrow r^2 + 4r + 5 = 0 \Rightarrow r = \frac{-4 \pm \sqrt{4^2 - 4(1)(5)}}{2(1)} = -2 \pm i \Rightarrow y_c = e^{-2x}(c_1 \cos x + c_2 \sin x)$$

$$y_p = Ax + B \Rightarrow y_p' = A \Rightarrow y_p'' = 0 \Rightarrow 0 + 4A + 5(Ax + B) = x + 2 \Rightarrow 5Ax + (4A + 5B) = x + 2 \Rightarrow 5A = 1,$$

$$4A + 5B = 2 \Rightarrow A = \frac{1}{5}, B = \frac{6}{25} \Rightarrow y = e^{-2x}(c_1 \cos x + c_2 \sin x) + \frac{1}{5}x + \frac{6}{25}$$

43. $y'' + 2y' = x^2 - e^x \Rightarrow r^2 + 2r = 0 \Rightarrow r(r+2) = 0 \Rightarrow r = 0$ or $r = -2 \Rightarrow y_c = c_1 e^{0 \cdot x} + c_2 e^{-2x} = c_1 + c_2 e^{-2x}$;
 $y_p = Ax^3 + Bx^2 + Cx + De^x \Rightarrow y'_p = 3Ax^2 + 2Bx + C + De^x \Rightarrow y''_p = 6Ax + 2B + De^x$
 $\Rightarrow (6Ax + 2B + De^x) + 2(3Ax^2 + 2Bx + C + De^x) = x^2 - e^x \Rightarrow 6Ax^2 + (6A + 4B)x + (2B + 2C) + 3De^x = x^2 - e^x$
 $\Rightarrow 6A = 1, 6A + 4B = 0, 2B + 2C = 0, 3D = -1 \Rightarrow A = \frac{1}{6}, B = -\frac{1}{4}, C = \frac{1}{8}, D = -\frac{1}{3}$
 $\Rightarrow y = c_1 + c_2 e^{-2x} + \frac{1}{6}x^3 - \frac{1}{4}x^2 + \frac{1}{8}x - \frac{1}{3}e^x$
44. $y'' + 9y = 9x - \cos x \Rightarrow r^2 + 9 = 0 \Rightarrow r = 0 \pm 3i \Rightarrow y_c = e^{0 \cdot x}(c_1 \cos 3x + c_2 \sin 3x) = c_1 \cos 3x + c_2 \sin 3x$;
 $y_p = Ax + B + C \sin x + D \cos x \Rightarrow y'_p = A + C \cos x - D \sin x \Rightarrow y''_p = -C \sin x - D \cos x$
 $\Rightarrow (-C \sin x - D \cos x) + 9(Ax + B + C \sin x + D \cos x) = 9x - \cos x \Rightarrow 9Ax + 9B + 8C \sin x + 8D \cos x = 9x - \cos x$
 $\Rightarrow 9A = 9, 9B = 0, 8C = 0, 8D = -1 \Rightarrow A = 1, B = 0, C = 0, D = -\frac{1}{8} \Rightarrow y = c_1 \cos 3x + c_2 \sin 3x + x - \frac{1}{8} \cos x$
45. $\frac{d^2 y}{dx^2} + y = \sec x \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2} \Rightarrow r^2 + 1 = 0 \Rightarrow r = 0 \pm i \Rightarrow y_c = c_1 \cos x + c_2 \sin x \Rightarrow y_1 = \cos x, y_2 = \sin x$
 $\Rightarrow v'_1 = \frac{\begin{vmatrix} 0 & \sin x \\ \sec x \tan x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{-\sin x \sec x \tan x}{1} = -\tan^2 x$ and $v'_2 = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \tan x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{\cos x \sec x \tan x}{1} = \tan x$
 $\Rightarrow v_1 = \int -\tan^2 x \, dx = \int (1 - \sec^2 x) \, dx = x - \tan x$ and $v_2 = \int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\ln|\cos x|$
 $\Rightarrow y_p = (x - \tan x)(\cos x) + (-\ln|\cos x|)(\sin x) = x \cos x - \sin x - (\sin x) \ln|\cos x|$
 $\Rightarrow y = c_1 \cos x + c_2 \sin x + x \cos x - (\sin x) \ln|\cos x|$
46. $y'' - 3y' + 2y = e^x - 2e^{2x} \Rightarrow r^2 - 3r + 2 = 0 \Rightarrow (r-1)(r-2) = 0 \Rightarrow r = 1$ or $r = 2 \Rightarrow y_c = c_1 e^x + c_2 e^{2x}$;
 $y_p = Ax e^{2x} + Bx e^x \Rightarrow y'_p = 2Ax e^{2x} + A e^{2x} + Bx e^x + B e^x \Rightarrow y''_p = 4Ax e^{2x} + 4A e^{2x} + Bx e^x + 2B e^x$
 $\Rightarrow (4Ax e^{2x} + 4A e^{2x} + Bx e^x + 2B e^x) - 3(2Ax e^{2x} + A e^{2x} + Bx e^x + B e^x) + 2(Ax e^{2x} + Bx e^x) = e^x - 2e^{2x}$
 $\Rightarrow A e^{2x} - B e^x = e^x - 2e^{2x} \Rightarrow A = -2, -B = 1 \Rightarrow A = -2, B = -1 \Rightarrow y = c_1 e^x + c_2 e^{2x} - 2x e^{2x} - x e^x$
47. $y' - 3y = e^x \Rightarrow r - 3 = 0 \Rightarrow r = 3 \Rightarrow y_c = c_1 e^{3x}; y_p = Ae^x \Rightarrow y'_p = Ae^x \Rightarrow Ae^x - 3Ae^x = e^x \Rightarrow -2Ae^x = e^x$
 $\Rightarrow -2A = 1 \Rightarrow A = -\frac{1}{2} \Rightarrow y = c_1 e^{3x} - \frac{1}{2} e^x$
48. $y' + 4y = x \Rightarrow r + 4 = 0 \Rightarrow r = -4 \Rightarrow y_c = c_1 e^{-4x}; y_p = Ax + B \Rightarrow y'_p = A \Rightarrow A + 4(Ax + B) = x$
 $\Rightarrow 4Ax + (A + 4B) = x \Rightarrow 4A = 1, A + 4B = 0 \Rightarrow A = \frac{1}{4}, B = -\frac{1}{16} \Rightarrow y = c_1 e^{-4x} + \frac{1}{4}x - \frac{1}{16}$
49. $y' - 3y = 5e^{3x} \Rightarrow r - 3 = 0 \Rightarrow r = 3 \Rightarrow y_c = c_1 e^{3x}; y_p = Ax e^{3x} \Rightarrow y'_p = 3Ax e^{3x} + A e^{3x}$
 $\Rightarrow (3Ax e^{3x} + A e^{3x}) - 3Ax e^{3x} = 5e^{3x} \Rightarrow A e^{3x} = 5e^{3x} \Rightarrow A = 5 \Rightarrow y = c_1 e^{3x} + 5x e^{3x}$
50. $y' + y = \sin x \Rightarrow r + 1 = 0 \Rightarrow r = -1 \Rightarrow y_c = c_1 e^{-x}; y_p = A \sin x + B \cos x \Rightarrow y'_p = A \cos x - B \sin x$
 $\Rightarrow (A \cos x - B \sin x) + (A \sin x + B \cos x) = \sin x \Rightarrow (A - B) \sin x + (A + B) \cos x = \sin x \Rightarrow A - B = 1, A + B = 0$
 $\Rightarrow A = \frac{1}{2}, B = -\frac{1}{2} \Rightarrow y = c_1 e^{-x} + \frac{1}{2} \sin x - \frac{1}{2} \cos x$
51. $\frac{d^2 y}{dx^2} + y = \sec^2 x, -\frac{\pi}{2} < x < \frac{\pi}{2}, y(0) = y'(0) = 1 \Rightarrow r^2 + 1 = 0 \Rightarrow r = 0 \pm i \Rightarrow y_c = c_1 \cos x + c_2 \sin x$
 $\Rightarrow y_1 = \cos x, y_2 = \sin x$
 $\Rightarrow v'_1 = \frac{\begin{vmatrix} 0 & \sin x \\ \sec^2 x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{-\sin x \sec^2 x}{1} = -\sec x \tan x$ and $v'_2 = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \sec^2 x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{\cos x \sec^2 x}{1} = \sec x$
 $\Rightarrow v_1 = \int -\sec x \tan x \, dx = -\sec x$ and $v_2 = \int \sec x \, dx = \ln|\sec x + \tan x|$
 $\Rightarrow y_p = (-\sec x)(\cos x) + (\ln|\sec x + \tan x|)(\sin x) = -1 + (\sin x) \ln|\sec x + \tan x|$
 $\Rightarrow y = c_1 \cos x + c_2 \sin x - 1 + (\sin x) \ln|\sec x + \tan x|; y(0) = 1 \Rightarrow 1 = c_1 - 1 \Rightarrow c_1 = 2;$

$$\frac{dy}{dx} = -c_1 \sin x + c_2 \cos x + (\cos x) \ln |\sec x + \tan x| + (\sin x) \sec x, y'(0) = 1 \Rightarrow 1 = c_2$$

$$\Rightarrow y = 2 \cos x + \sin x - 1 + (\sin x) \ln |\sec x + \tan x|$$

52. $\frac{d^2y}{dx^2} + y = e^{2x}, y(0) = 0, y'(0) = \frac{2}{5} \Rightarrow r^2 + 1 = 0 \Rightarrow r = 0 \pm i \Rightarrow y_c = c_1 \cos x + c_2 \sin x$
 $y_p = A e^{2x} \Rightarrow y_p' = 2A e^{2x} \Rightarrow y_p'' = 4A e^{2x} \Rightarrow 4A e^{2x} + A e^{2x} = e^{2x} \Rightarrow 5A e^{2x} = e^{2x} \Rightarrow 5A = 1 \Rightarrow A = \frac{1}{5}$
 $\Rightarrow y = c_1 \cos x + c_2 \sin x + \frac{1}{5} e^{2x} \Rightarrow y' = -c_1 \sin x + c_2 \cos x + \frac{2}{5} e^{2x}; y(0) = 0 \Rightarrow c_1 + \frac{1}{5} = 0 \Rightarrow c_1 = -\frac{1}{5};$
 $y'(0) = \frac{2}{5} \Rightarrow c_2 + \frac{2}{5} = \frac{2}{5} \Rightarrow c_2 = 0 \Rightarrow y = -\frac{1}{5} \cos x + \frac{1}{5} e^{2x}$
53. $y'' + y' = x, y_p = \frac{x^2}{2} - x, y(0) = 0, y'(0) = 0; y_p = \frac{x^2}{2} - x \Rightarrow y_p' = x - 1 \Rightarrow y_p'' = 1 \Rightarrow y'' + y' = 1 + x - 1 = x$
 $\Rightarrow x = x \Rightarrow y_p$ satisfies the differential equation. $y'' + y' = 0 \Rightarrow r^2 + r = 0 \Rightarrow r(r+1) = 0 \Rightarrow r = 0$ or $r = -1$
 $\Rightarrow y_c = c_1 e^{0x} + c_2 e^{-x} = c_1 + c_2 e^{-x} \Rightarrow y = c_1 + c_2 e^{-x} + \frac{x^2}{2} - x \Rightarrow y' = -c_2 e^{-x} + x - 1;$
 $y(0) = 0 \Rightarrow 0 = c_1 + c_2, y'(0) = 0 \Rightarrow -c_2 - 1 = 0 \Rightarrow c_1 = 1, c_2 = -1 \Rightarrow y = 1 - e^{-x} + \frac{x^2}{2} - x$
54. $y'' + y = x, y_p = 2 \sin x + x, y(0) = 0, y'(0) = 0; y_p = 2 \sin x + x \Rightarrow y_p' = 2 \cos x + 1 \Rightarrow y_p'' = -2 \sin x$
 $\Rightarrow y'' + y = (-2 \sin x) + (2 \sin x + x) = x \Rightarrow x = x \Rightarrow y_p$ satisfies the differential equation. $y'' + y = 0 \Rightarrow r^2 + 1 = 0$
 $\Rightarrow r = 0 \pm i \Rightarrow y_c = c_1 \cos x + c_2 \sin x \Rightarrow y = c_1 \cos x + c_2 \sin x + 2 \sin x + x \Rightarrow y' = -c_1 \sin x + c_2 \cos x + 2 \cos x + 1;$
 $y(0) = 0 \Rightarrow 0 = c_1; y'(0) = 0 \Rightarrow 0 = c_2 + 3 \Rightarrow c_2 = -3 \Rightarrow y = -3 \sin x + 2 \sin x + x = -\sin x + x$
55. $\frac{1}{2} y'' + y' + y = 4e^x (\cos x - \sin x), y_p = 2e^x \cos x, y(0) = 0, y'(0) = 1; y_p = 2e^x \cos x \Rightarrow y_p' = 2e^x \cos x - 2e^x \sin x$
 $\Rightarrow y_p'' = -4e^x \sin x \Rightarrow \frac{1}{2} y'' + y' + y = \frac{1}{2} (-4e^x \sin x) + (2e^x \cos x - 2e^x \sin x) + 2e^x \cos x = 4e^x \cos x - 4e^x \sin x$
 $\Rightarrow 4e^x \cos x - 4e^x \sin x = 4e^x (\cos x - \sin x) \Rightarrow y_p$ satisfies the differential equation. $\frac{1}{2} y'' + y' + y = 0 \Rightarrow \frac{1}{2} r^2 + r + 1 = 0$
 $\Rightarrow r = \frac{-1 \pm \sqrt{1^2 - 4(\frac{1}{2})(1)}}{2(\frac{1}{2})} = -1 \pm i \Rightarrow y_c = e^{-x} (c_1 \cos x + c_2 \sin x) \Rightarrow y = e^{-x} (c_1 \cos x + c_2 \sin x) + 2e^x \cos x$
 $\Rightarrow y' = -e^{-x} (c_1 \cos x + c_2 \sin x) + e^{-x} (-c_1 \sin x + c_2 \cos x) + 2e^x \cos x - 2e^x \sin x; y(0) = 0 \Rightarrow c_1 + 2 = 0 \Rightarrow c_1 = -2;$
 $y'(0) = 1 \Rightarrow -c_1 + c_2 + 2 = 1 \Rightarrow c_2 = -3 \Rightarrow y = e^{-x} (-2 \cos x - 3 \sin x) + 2e^x \cos x = 2(e^x - e^{-x}) \cos x - 3e^{-x} \sin x$
56. $y'' - y' - 2y = 1 - 2x, y_p = x - 1, y(0) = 0, y'(0) = 1; y_p = x - 1 \Rightarrow y_p' = 1 \Rightarrow y_p'' = 0$
 $\Rightarrow y'' - y' - 2y = 0 - 1 - 2(x - 1) = 1 - 2x \Rightarrow 1 - 2x = 1 - 2x \Rightarrow y_p$ satisfies the differential equation.
 $y'' - y' - 2y = 0 \Rightarrow r^2 - r - 2 = 0 \Rightarrow (r - 2)(r + 1) = 0 \Rightarrow r = 2$ or $r = -1 \Rightarrow y_c = c_1 e^{2x} + c_2 e^{-x}$
 $\Rightarrow y = c_1 e^{2x} + c_2 e^{-x} + x - 1 \Rightarrow y' = 2c_1 e^{2x} - c_2 e^{-x} + 1; y(0) = 0 \Rightarrow c_1 + c_2 - 1 = 0 \Rightarrow c_1 + c_2 = 1,$
 $y'(0) = 1 \Rightarrow 2c_1 - c_2 + 1 = 1 \Rightarrow 2c_1 - c_2 = 0. \text{ Thus } c_1 + c_2 = 1, 2c_1 - c_2 = 0 \Rightarrow c_1 = \frac{1}{3}, c_2 = \frac{2}{3}$
 $\Rightarrow y = \frac{1}{3} e^{2x} + \frac{2}{3} e^{-x} + x - 1$
57. $y'' - 2y' + y = 2e^x, y_p = x^2 e^x, y(0) = 1, y'(0) = 0; y_p = x^2 e^x \Rightarrow y_p' = x^2 e^x + 2x e^x \Rightarrow y_p'' = x^2 e^x + 4x e^x + 2e^x$
 $\Rightarrow y'' - 2y' + y = (x^2 e^x + 4x e^x + 2e^x) - 2(x^2 e^x + 2x e^x) + x^2 e^x = x^2 e^x \Rightarrow x^2 e^x = x^2 e^x \Rightarrow y_p$ satisfies the differential equation. $y'' - 2y' + y = 0 \Rightarrow r^2 - 2r + 1 = 0 \Rightarrow (r - 1)^2 = 0 \Rightarrow r = 1, \text{ repeated twice} \Rightarrow y_c = c_1 e^x + c_2 x e^x$
 $\Rightarrow y = c_1 e^x + c_2 x e^x + x^2 e^x \Rightarrow y' = c_1 e^x + c_2 x e^x + c_2 e^x + x^2 e^x + 2x e^x; y(0) = 1 \Rightarrow c_1 = 1; y'(0) = 0 \Rightarrow c_1 + c_2 = 0$
 $\Rightarrow c_2 = -1 \Rightarrow y = e^x - x e^x + x^2 e^x$
58. $y'' - 2y' + y = x^{-1} e^x, x > 0, y_p = x e^x \ln x, y(1) = e, y'(1) = 0; y_p = x e^x \ln x \Rightarrow y_p' = e^x + x e^x \ln x + e^x \ln x$
 $\Rightarrow y_p'' = 2e^x + x e^x \ln x + e^x \ln x + \frac{e^x}{x}$
 $\Rightarrow y'' - 2y' + y = (2e^x + x e^x \ln x + e^x \ln x + \frac{e^x}{x}) - 2(e^x + x e^x \ln x + e^x \ln x) + x e^x \ln x = \frac{e^x}{x} \Rightarrow \frac{e^x}{x} = x^{-1} e^x$
 $\Rightarrow y_p$ satisfies the differential equation. $y'' - 2y' + y = 0 \Rightarrow r^2 - 2r + 1 = 0 \Rightarrow (r - 1)^2 = 0 \Rightarrow r = 1, \text{ repeated twice}$
 $\Rightarrow y_c = c_1 e^x + c_2 x e^x \Rightarrow y = c_1 e^x + c_2 x e^x + x e^x \ln x \Rightarrow y' = c_1 e^x + c_2 x e^x + c_2 e^x + e^x + x e^x \ln x + e^x \ln x;$

$$y(1) = e \Rightarrow c_1 e + c_2 e = e, y'(1) = 0 \Rightarrow c_1 e + 2c_2 e + e = 0 \Rightarrow c_1 e + 2c_2 e = -e \Rightarrow c_1 + c_2 = 1, c_1 + 2c_2 = -1 \\ \Rightarrow c_1 = 3, c_2 = -2 \Rightarrow y = 3e^x - 2xe^x + x e^x \ln x$$

$$59. x^2 y'' + 2xy' - 2y = x^2, y_1 = x^{-2}, y_2 = x$$

$$\Rightarrow v'_1 = \frac{\begin{vmatrix} 0 & x \\ \frac{x^2}{x^2} & 1 \end{vmatrix}}{\begin{vmatrix} x^{-2} & x \\ -2x^{-3} & 1 \end{vmatrix}} = \frac{-x}{3x^{-2}} = -\frac{1}{3}x^3 \text{ and } v'_2 = \frac{\begin{vmatrix} x^{-2} & 0 \\ -2x^{-3} & \frac{x^2}{x^2} \end{vmatrix}}{\begin{vmatrix} x^{-2} & x \\ -2x^{-3} & 1 \end{vmatrix}} = \frac{x^{-2}}{3x^{-2}} = \frac{1}{3}$$

$$\Rightarrow v_1 = \int -\frac{1}{3}x^3 dx = -\frac{1}{12}x^4 \text{ and } v_2 = \int \frac{1}{3}dx = \frac{1}{3}x \Rightarrow y_p = \left(-\frac{1}{12}x^4\right)(x^{-2}) + \left(\frac{1}{3}x\right)(x) = \frac{1}{4}x^2$$

$$60. x^2 y'' + xy' - y = x, y_1 = x^{-1}, y_2 = x$$

$$\Rightarrow v'_1 = \frac{\begin{vmatrix} 0 & x \\ \frac{x}{x^2} & 1 \end{vmatrix}}{\begin{vmatrix} x^{-1} & x \\ -x^{-2} & 1 \end{vmatrix}} = \frac{-1}{2x^{-1}} = -\frac{1}{2}x \text{ and } v'_2 = \frac{\begin{vmatrix} x^{-1} & 0 \\ -x^{-2} & \frac{x}{x^2} \end{vmatrix}}{\begin{vmatrix} x^{-1} & x \\ -x^{-2} & 1 \end{vmatrix}} = \frac{x^{-2}}{2x^{-1}} = \frac{1}{2}x^{-1}$$

$$\Rightarrow v_1 = \int -\frac{1}{2}x dx = -\frac{1}{4}x^2 \text{ and } v_2 = \int \frac{1}{2}x^{-1} dx = \frac{1}{2}\ln|x| \Rightarrow y_p = \left(-\frac{1}{4}x^2\right)(x^{-1}) + \left(\frac{1}{2}\ln|x|\right)(x) = -\frac{1}{4}x + \frac{1}{2}x \ln|x|$$

17.3 APPLICATIONS

- $mg = 16 \Rightarrow m = \frac{16}{32}; k = 1; \delta = 1 \Rightarrow \frac{16}{32} \frac{d^2 y}{dt^2} + 1 \frac{dy}{dt} + 1y = 0 \Rightarrow \frac{1}{2} \frac{d^2 y}{dt^2} + \frac{dy}{dt} + y = 0, y(0) = 2, y'(0) = 2$
- $mg = 8 \text{ lb} \Rightarrow m = \frac{8}{32}; 8 = k \cdot 4 \Rightarrow k = 2; \delta = 1.5 \Rightarrow \frac{8}{32} \frac{d^2 y}{dt^2} + 1.5 \frac{dy}{dt} + 2y = 0 \Rightarrow \frac{1}{4} \frac{d^2 y}{dt^2} + \frac{3}{2} \frac{dy}{dt} + 2y = 0, y(0) = -2, y'(0) = 3$
- $20 = k \cdot \frac{1}{2} \Rightarrow k = 40; w = 25 \text{ lb} \Rightarrow m = \frac{25}{32}; \delta = 0 \Rightarrow \frac{25}{32} \frac{d^2 y}{dt^2} + 0 \cdot \frac{dy}{dt} + 40y = 0$. If $w = 25 \text{ lb}$, it stretches the spring $25 = 40x \Rightarrow x = \frac{5}{8} \text{ ft} \Rightarrow$ spring is now stretched $\frac{6+5}{12} - \frac{5}{8} = \frac{7}{24} \text{ ft}$ below equilibrium $\Rightarrow y(0) = \frac{7}{24}$; initial velocity is $v_0 \frac{\text{in}}{\text{sec}} = \frac{v_0}{12} \frac{\text{ft}}{\text{sec}} \Rightarrow y'(0) = \frac{v_0}{12}$. Thus we have $\frac{25}{32} \frac{d^2 y}{dt^2} + 40y = 0, y(0) = \frac{7}{24}, y'(0) = \frac{v_0}{12}$
- $w = 10 \text{ lb} \Rightarrow m = \frac{10}{32}; 10 = k \cdot \left(\frac{2}{12}\right) \Rightarrow k = 60 \frac{\text{lb}}{\text{ft}}; \delta = \frac{20}{\sqrt{g}} = \frac{20}{\sqrt{32}} \Rightarrow \frac{10}{32} \frac{d^2 y}{dt^2} + \frac{20}{\sqrt{32}} \frac{dy}{dt} + 60y = 0 \\ \Rightarrow \frac{10}{32} \frac{d^2 y}{dt^2} + \frac{5}{\sqrt{2}} \frac{dy}{dt} + 60y = 0, y(0) = \frac{1}{4}, y'(0) = 0$
- $E(t) = 20 \cos t; R \frac{dq}{dt} = 4 \frac{dq}{dt}; \frac{1}{C} q = 10q; L \frac{di}{dt} = 2 \frac{d^2 q}{dt^2} \Rightarrow 2 \frac{d^2 q}{dt^2} + 4 \frac{dq}{dt} + 10q = 20 \cos t, q(0) = 2, q'(0) = 3$
- $L = 2, R = 12, C = \frac{1}{16}, E(t) = 300 \Rightarrow 2 \frac{d^2 q}{dt^2} + 12 \frac{dq}{dt} + 16q = 300, q(0) = 0, q'(0) = 0$
- $mg = 16 \Rightarrow m = \frac{16}{32}; k = 1; \text{resistance} = \text{velocity} \Rightarrow \delta = 1 \Rightarrow \frac{1}{2} \frac{d^2 y}{dt^2} + 1 \frac{dy}{dt} + 1y = 0, y(0) = 2, y'(0) = 2 \\ \Rightarrow \frac{1}{2}r^2 + r + 1 = 0 \Rightarrow r^2 + 2r + 2 = 0 \Rightarrow r = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2(1)} = -1 \pm i \Rightarrow y = e^{-t}(c_1 \cos t + c_2 \sin t), \\ \Rightarrow y' = e^{-t}(-c_1 \sin t + c_2 \cos t) - e^{-t}(c_1 \cos t + c_2 \sin t); y(0) = 2 \Rightarrow c_1 = 2; y'(0) = 2 \Rightarrow c_2 - c_1 = 2 \Rightarrow c_2 = 4 \\ \Rightarrow y(t) = e^{-t}(2 \cos t + 4 \sin t). \text{ At } t = \pi, y = e^{-\pi}(2 \cos \pi + 4 \sin \pi) = -2e^{-\pi} \approx -0.0864 \Rightarrow 0.0864 \text{ ft above equilibrium.}$
- $w = 8 \Rightarrow m = \frac{8}{32}; 8 = k \cdot 4 \Rightarrow k = 2; \text{resistance} = 1.5 v \Rightarrow \delta = 1.5 \Rightarrow \frac{1}{4} \frac{d^2 y}{dt^2} + 1.5 \frac{dy}{dt} + 2y = 0, y(0) = -2, y'(0) = 3 \\ \Rightarrow \frac{1}{4}r^2 + 1.5r + 2 = 0 \Rightarrow r^2 + 6r + 8 = 0 \Rightarrow (r+4)(r+2) \Rightarrow r = -4 \text{ or } r = -2 \Rightarrow y = c_1 e^{-4t} + c_2 e^{-2t} \\ \Rightarrow y' = -4c_1 e^{-4t} - 2c_2 e^{-2t}; y(0) = -2 \Rightarrow c_1 + c_2 = -2; \Rightarrow y'(0) = 3 \Rightarrow -4c_1 - 2c_2 = 3 \Rightarrow c_1 = \frac{1}{2}, c_2 = -\frac{5}{2} \\ \Rightarrow y(t) = \frac{1}{2}e^{-4t} - \frac{5}{2}e^{-2t}. \text{ At } t = 2, y = \frac{1}{2}e^{-8} - \frac{5}{2}e^{-2} \approx -0.0456 \Rightarrow 0.0456 \text{ ft above equilibrium.}$

9. $20 = k \cdot \frac{1}{2} \Rightarrow k = 40$; $w = 25 \text{ lb} \Rightarrow m = \frac{25}{32}$; $\delta = 0 \Rightarrow \frac{25}{32} \frac{d^2y}{dt^2} + 0 \cdot \frac{dy}{dt} + 40y = 0$. If $w = 25 \text{ lb}$, it stretches the spring $25 = 40x \Rightarrow x = \frac{5}{8} \text{ ft} \Rightarrow$ spring is now stretched $\frac{6+5}{12} - \frac{5}{8} = \frac{7}{24} \text{ ft}$ below equilibrium $\Rightarrow y(0) = \frac{7}{24}$; initial velocity is $v_0 \frac{\text{in}}{\text{sec}} = \frac{v_0}{12} \frac{\text{ft}}{\text{sec}} \Rightarrow y'(0) = \frac{v_0}{12}$. Thus we have $\frac{25}{32} \frac{d^2y}{dt^2} + 40y = 0$, $y(0) = \frac{7}{24}$, $y'(0) = \frac{v_0}{12} \Rightarrow \frac{25}{32} r^2 + 40 = 0 \Rightarrow r^2 + \frac{256}{5} = 0$
 $\Rightarrow r = 0 \pm \frac{16}{\sqrt{5}}i \Rightarrow y = e^{0 \cdot t} \left(c_1 \cos\left(\frac{16}{\sqrt{5}}t\right) + c_2 \sin\left(\frac{16}{\sqrt{5}}t\right) \right) = c_1 \cos\left(\frac{16}{\sqrt{5}}t\right) + c_2 \sin\left(\frac{16}{\sqrt{5}}t\right)$, $y(0) = \frac{7}{24} \Rightarrow c_1 = \frac{7}{24}$;
 $y' = -\frac{16}{\sqrt{5}}c_1 \sin\left(\frac{16}{\sqrt{5}}t\right) + \frac{16}{\sqrt{5}}c_2 \cos\left(\frac{16}{\sqrt{5}}t\right)$, $y'(0) = \frac{v_0}{12} \Rightarrow \frac{16}{\sqrt{5}}c_2 = \frac{v_0}{12} \Rightarrow c_2 = \frac{v_0\sqrt{5}}{192}$
 $\Rightarrow y(t) = \frac{7}{24} \cos\left(\frac{16}{\sqrt{5}}t\right) + \frac{v_0\sqrt{5}}{192} \sin\left(\frac{16}{\sqrt{5}}t\right)$ (in feet) or $y(t) = \frac{7}{2} \cos\left(\frac{16}{\sqrt{5}}t\right) + \frac{v_0\sqrt{5}}{16} \sin\left(\frac{16}{\sqrt{5}}t\right)$ (in inches)
10. $m = 1$, $k = \frac{25}{4}$, $\delta = 3 \Rightarrow 1 \cdot \frac{d^2y}{dt^2} + 3 \frac{dy}{dt} + \frac{25}{4}y = 0$, $y(0) = -1$, $y'(0) = 3 \Rightarrow r^2 + 3r + \frac{25}{4} = 0$
 $\Rightarrow 4r^2 + 12r + 25 = 0 \Rightarrow r = \frac{-12 \pm \sqrt{12^2 - 4(4)(25)}}{2(4)} = -\frac{3}{2} \pm 2i \Rightarrow y = e^{-\frac{3}{2}t} (c_1 \cos 2t + c_2 \sin 2t)$
 $\Rightarrow y' = e^{-\frac{3}{2}t} \left(\left(-\frac{3}{2}c_1 + 2c_2\right) \cos 2t + \left(-2c_1 - \frac{3}{2}c_2\right) \sin 2t \right)$; $y(0) = -1 \Rightarrow c_1 = -1$, $y'(0) = 3 \Rightarrow -\frac{3}{2}c_1 + 2c_2 = 3$
 $\Rightarrow c_1 = -1$, $c_2 = \frac{3}{4} \Rightarrow y(t) = e^{-\frac{3}{2}t} (-\cos 2t + \frac{3}{4} \sin 2t)$
11. $mg = 10 \Rightarrow m = \frac{5}{16}$; $10 = k \cdot \frac{1}{6} \Rightarrow k = 60$; $\delta = \frac{40}{\sqrt{32}} = 5\sqrt{2} \Rightarrow \frac{5}{16} \frac{d^2y}{dt^2} + 5\sqrt{2} \frac{dy}{dt} + 60y = 0$, $y(0) = \frac{1}{4}$, $y'(0) = 0$
 $\Rightarrow \frac{5}{16}r^2 + 5\sqrt{2}r + 60 = 0 \Rightarrow r^2 + 16\sqrt{2}r + 192 = 0 \Rightarrow r = \frac{-16\sqrt{2} \pm \sqrt{(16\sqrt{2})^2 - 4(1)(192)}}{2(1)} = -8\sqrt{2} \pm 8i$
 $\Rightarrow y = e^{-8\sqrt{2}t} (c_1 \cos 8t + c_2 \sin 8t) \Rightarrow y' = e^{-8\sqrt{2}t} \left((-8\sqrt{2}c_1 + 8c_2) \cos 8t + (-8c_1 - 8\sqrt{2}c_2) \sin 8t \right)$;
 $y(0) = \frac{1}{4} \Rightarrow c_1 = \frac{1}{4}$, $y'(0) = 0 \Rightarrow -8\sqrt{2}c_1 + 8c_2 = 0 \Rightarrow c_1 = \frac{1}{4}$, $c_2 = \frac{\sqrt{2}}{4} \Rightarrow y = e^{-8\sqrt{2}t} \left(\frac{1}{4} \cos 8t + \frac{\sqrt{2}}{4} \sin 8t \right)$
Solve $y(t) = 0 \Rightarrow e^{-8\sqrt{2}t} \left(\frac{1}{4} \cos 8t + \frac{\sqrt{2}}{4} \sin 8t \right) = 0 \Rightarrow \frac{1}{4} \cos 8t + \frac{\sqrt{2}}{4} \sin 8t = 0 \Rightarrow \tan 8t = -\frac{1}{\sqrt{2}} \Rightarrow t \approx 0.3157 \text{ sec}$
12. $w = mg$; $mg = k \cdot \frac{1}{2} \Rightarrow k = 2mg = 64m$; $\delta = 0 \Rightarrow m \frac{d^2y}{dt^2} + 0 \cdot \frac{dy}{dt} + 64my = 0$, $y(0) = \frac{1}{6}$, $y'(0) = \frac{1}{6} \Rightarrow mr^2 + 64m = 0$,
 $\Rightarrow r^2 + 64 = 0 \Rightarrow r = 0 \pm 8i \Rightarrow y = e^{0 \cdot t} (c_1 \cos 8t + c_2 \sin 8t) = c_1 \cos 8t + c_2 \sin 8t \Rightarrow y' = -8c_1 \sin 8t + 8c_2 \cos 8t$
 $y(0) = \frac{1}{6} \Rightarrow c_1 = \frac{1}{6}$, $y'(0) = \frac{1}{6} \Rightarrow 8c_2 = \frac{1}{6} \Rightarrow c_1 = \frac{1}{6}$, $c_2 = \frac{1}{48} \Rightarrow y(t) = \frac{1}{6} \cos 8t + \frac{1}{48} \sin 8t$
(a) Solve $y(t) = 0 \Rightarrow \frac{1}{6} \cos 8t + \frac{1}{48} \sin 8t = 0 \Rightarrow \tan 8t = -8 \Rightarrow t \approx -0.1808$, first positive solution is $t = -0.1808 + \frac{\pi}{8}$
 $\approx 0.2119 \text{ sec}$
(b) $y' = -\frac{4}{3} \sin 8t + \frac{1}{6} \cos 8t$, solve $y'(t) = 0 \Rightarrow -\frac{4}{3} \sin 8t + \frac{1}{6} \cos 8t = 0 \Rightarrow \tan 8t = \frac{1}{8} \Rightarrow t \approx 0.4082 \text{ sec}$
 $\Rightarrow y(0.4082) \approx 0.1680 \text{ ft}$
(c) $y'' = -\frac{32}{3} \cos 8t - \frac{4}{3} \sin 8t$, solve $y''(t) = 0 \Rightarrow -\frac{32}{3} \cos 8t - \frac{4}{3} \sin 8t = 0 \Rightarrow \tan 8t = -8 \Rightarrow t = \frac{\tan^{-1}(-8)}{8}$, first positive solution where maximum occurs is at $t = \frac{\tan^{-1}(-8)}{8} + \frac{\pi}{4} \Rightarrow y' \left(\frac{\tan^{-1}(-8)}{8} + \frac{\pi}{4} \right)$
 $= -\frac{4}{3} \sin 8 \left(\frac{\tan^{-1}(-8)}{8} + \frac{\pi}{4} \right) + \frac{1}{6} \cos 8 \left(\frac{\tan^{-1}(-8)}{8} + \frac{\pi}{4} \right) = -\frac{4}{3} \sin (\tan^{-1}(-8) + 2\pi) + \frac{1}{6} \cos ((\tan^{-1}(-8) + 2\pi))$
 $= -\frac{4}{3} [\sin (\tan^{-1}(-8)) \cos 2\pi + \cos (\tan^{-1}(-8)) \sin 2\pi] + \frac{1}{6} [\cos (\tan^{-1}(-8)) \cos 2\pi - \sin (\tan^{-1}(-8)) \sin 2\pi]$
 $= -\frac{4}{3} \left[-\frac{8}{\sqrt{65}} + 0 \right] + \frac{1}{6} \left[\frac{1}{\sqrt{65}} - 0 \right] = \frac{65}{6\sqrt{65}} = \frac{1}{6} \sqrt{65} \frac{\text{ft}}{\text{sec}} \Rightarrow 12 \left(\frac{1}{6} \sqrt{65} \right) = 2\sqrt{65} \frac{\text{in}}{\text{sec}}$. Note that $2\sqrt{2g+1} = 2\sqrt{65}$ when $g = 32$.
13. First weight: $w = 10 \text{ lb} \Rightarrow m = \frac{5}{16}$; $10 = k \cdot \left(\frac{5}{6}\right) \Rightarrow k = 12 \frac{\text{lb}}{\text{ft}}$; $\delta = 0 \Rightarrow \frac{5}{16} \frac{d^2y}{dt^2} + 0 \cdot \frac{dy}{dt} + 12y = 0$, $y(0) = \frac{1}{6}$, $y'(0) = -\frac{1}{3}$
 $\Rightarrow \frac{5}{16}r^2 + 12 = 0 \Rightarrow 5r^2 + 192 = 0 \Rightarrow r = 0 \pm \frac{8\sqrt{15}}{5}i \Rightarrow y = e^{0 \cdot t} \left(c_1 \cos \frac{8\sqrt{15}}{5}t + c_2 \sin \frac{8\sqrt{15}}{5}t \right)$
 $= c_1 \cos \frac{8\sqrt{15}}{5}t + c_2 \sin \frac{8\sqrt{15}}{5}t \Rightarrow y' = -\frac{8\sqrt{15}}{5}c_1 \sin \frac{8\sqrt{15}}{5}t + \frac{8\sqrt{15}}{5}c_2 \cos \frac{8\sqrt{15}}{5}t$; $y(0) = \frac{1}{6} \Rightarrow c_1 = \frac{1}{6}$, $y'(0) = -\frac{1}{3}$
 $\Rightarrow \frac{8\sqrt{15}}{5}c_2 = -\frac{1}{3} \Rightarrow c_1 = \frac{1}{6}$, $c_2 = -\frac{\sqrt{15}}{72} \Rightarrow y = \frac{1}{6} \cos \frac{8\sqrt{15}}{5}t - \frac{\sqrt{15}}{72} \sin \frac{8\sqrt{15}}{5}t$. The amplitude is $C = \sqrt{\left(\frac{1}{6}\right)^2 + \left(-\frac{\sqrt{15}}{72}\right)^2}$
 $= \frac{\sqrt{159}}{72}$.

Second weight: $x = c_3 \cos \omega t + c_4 \sin \omega t$, $x(0) = \frac{\sqrt{159}}{72}$, $x'(0) = 2 \Rightarrow x' = -\omega c_3 \sin \omega t + \omega c_4 \cos \omega t$; $x(0) = \frac{\sqrt{159}}{72} \Rightarrow c_3 = \frac{\sqrt{159}}{72}$, $x'(0) = 2 \Rightarrow \omega c_4 = 2 \Rightarrow c_3 = \frac{\sqrt{159}}{72}$, $c_4 = \frac{2}{\omega} \Rightarrow x = \frac{\sqrt{159}}{72} \cos \omega t + \frac{2}{\omega} \sin \omega t$. Since amplitude of second spring $= 2C \Rightarrow 2\left(\frac{\sqrt{159}}{72}\right) = \sqrt{\left(\frac{\sqrt{159}}{72}\right)^2 + \left(\frac{2}{\omega}\right)^2} \Rightarrow \omega = \frac{48}{\sqrt{53}} \Rightarrow \frac{48}{\sqrt{53}} = \sqrt{\frac{k}{m}} = \sqrt{\frac{12}{m}} \Rightarrow m = \frac{53}{192} \Rightarrow mg = \left(\frac{53}{192}\right)32 = 8.8333 \text{ lbs}$

14. First spring: $m_1 g = k_1 \cdot \frac{1}{4} \Rightarrow k_1 = 128m_1$; $\delta_1 = 0 \Rightarrow m_1 \frac{d^2 y}{dt^2} + 0 \cdot \frac{dy}{dt} + 128m_1 y = 0$, $y(0) = \frac{1}{12}$, $y'(0) = 0$

$$\Rightarrow m_1 r^2 + 128m_1 = 0 \Rightarrow r^2 + 128 = 0 \Rightarrow r = 0 \pm 8\sqrt{2}i \Rightarrow y = e^{0t} \left(c_1 \cos 8\sqrt{2}t + c_2 \sin 8\sqrt{2}t \right)$$

$$= c_1 \cos 8\sqrt{2}t + c_2 \sin 8\sqrt{2}t \Rightarrow y' = -8\sqrt{2}c_1 \sin 8\sqrt{2}t + 8\sqrt{2}c_2 \cos 8\sqrt{2}t$$
; $y(0) = \frac{1}{12} \Rightarrow c_1 = \frac{1}{12}$, $y'(0) = 0 \Rightarrow 8\sqrt{2}c_2 = 0 \Rightarrow c_2 = 0$; $c_1 = \frac{1}{12}$, $c_2 = 0 \Rightarrow y(t) = \frac{1}{12} \cos 8\sqrt{2}t$

Second spring: $m_2 g = k_2 \cdot \frac{3}{4} \Rightarrow k_2 = \frac{128}{3}m_2$; $\delta_2 = 0 \Rightarrow m_2 \frac{d^2 x}{dt^2} + 0 \cdot \frac{dx}{dt} + \frac{128}{3}m_2 x = 0$, $x(0) = \frac{1}{12}$, $x'(0) = 0$

$$\Rightarrow \frac{d^2 x}{dt^2} + \frac{128}{3}x = 0 \Rightarrow r^2 + \frac{128}{3} = 0 \Rightarrow r = 0 \pm 8\sqrt{\frac{2}{3}}i \Rightarrow x = e^{0t} \left(c_3 \cos 8\sqrt{\frac{2}{3}}t + c_4 \sin 8\sqrt{\frac{2}{3}}t \right)$$

$$= c_3 \cos 8\sqrt{\frac{2}{3}}t + c_4 \sin 8\sqrt{\frac{2}{3}}t \Rightarrow y' = -8\sqrt{\frac{2}{3}}c_3 \sin 8\sqrt{\frac{2}{3}}t + 8\sqrt{\frac{2}{3}}c_4 \cos 8\sqrt{\frac{2}{3}}t$$
; $x(0) = \frac{1}{12} \Rightarrow c_3 = \frac{1}{12}$, $x'(0) = 0 \Rightarrow 8\sqrt{\frac{2}{3}}c_4 = 0 \Rightarrow c_4 = 0$; $c_3 = \frac{1}{12}$, $c_4 = 0 \Rightarrow x(t) = \frac{1}{12} \cos 8\sqrt{\frac{2}{3}}t$

Equal velocities $\Rightarrow y' = x' \Rightarrow -\frac{2\sqrt{2}}{3} \sin 8\sqrt{2}t = -\frac{2}{3} \sqrt{\frac{2}{3}} \sin 8\sqrt{\frac{2}{3}}t \Rightarrow t \approx 0.2237 \text{ sec}$

15. $mg = 16 \Rightarrow m = \frac{1}{2}$; $16 = k \cdot 4 \Rightarrow k = 4$; $\delta = 0 \Rightarrow \frac{1}{2} \frac{d^2 y}{dt^2} + 0 \cdot \frac{dy}{dt} + 4y = 0$, $y(0) = 5$, $y'(0) = 0 \Rightarrow \frac{1}{2}r^2 + 4 = 0$

$$\Rightarrow r^2 + 8 = 0 \Rightarrow r = 0 \pm 2\sqrt{2}i \Rightarrow y = e^{0t} \left(c_1 \cos 2\sqrt{2}t + c_2 \sin 2\sqrt{2}t \right) = c_1 \cos 2\sqrt{2}t + c_2 \sin 2\sqrt{2}t$$

$$\Rightarrow y' = -2\sqrt{2}c_1 \sin 2\sqrt{2}t + 2\sqrt{2}c_2 \cos 2\sqrt{2}t$$
; $y(0) = 5 \Rightarrow c_1 = 5$, $y'(0) = 0 \Rightarrow 2\sqrt{2}c_2 = 0 \Rightarrow c_2 = 0$

$$\Rightarrow y(t) = 5 \cos 2\sqrt{2}t. \text{ The amplitude is } C = \sqrt{5^2 + 0^2} = 5$$

$y = c_3 \cos 2\sqrt{2}t + c_4 \sin 2\sqrt{2}t$, $y(0) = 5$, $y'(0) = v_0 \Rightarrow y' = -2\sqrt{2}c_3 \sin 2\sqrt{2}t + 2\sqrt{2}c_4 \cos 2\sqrt{2}t$; $y(0) = 5 \Rightarrow c_3 = 5$

$$y'(0) = v_0 \Rightarrow 2\sqrt{2}c_4 = v_0 \Rightarrow c_4 = \frac{v_0}{2\sqrt{2}} \Rightarrow y(t) = 5 \cos 2\sqrt{2}t + \frac{v_0}{2\sqrt{2}} \sin 2\sqrt{2}t, \text{ and the new amplitude is } 2 \cdot 5$$

$$\Rightarrow 10 = \sqrt{5^2 + \left(\frac{v_0}{2\sqrt{2}}\right)^2} \Rightarrow v_0 = 10\sqrt{6} \approx 24.4949 \frac{\text{ft}}{\text{sec}}$$

16. $mg = 8 \Rightarrow m = \frac{1}{4}$; $8 = k \cdot \frac{3}{12} \Rightarrow k = 32$; $\delta = 2 \Rightarrow \frac{1}{4} \frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} + 32y = 0$, $y(0) = 0$, $y'(0) = \frac{1}{3} \Rightarrow \frac{1}{4}r^2 + 2r + 32 = 0$

$$\Rightarrow r^2 + 8r + 128 = 0 \Rightarrow r = \frac{-8 \pm \sqrt{8^2 - 4(1)(128)}}{2(1)} = -4 \pm 4\sqrt{7}i \Rightarrow y = e^{-4t} \left(c_1 \cos 4\sqrt{7}t + c_2 \sin 4\sqrt{7}t \right)$$

$$\Rightarrow y' = e^{-4t} \left[(-4c_1 + 4\sqrt{7}c_2) \cos 4\sqrt{7}t + (-4\sqrt{7}c_1 - 4c_2) \sin 4\sqrt{7}t \right]$$
; $y(0) = 0 \Rightarrow c_1 = 0$, $y'(0) = \frac{1}{3}$

$$\Rightarrow -4c_1 + 4\sqrt{7}c_2 = \frac{1}{3} \Rightarrow c_1 = 0$$
, $c_2 = \frac{1}{12\sqrt{7}} \Rightarrow y(t) = \frac{1}{12\sqrt{7}} e^{-4t} \sin 4\sqrt{7}t$. Solve $y(t) = 0 \Rightarrow \frac{1}{12\sqrt{7}} e^{-4t} \sin 4\sqrt{7}t = 0$

$$\Rightarrow \sin 4\sqrt{7}t = 0 \Rightarrow t = \frac{\pi}{4\sqrt{7}} \approx 0.2969 \text{ sec}$$

17. δ decreases by 90% in 10 sec $\Rightarrow$ 10% remains $\Rightarrow e^{-10b} = \frac{1}{10} \Rightarrow b = -\frac{1}{10} \ln\left(\frac{1}{10}\right) = \frac{\ln 10}{10} \Rightarrow 2b = \frac{\delta}{m} \Rightarrow \delta = \frac{\ln 10}{5}m$

period $= 2 \text{ sec} \Rightarrow 2 = \frac{2\pi}{\sqrt{\omega^2 - b^2}} \Rightarrow 4 = \frac{4\pi^2}{\omega^2 - b^2} \Rightarrow \omega^2 = \pi^2 + b^2 = \pi^2 + \left(\frac{\ln 10}{10}\right)^2 = \frac{100\pi^2 + (\ln 10)^2}{100} \Rightarrow \frac{k}{m} = \frac{100\pi^2 + (\ln 10)^2}{100}$

$k = \frac{100\pi^2 + (\ln 10)^2}{100}m \Rightarrow m \frac{d^2 y}{dt^2} + \left(\frac{\ln 10}{5}m\right) \frac{dy}{dt} + \left(\frac{100\pi^2 + (\ln 10)^2}{100}m\right)y = 0$. When $y = \frac{1}{4}$ and $y' = -2$, then

$$m \frac{d^2 y}{dt^2} + \left(\frac{\ln 10}{5}m\right)(-2) + \left(\frac{100\pi^2 + (\ln 10)^2}{100}m\right)\left(\frac{1}{4}\right) = 0 \Rightarrow \frac{d^2 y}{dt^2} = \frac{2 \ln 10}{5} - \frac{100\pi^2 + (\ln 10)^2}{400} \approx -1.5596 \frac{\text{ft}}{\text{sec}^2}$$

18. $mg = 10 \Rightarrow m = \frac{5}{16}$; $10 = k \cdot 2 \Rightarrow k = 5$; $\delta = \frac{10}{\sqrt{32}} = \frac{5\sqrt{2}}{4} \Rightarrow \frac{5}{16} \frac{d^2y}{dt^2} + \frac{5\sqrt{2}}{4} \frac{dy}{dt} + 5y = 0$, $y(0) = \frac{1}{2}$, $y'(0) = 0$
 $\Rightarrow \frac{5}{16}r^2 + \frac{5\sqrt{2}}{4}r + 5 = 0 \Rightarrow r^2 + 4\sqrt{2}r + 16 = 0 \Rightarrow r = \frac{-4\sqrt{2} \pm \sqrt{(4\sqrt{2})^2 - 4(1)(16)}}{2(1)} = -2\sqrt{2} \pm 2\sqrt{2}i$
 $\Rightarrow y(t) = e^{-2\sqrt{2}t} (c_1 \cos 2\sqrt{2}t + c_2 \sin 2\sqrt{2}t)$
 $\Rightarrow y' = e^{-2\sqrt{2}t} [(-2\sqrt{2}c_1 + 2\sqrt{2}c_2) \cos 2\sqrt{2}t + (-2\sqrt{2}c_1 - 2\sqrt{2}c_2) \sin 2\sqrt{2}t]$; $y(0) = \frac{1}{2} \Rightarrow c_1 = \frac{1}{2}$,
 $y'(0) = 0 \Rightarrow -2\sqrt{2}c_1 + 2\sqrt{2}c_2 = 0 \Rightarrow c_1 = \frac{1}{2}$, $c_2 = \frac{1}{2} \Rightarrow y = e^{-2\sqrt{2}t} (\frac{1}{2} \cos 2\sqrt{2}t + \frac{1}{2} \sin 2\sqrt{2}t)$
 To find maximum, solve $y'(t) = 0 \Rightarrow y'(t) = -2\sqrt{2}e^{-2\sqrt{2}t} \sin 2\sqrt{2}t \Rightarrow -2\sqrt{2}e^{-2\sqrt{2}t} \sin 2\sqrt{2}t = 0$
 $\Rightarrow \sin 2\sqrt{2}t = 0 \Rightarrow t = 0, \frac{\pi}{2\sqrt{2}}, \frac{\pi}{\sqrt{2}}$, and at $t = \frac{\pi}{2\sqrt{2}}$, $y\left(\frac{\pi}{2\sqrt{2}}\right) \approx -0.02161 \text{ ft} \approx -0.2593 \text{ in (above equilibrium)}$
19. $L = \frac{1}{5}$, $R = 1$, $C = \frac{5}{6}$, $E(t) = 0 \Rightarrow \frac{1}{5} \frac{d^2q}{dt^2} + 1 \cdot \frac{dq}{dt} + \frac{6}{5}q = 0$, $q(0) = 2$, $q'(0) = 4 \Rightarrow \frac{1}{5}r^2 + r + \frac{6}{5} = 0$
 $\Rightarrow r^2 + 5r + 6 = 0 \Rightarrow (r+3)(r+2) = 0 \Rightarrow r = -3 \text{ or } r = -2 \Rightarrow q(t) = c_1 e^{-3t} + c_2 e^{-2t} \Rightarrow q' = -3c_1 e^{-3t} - 2c_2 e^{-2t}$
 $q(0) = 2 \Rightarrow c_1 + c_2 = 2$; $q'(0) = 4 \Rightarrow -3c_1 - 2c_2 = 4 \Rightarrow c_1 = -8$, $c_2 = 10 \Rightarrow q = -8e^{-3t} + 10e^{-2t}$
 $\lim_{t \rightarrow \infty} q = \lim_{t \rightarrow \infty} (-8e^{-3t} + 10e^{-2t}) = 0$
20. $\frac{1}{C}q = 10q$; $R \frac{dq}{dt} = 4 \frac{dq}{dt}$; $L \frac{d^2q}{dt^2} = 2 \frac{d^2q}{dt^2}$, $E(t) = 0 \Rightarrow 2 \frac{d^2q}{dt^2} + 4 \frac{dq}{dt} + 10q = 0$, $q(0) = 2$, $q'(0) = 3 \Rightarrow 2r^2 + 4r + 10 = 0$
 $\Rightarrow r^2 + 2r + 5 = 0 \Rightarrow r = \frac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2(1)} = -1 \pm 2i \Rightarrow q(t) = e^{-t}(c_1 \cos 2t + c_2 \sin 2t)$
 $\Rightarrow q' = e^{-t}((-c_1 + 2c_2) \cos 2t + (-2c_1 - c_2) \sin 2t)$; $q(0) = 2 \Rightarrow c_1 + c_2 = 2$; $q'(0) = 3 \Rightarrow -c_1 + 2c_2 = 3$
 $\Rightarrow c_1 = \frac{1}{3}$, $c_2 = \frac{5}{3} \Rightarrow q(t) = e^{-t}(\frac{1}{3} \cos 2t + \frac{5}{3} \sin 2t)$
21. $mg = 16 \Rightarrow m = \frac{16}{32}$; $16 = k \cdot 4 \Rightarrow k = 4$; $\delta = 4.5$; $f(t) = 4 + e^{-2t} \Rightarrow \frac{1}{2} \frac{d^2y}{dt^2} + 4.5 \frac{dy}{dt} + 4y = 4 + e^{-2t}$, $y(0) = 2$, $y'(0) = 4$
 $\Rightarrow \frac{1}{2}r^2 + 4.5r + 4 = 0 \Rightarrow r^2 + 9r + 8 = 0 \Rightarrow (r+8)(r+1) = 0 \Rightarrow r = -8 \text{ or } r = -1 \Rightarrow y_c = c_1 e^{-8t} + c_2 e^{-t}$;
 $y_p = A + Be^{-2t} \Rightarrow y'_p = -2Be^{-2t} \Rightarrow y''_p = 4Be^{-2t} \Rightarrow \frac{1}{2}(4Be^{-2t}) + 4.5(-2Be^{-2t}) + 4(A + Be^{-2t}) = 4 + e^{-2t}$
 $\Rightarrow 4A - 3Be^{-2t} = 4 + e^{-2t} \Rightarrow 4A = 4$, $-3B = 1 \Rightarrow A = 1$, $B = -\frac{1}{3} \Rightarrow y(t) = c_1 e^{-8t} + c_2 e^{-t} + 1 - \frac{1}{3}e^{-2t}$
 $\Rightarrow y' = -8c_1 e^{-8t} - c_2 e^{-t} + \frac{2}{3}e^{-2t}$; $y(0) = 2 \Rightarrow c_1 + c_2 + \frac{2}{3} = 2 \Rightarrow c_1 + c_2 = \frac{4}{3}$, $y'(0) = 4 \Rightarrow -8c_1 - c_2 + \frac{2}{3} = 4$
 $\Rightarrow -8c_1 - c_2 = \frac{10}{3} \Rightarrow c_1 = -\frac{2}{3}$, $c_2 = 2 \Rightarrow y(t) = -\frac{2}{3}e^{-8t} + 2e^{-t} + 1 - \frac{1}{3}e^{-2t}$
22. $m = 10$; $k = 140$; $\delta = 90$; $f(t) = 5 \sin t \Rightarrow 10 \frac{d^2y}{dt^2} + 90 \frac{dy}{dt} + 140y = 5 \sin t$, $y(0) = 0$, $y'(0) = -1 \Rightarrow 10r^2 + 90r + 140 = 0$
 $\Rightarrow r^2 + 9r + 14 = 0 \Rightarrow (r+2)(r+7) = 0 \Rightarrow r = -2 \text{ or } r = -7 \Rightarrow y_c = c_1 e^{-2t} + c_2 e^{-7t}$; $y_p = A \sin t + B \cos t$
 $\Rightarrow y'_p = A \cos t - B \sin t \Rightarrow y''_p = -A \sin t - B \cos t$
 $\Rightarrow 10(-A \sin t - B \cos t) + 90(A \cos t - B \sin t) + 140(A \sin t + B \cos t) = 5 \sin t$
 $\Rightarrow (130A - 90B) \sin t + (90A + 130B) \cos t = 5 \sin t \Rightarrow A = \frac{13}{500}$, $B = -\frac{9}{500}$
 $\Rightarrow y(t) = c_1 e^{-2t} + c_2 e^{-7t} + \frac{13}{500} \sin t - \frac{9}{500} \cos t \Rightarrow y' = -2c_1 e^{-2t} - 7c_2 e^{-7t} + \frac{13}{500} \cos t + \frac{9}{500} \sin t$;
 $y(0) = 0 \Rightarrow c_1 + c_2 - \frac{9}{500} = 0$, $y'(0) = -1 \Rightarrow -2c_1 - 7c_2 + \frac{13}{500} = -1 \Rightarrow c_1 + c_2 = \frac{9}{500}$, $-2c_1 - 7c_2 = -\frac{513}{500}$
 $\Rightarrow c_1 = -\frac{9}{50}$, $c_2 = \frac{99}{500} \Rightarrow y(t) = -\frac{9}{50}e^{-2t} + \frac{99}{500}e^{-7t} + \frac{13}{500} \sin t - \frac{9}{500} \cos t$
23. $m = 2 \Rightarrow mg = 2(9.8) = 19.6$; $19.6 = k \cdot 1.96 \Rightarrow k = 10$; $\delta = 4$; $f(t) = 20 \cos t \Rightarrow 2 \frac{d^2y}{dt^2} + 4 \frac{dy}{dt} + 10y = 20 \cos t$,
 $y(0) = 2$, $y'(0) = 3 \Rightarrow 2r^2 + 4r + 10 = 0 \Rightarrow r = \frac{-4 \pm \sqrt{4^2 - 4(2)(10)}}{2(2)} = -1 \pm 2i \Rightarrow y_c = e^{-t}(c_1 \cos 2t + c_2 \sin 2t)$
 $y_p = A \sin t + B \cos t \Rightarrow y'_p = A \cos t - B \sin t \Rightarrow y''_p = -A \sin t - B \cos t$
 $\Rightarrow 2(-A \sin t - B \cos t) + 4(A \cos t - B \sin t) + 10(A \sin t + B \cos t) = 20 \cos t$
 $\Rightarrow (8A - 4B) \sin t + (4A + 8B) \cos t = 20 \cos t \Rightarrow 8A - 4B = 0$, $4A + 8B = 20 \Rightarrow A = 1$, $B = 2$
 $\Rightarrow y(t) = e^{-t}(c_1 \cos 2t + c_2 \sin 2t) + \sin t + 2 \cos t \Rightarrow y' = e^{-t}((-c_1 + 2c_2) \cos 2t + (-2c_1 - c_2) \sin 2t) + \cos t - 2 \sin t$;

$$y(0) = 2 \Rightarrow c_1 + 2 = 2, y'(0) = 3 \Rightarrow -c_1 + 2c_2 + 1 = 3 \Rightarrow c_1 = 0, -c_1 + 2c_2 = 2 \Rightarrow c_1 = 0, c_2 = 1$$

$$\Rightarrow y(t) = e^{-t} \sin 2t + \sin t + 2 \cos t; y(\pi) = -2 \Rightarrow 2 \text{ m above equilibrium}$$

24. $mg = 8 \Rightarrow m = \frac{8}{32}; 8 = k \cdot 4 \Rightarrow k = 2; \delta = 1.5; f(t) = 6 + e^{-t} \Rightarrow \frac{1}{4} \frac{d^2 y}{dt^2} + 1.5 \frac{dy}{dt} + 2y = 6 + e^{-t}, y(0) = -2, y'(0) = 3$
 $\Rightarrow \frac{1}{4} r^2 + 1.5r + 2 = 0 \Rightarrow r^2 + 6r + 8 = 0 \Rightarrow (r+2)(r+4) = 0 \Rightarrow r = -2 \text{ or } r = -4 \Rightarrow y_c = c_1 e^{-2t} + c_2 e^{-4t};$
 $y_p = A + B e^{-t} \Rightarrow y_p' = -B e^{-t} \Rightarrow y_p'' = B e^{-t} \Rightarrow \frac{1}{4}(B e^{-t}) + 1.5(-B e^{-t}) + 2(A + B e^{-t}) = 6 + e^{-t}$
 $\Rightarrow 2A + \frac{3}{4} B e^{-t} = 6 + e^{-t} \Rightarrow 2A = 6, \frac{3}{4} B = 1 \Rightarrow A = 3, B = \frac{4}{3} \Rightarrow y(t) = c_1 e^{-2t} + c_2 e^{-4t} + 3 + \frac{4}{3} e^{-t}$
 $\Rightarrow y' = -2c_1 e^{-2t} - 4c_2 e^{-4t} - \frac{4}{3} e^{-t}; y(0) = -2 \Rightarrow c_1 + c_2 + \frac{13}{3} = -2 \Rightarrow c_1 + c_2 = -\frac{19}{3}, y'(0) = 3$
 $\Rightarrow -2c_1 - 4c_2 - \frac{4}{3} = 3 \Rightarrow -2c_1 - 4c_2 = \frac{13}{3} \Rightarrow c_1 = -\frac{21}{2}, c_2 = \frac{25}{6} \Rightarrow y(t) = -\frac{21}{2} e^{-2t} + \frac{25}{6} e^{-4t} + 3 + \frac{4}{3} e^{-t};$
 $y(2) \approx 2.98953 \Rightarrow 2.99 \text{ ft below equilibrium}$

25. $L = 10, R = 10, C = \frac{1}{500}, E(t) = 100 \Rightarrow 10 \frac{d^2 q}{dt^2} + 10 \frac{dq}{dt} + 500q = 100, q(0) = 10, q'(0) = 0 \Rightarrow 10r^2 + 10r + 500 = 0$
 $\Rightarrow r^2 + r + 50 = 0 \Rightarrow r = \frac{-1 \pm \sqrt{1^2 - 4(1)(50)}}{2(1)} = -\frac{1}{2} \pm \frac{\sqrt{199}}{2} i \Rightarrow q_c = e^{-\frac{1}{2}t} \left(c_1 \cos \frac{\sqrt{199}}{2} t + c_2 \sin \frac{\sqrt{199}}{2} t \right)$
 $q_p = A \Rightarrow q_p' = 0 \Rightarrow q_p'' = 0 \Rightarrow 10(0) + 10(0) + 500A = 100 \Rightarrow 500A = 100 \Rightarrow A = \frac{1}{5}$
 $\Rightarrow q(t) = e^{-\frac{1}{2}t} \left(c_1 \cos \frac{\sqrt{199}}{2} t + c_2 \sin \frac{\sqrt{199}}{2} t \right) + \frac{1}{5} \Rightarrow q' = e^{-\frac{1}{2}t} \left[\left(-\frac{1}{2} c_1 + \frac{\sqrt{199}}{2} c_2 \right) \cos \frac{\sqrt{199}}{2} t + \left(-\frac{\sqrt{199}}{2} c_1 - \frac{1}{2} c_2 \right) \sin \frac{\sqrt{199}}{2} t \right]$
 $q(0) = 10 \Rightarrow c_1 + \frac{1}{5} = 10 \Rightarrow c_1 = \frac{49}{5}, q'(0) = 0 \Rightarrow -\frac{1}{2} c_1 + \frac{\sqrt{199}}{2} c_2 = 0 \Rightarrow c_1 = \frac{49}{5}, c_2 = \frac{49\sqrt{199}}{995}$
 $\Rightarrow q(t) = e^{-\frac{1}{2}t} \left(\frac{49}{5} \cos \frac{\sqrt{199}}{2} t + \frac{49\sqrt{199}}{995} \sin \frac{\sqrt{199}}{2} t \right) + \frac{1}{5}$

26. $R \frac{dq}{dt} = 4 \frac{dq}{dt}; \frac{1}{C} q = 10q; L \frac{di}{dt} = 2 \frac{d^2 q}{dt^2}, E(t) = 20 \cos t \Rightarrow 2 \frac{d^2 q}{dt^2} + 4 \frac{dq}{dt} + 10q = 0, q(0) = 2, q'(0) = 3 \Rightarrow 2r^2 + 4r + 10 = 0$
 $\Rightarrow r^2 + 2r + 5 = 0 \Rightarrow r = \frac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2(1)} = -1 \pm 2i \Rightarrow q_c = e^{-t} (c_1 \cos 2t + c_2 \sin 2t); q_p = A \sin t + B \cos t$
 $\Rightarrow q_p' = A \cos t - B \sin t \Rightarrow q_p'' = -A \sin t - B \cos t$
 $\Rightarrow 2(-A \sin t - B \cos t) + 4(A \cos t - B \sin t) + 10(A \sin t + B \cos t) = 20 \cos t$
 $\Rightarrow (8A - 4B) \sin t + (4A + 8B) \cos t = 20 \cos t \Rightarrow 8A - 4B = 0, 4A + 8B = 20 \Rightarrow A = 1, B = 2$
 $\Rightarrow q(t) = e^{-t} (c_1 \cos 2t + c_2 \sin 2t) + \sin t + 2 \cos t \Rightarrow q' = e^{-t} ((-c_1 + 2c_2) \cos 2t + (-2c_1 - c_2) \sin 2t) + \cos t - 2 \sin t;$
 $q(0) = 2 \Rightarrow c_1 + 2 = 2; q'(0) = 3 \Rightarrow -c_1 + 2c_2 + 1 = 3 \Rightarrow c_2 = 1 \Rightarrow q(t) = 2e^{-t} \sin 2t + \sin t + 2 \cos t; y(10) \approx -2.222$
 $\Rightarrow 2.222 \text{ ft above equilibrium}$

17.4 EULER EQUATIONS

- $x^2 y'' + 2x y' - 2y = 0 \Rightarrow r^2 + (2-1)r - 2 = 0 \Rightarrow r^2 + r - 2 = 0 \Rightarrow (r-1)(r+2) = 0 \Rightarrow r = 1 \text{ or } r = -2$
 $\Rightarrow y = c_1 e^x + c_2 e^{-2x} = c_1 e^{\ln x} + c_2 e^{-2 \ln x} \Rightarrow y = c_1 x + \frac{c_2}{x^2}$
- $x^2 y'' + x y' - 4y = 0 \Rightarrow r^2 + (1-1)r - 4 = 0 \Rightarrow r^2 - 4 = 0 \Rightarrow (r-2)(r+2) = 0 \Rightarrow r = 2 \text{ or } r = -2$
 $\Rightarrow y = c_1 e^{2x} + c_2 e^{-2x} = c_1 e^{2 \ln x} + c_2 e^{-2 \ln x} \Rightarrow y = c_1 x^2 + \frac{c_2}{x^2}$
- $x^2 y'' - 6y = 0 \Rightarrow r^2 + (0-1)r - 6 = 0 \Rightarrow r^2 - r - 6 = 0 \Rightarrow (r-3)(r+2) = 0 \Rightarrow r = 3 \text{ or } r = -2$
 $\Rightarrow y = c_1 e^{3x} + c_2 e^{-2x} = c_1 e^{3 \ln x} + c_2 e^{-2 \ln x} \Rightarrow y = c_1 x^3 + \frac{c_2}{x^2}$
- $x^2 y'' + x y' - y = 0 \Rightarrow r^2 + (1-1)r - 1 = 0 \Rightarrow r^2 - 1 = 0 \Rightarrow (r-1)(r+1) = 0 \Rightarrow r = 1 \text{ or } r = -1$
 $\Rightarrow y = c_1 e^x + c_2 e^{-x} = c_1 e^{\ln x} + c_2 e^{-\ln x} \Rightarrow y = c_1 x + \frac{c_2}{x}$
- $x^2 y'' - 5x y' + 8y = 0 \Rightarrow r^2 + (-5-1)r + 8 = 0 \Rightarrow r^2 - 6r + 8 = 0 \Rightarrow (r-4)(r-2) = 0 \Rightarrow r = 4 \text{ or } r = 2$
 $\Rightarrow y = c_1 e^{4x} + c_2 e^{2x} = c_1 e^{4 \ln x} + c_2 e^{2 \ln x} \Rightarrow y = c_1 x^4 + c_2 x^2$

6. $2x^2y'' + 7xy' + 2y = 0 \Rightarrow 2r^2 + (7-2)r + 2 = 0 \Rightarrow 2r^2 + 5r + 2 = 0 \Rightarrow (2r+1)(r+2) = 0 \Rightarrow r = -\frac{1}{2}$ or $r = -2$
 $\Rightarrow y = c_1e^{-\frac{1}{2}z} + c_2e^{-2z} = c_1e^{-\frac{1}{2}\ln x} + c_2e^{-2\ln x} \Rightarrow y = \frac{c_1}{\sqrt{x}} + \frac{c_2}{x^2}$
7. $3x^2y'' + 4xy' = 0 \Rightarrow 3r^2 + (4-3)r = 0 \Rightarrow 3r^2 + r = 0 \Rightarrow r(3r+1) = 0 \Rightarrow r = 0$ or $r = -\frac{1}{3} \Rightarrow y = c_1e^{0 \cdot z} + c_2e^{-\frac{1}{3}z}$
 $= c_1e^{0 \cdot \ln x} + c_2e^{-\frac{1}{3}\ln x} \Rightarrow y = c_1 + \frac{c_2}{\sqrt[3]{x}}$
8. $x^2y'' + 6xy' + 4y = 0 \Rightarrow r^2 + (6-1)r + 4 = 0 \Rightarrow r^2 + 5r + 4 = 0 \Rightarrow (r+1)(r+4) = 0 \Rightarrow r = -1$ or $r = -4$
 $\Rightarrow y = c_1e^{-z} + c_2e^{-4z} = c_1e^{-\ln x} + c_2e^{-4\ln x} \Rightarrow y = \frac{c_1}{x} + \frac{c_2}{x^4}$
9. $x^2y'' - xy' + y = 0 \Rightarrow r^2 + (-1-1)r + 1 = 0 \Rightarrow r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0 \Rightarrow r = 1$, repeated twice
 $\Rightarrow y = c_1e^z + c_2ze^z = c_1e^{\ln x} + c_2 \ln x e^{\ln x} \Rightarrow y = c_1x + c_2x \ln x$
10. $x^2y'' - xy' + 2y = 0 \Rightarrow r^2 + (-1-1)r + 2 = 0 \Rightarrow r^2 - 2r + 2 = 0 \Rightarrow r = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} = 1 \pm i$
 $\Rightarrow y = e^z(c_1 \cos z + c_2 \sin z) = e^{\ln x}(c_1 \cos(\ln x) + c_2 \sin(\ln x)) \Rightarrow y = x(c_1 \cos(\ln x) + c_2 \sin(\ln x))$
11. $x^2y'' - xy' + 5y = 0 \Rightarrow r^2 + (-1-1)r + 5 = 0 \Rightarrow r^2 - 2r + 5 = 0 \Rightarrow r = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} = 1 \pm 2i$
 $\Rightarrow y = e^z(c_1 \cos 2z + c_2 \sin 2z) = e^{\ln x}(c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x)) \Rightarrow y = x(c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x))$
12. $x^2y'' + 7xy' + 13y = 0 \Rightarrow r^2 + (7-1)r + 13 = 0 \Rightarrow r^2 + 6r + 13 = 0 \Rightarrow r = \frac{-6 \pm \sqrt{(6)^2 - 4(1)(13)}}{2(1)} = -3 \pm 2i$
 $\Rightarrow y = e^{-3z}(c_1 \cos 2z + c_2 \sin 2z) = e^{-3 \ln x}(c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x)) \Rightarrow y = \frac{1}{x^3}(c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x))$
13. $x^2y'' + 3xy' + 10y = 0 \Rightarrow r^2 + (3-1)r + 10 = 0 \Rightarrow r^2 + 2r + 10 = 0 \Rightarrow r = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(10)}}{2(1)} = -1 \pm 3i$
 $\Rightarrow y = e^{-z}(c_1 \cos 3z + c_2 \sin 3z) = e^{-\ln x}(c_1 \cos(3 \ln x) + c_2 \sin(3 \ln x)) \Rightarrow y = \frac{1}{x}(c_1 \cos(3 \ln x) + c_2 \sin(3 \ln x))$
14. $x^2y'' - 5xy' + 10y = 0 \Rightarrow r^2 + (-5-1)r + 10 = 0 \Rightarrow r^2 - 6r + 10 = 0 \Rightarrow r = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(10)}}{2(1)} = 3 \pm i$
 $\Rightarrow y = e^{3z}(c_1 \cos z + c_2 \sin z) = e^{3 \ln x}(c_1 \cos(\ln x) + c_2 \sin(\ln x)) \Rightarrow y = x^3(c_1 \cos(\ln x) + c_2 \sin(\ln x))$
15. $4x^2y'' + 8xy' + 5y = 0 \Rightarrow r^2 + (8-4)r + 5 = 0 \Rightarrow 4r^2 + 4r + 5 = 0 \Rightarrow r = \frac{-4 \pm \sqrt{(4)^2 - 4(4)(5)}}{2(4)} = -\frac{1}{2} \pm i$
 $\Rightarrow y = e^{-\frac{1}{2}z}(c_1 \cos z + c_2 \sin z) = e^{-\frac{1}{2} \ln x}(c_1 \cos(\ln x) + c_2 \sin(\ln x)) \Rightarrow y = \frac{1}{\sqrt{x}}(c_1 \cos(\ln x) + c_2 \sin(\ln x))$
16. $4x^2y'' - 4xy' + 5y = 0 \Rightarrow 4r^2 + (-4-4)r + 5 = 0 \Rightarrow 4r^2 - 8r + 5 = 0 \Rightarrow r = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(4)(5)}}{2(4)} = 1 \pm \frac{1}{2}i$
 $\Rightarrow y = e^z(c_1 \cos(\frac{1}{2}z) + c_2 \sin(\frac{1}{2}z)) = e^{\ln x}(c_1 \cos(\frac{1}{2} \ln x) + c_2 \sin(\frac{1}{2} \ln x)) \Rightarrow y = x(c_1 \cos(\frac{1}{2} \ln x) + c_2 \sin(\frac{1}{2} \ln x))$
17. $x^2y'' + 3xy' + y = 0 \Rightarrow r^2 + (3-1)r + 1 = 0 \Rightarrow r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0 \Rightarrow r = -1$, repeated twice
 $\Rightarrow y = c_1e^{-z} + c_2ze^{-z} = c_1e^{-\ln x} + c_2 \ln x e^{-\ln x} \Rightarrow y = \frac{c_1}{x} + \frac{c_2 \ln x}{x}$
18. $x^2y'' - 3xy' + 9y = 0 \Rightarrow r^2 + (-3-1)r + 9 = 0 \Rightarrow r^2 - 4r + 9 = 0 \Rightarrow r = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(9)}}{2(1)} = 2 \pm \sqrt{5}i$
 $\Rightarrow y = e^{2z}(c_1 \cos(\sqrt{5}z) + c_2 \sin(\sqrt{5}z)) = e^{2 \ln x}(c_1 \cos(\sqrt{5} \ln x) + c_2 \sin(\sqrt{5} \ln x))$
 $\Rightarrow y = x^2(c_1 \cos(\sqrt{5} \ln x) + c_2 \sin(\sqrt{5} \ln x))$

1022 Chapter 17 Second-Order Differential Equations

19. $x^2 y'' + x y' = 0 \Rightarrow r^2 + (1 - 1)r = 0 \Rightarrow r^2 = 0 \Rightarrow r = 0$, repeated twice $\Rightarrow y = c_1 e^{0 \cdot z} + c_2 z e^{0 \cdot z} = c_1 e^{0 \cdot \ln x} + c_2 \ln x e^{0 \cdot \ln x}$
 $\Rightarrow y = c_1 + c_2 \ln x$
20. $4x^2 y'' + y = 0 \Rightarrow 4r^2 + (0 - 4)r + 1 = 0 \Rightarrow 4r^2 - 4r + 1 = 0 \Rightarrow (2r - 1)^2 = 0 \Rightarrow r = \frac{1}{2}$, repeated twice
 $\Rightarrow y = c_1 e^{\frac{1}{2}z} + c_2 z e^{\frac{1}{2}z} = c_1 e^{\frac{1}{2} \ln x} + c_2 \ln x e^{\frac{1}{2} \ln x} \Rightarrow y = c_1 \sqrt{x} + c_2 \sqrt{x} \ln x$
21. $9x^2 y'' + 15x y' + y = 0 \Rightarrow 9r^2 + (15 - 9)r + 1 = 0 \Rightarrow 9r^2 + 6r + 1 = 0 \Rightarrow (3r + 1)^2 = 0 \Rightarrow r = -\frac{1}{3}$, repeated twice
 $\Rightarrow y = c_1 e^{-\frac{1}{3}z} + c_2 z e^{-\frac{1}{3}z} = c_1 e^{-\frac{1}{3} \ln x} + c_2 \ln x e^{-\frac{1}{3} \ln x} \Rightarrow y = \frac{c_1}{\sqrt[3]{x}} + \frac{c_2 \ln x}{\sqrt[3]{x}}$
22. $16x^2 y'' - 8x y' + 9y = 0 \Rightarrow 16r^2 + (-8 - 16)r + 9 = 0 \Rightarrow 16r^2 - 24r + 9 = 0 \Rightarrow (4r - 3)^2 = 0 \Rightarrow r = \frac{3}{4}$, repeated twice
 $\Rightarrow y = c_1 e^{\frac{3}{4}z} + c_2 z e^{\frac{3}{4}z} = c_1 e^{\frac{3}{4} \ln x} + c_2 \ln x e^{\frac{3}{4} \ln x} \Rightarrow y = c_1 x^{3/4} + c_2 x^{3/4} \ln x$
23. $16x^2 y'' + 56x y' + 25y = 0 \Rightarrow 16r^2 + (56 - 16)r + 25 = 0 \Rightarrow 16r^2 + 40r + 25 = 0 \Rightarrow (4r + 5)^2 = 0 \Rightarrow r = -\frac{5}{4}$, repeated twice
 $\Rightarrow y = c_1 e^{-\frac{5}{4}z} + c_2 z e^{-\frac{5}{4}z} = c_1 e^{-\frac{5}{4} \ln x} + c_2 \ln x e^{-\frac{5}{4} \ln x} \Rightarrow y = \frac{c_1}{x^{5/4}} + \frac{c_2 \ln x}{x^{5/4}}$
24. $4x^2 y'' - 16x y' + 25y = 0 \Rightarrow 4r^2 + (-16 - 4)r + 25 = 0 \Rightarrow 4r^2 - 20r + 25 = 0 \Rightarrow (2r - 5)^2 = 0 \Rightarrow r = \frac{5}{2}$, repeated twice
 $\Rightarrow y = c_1 e^{\frac{5}{2}z} + c_2 z e^{\frac{5}{2}z} = c_1 e^{\frac{5}{2} \ln x} + c_2 \ln x e^{\frac{5}{2} \ln x} \Rightarrow y = c_1 x^{5/2} + c_2 x^{5/2} \ln x$
25. $x^2 y'' + 3x y' - 3y = 0, y(1) = 1, y'(1) = -1 \Rightarrow r^2 + (3 - 1)r - 3 = 0 \Rightarrow r^2 + 2r - 3 = 0 \Rightarrow (r - 1)(r + 3) = 0$
 $\Rightarrow r = 1$ or $r = -3 \Rightarrow y = c_1 e^z + c_2 e^{-3z} = c_1 e^{\ln x} + c_2 e^{-3 \ln x} \Rightarrow y = c_1 x + \frac{c_2}{x^3} \Rightarrow y' = c_1 - \frac{3c_2}{x^4};$
 $y(1) = 1 \Rightarrow c_1 + c_2 = 1; y'(1) = -1 \Rightarrow c_1 - 3c_2 = -1 \Rightarrow c_1 = \frac{1}{2}, c_2 = \frac{1}{2} \Rightarrow y = \frac{1}{2}x + \frac{1}{2x^3}$
26. $6x^2 y'' + 7x y' - 2y = 0, y(1) = 0, y'(1) = 1 \Rightarrow 6r^2 + (7 - 6)r - 2 = 0 \Rightarrow 6r^2 + r - 2 = 0 \Rightarrow (2r - 1)(3r + 2) = 0$
 $\Rightarrow r = \frac{1}{2}$ or $r = -\frac{2}{3} \Rightarrow y = c_1 e^{\frac{1}{2}z} + c_2 e^{-\frac{2}{3}z} = c_1 e^{\frac{1}{2} \ln x} + c_2 e^{-\frac{2}{3} \ln x} \Rightarrow y = c_1 \sqrt{x} + \frac{c_2}{x^{2/3}} \Rightarrow y' = \frac{c_1}{2\sqrt{x}} - \frac{2c_2}{3x^{5/3}};$
 $y(1) = 0 \Rightarrow c_1 + c_2 = 0; y'(1) = 1 \Rightarrow \frac{1}{2}c_1 - \frac{2}{3}c_2 = 1 \Rightarrow c_1 = \frac{6}{7}, c_2 = -\frac{6}{7} \Rightarrow y = \frac{6}{7}\sqrt{x} - \frac{6}{7x^{2/3}}$
27. $x^2 y'' - x y' + y = 0, y(1) = 1, y'(1) = 1 \Rightarrow r^2 + (-1 - 1)r + 1 = 0 \Rightarrow r^2 - 2r + 1 = 0 \Rightarrow (r - 1)^2 = 0 \Rightarrow r = 1$, repeated twice
 $\Rightarrow y = c_1 e^z + c_2 z e^z = c_1 e^{\ln x} + c_2 \ln x e^{\ln x} \Rightarrow y = c_1 x + c_2 x \ln x \Rightarrow y' = c_1 + c_2 \ln x + c_2;$
 $y(1) = 1 \Rightarrow c_1 = 1; y'(1) = 1 \Rightarrow c_1 + c_2 = 1 \Rightarrow c_1 = 1, c_2 = 0 \Rightarrow y = x$
28. $x^2 y'' + 7x y' + 9y = 0, y(1) = 1, y'(1) = 0 \Rightarrow r^2 + (7 - 1)r + 9 = 0 \Rightarrow r^2 + 6r + 9 = 0 \Rightarrow (r + 3)^2 = 0 \Rightarrow r = -3$, repeated twice
 $\Rightarrow y = c_1 e^{-3z} + c_2 z e^{-3z} = c_1 e^{-3 \ln x} + c_2 \ln x e^{-3 \ln x} \Rightarrow y = \frac{c_1}{x^3} + \frac{c_2 \ln x}{x^3} \Rightarrow y' = -\frac{3c_1}{x^4} + c_2 \frac{1 - 3 \ln x}{x^4};$
 $y(1) = 1 \Rightarrow c_1 = 1; y'(1) = 0 \Rightarrow -3c_1 + c_2 = 0 \Rightarrow c_1 = 1, c_2 = 3 \Rightarrow y = \frac{1}{x^3} + \frac{3 \ln x}{x^3}$
29. $x^2 y'' - x y' + 2y = 0, y(1) = -1, y'(1) = 1 \Rightarrow r^2 + (-1 - 1)r + 2 = 0 \Rightarrow r^2 - 2r + 2 = 0 \Rightarrow r = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)}$
 $= 1 \pm i \Rightarrow y = e^z (c_1 \cos z + c_2 \sin z) = e^{\ln x} (c_1 \cos(\ln x) + c_2 \sin(\ln x)) \Rightarrow y = x (c_1 \cos(\ln x) + c_2 \sin(\ln x))$
 $\Rightarrow y' = (c_1 + c_2) \cos(\ln x) + (c_2 - c_1) \sin(\ln x); y(1) = -1 \Rightarrow c_1 = -1; y'(1) = 1 \Rightarrow c_1 + c_2 = 1 \Rightarrow c_1 = -1, c_2 = 2$
 $\Rightarrow y = x (-\cos(\ln x) + 2 \sin(\ln x))$
30. $x^2 y'' + 3x y' + 5y = 0, y(1) = 1, y'(1) = 0 \Rightarrow r^2 + (3 - 1)r + 5 = 0 \Rightarrow r^2 + 2r + 5 = 0 \Rightarrow r = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(5)}}{2(1)}$
 $= -1 \pm 2i \Rightarrow y = e^{-z} (c_1 \cos 2z + c_2 \sin 2z) = e^{-\ln x} (c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x)) \Rightarrow y = \frac{1}{x} (c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x))$
 $\Rightarrow y' = -\frac{1}{x^2} ((c_1 - 2c_2) \cos(2 \ln x) + (2c_1 + c_2) \sin(2 \ln x)); y(1) = 1 \Rightarrow c_1 = 1; y'(1) = 0 \Rightarrow -c_1 + 2c_2 = 0 \Rightarrow c_1 = 1,$
 $c_2 = \frac{1}{2} \Rightarrow y = \frac{1}{x} (\cos(2 \ln x) + \frac{1}{2} \sin(2 \ln x))$

17.5 POWER-SERIES SOLUTIONS

$$1. \quad y'' + 2y' = 0 \Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 2 \sum_{n=1}^{\infty} n c_n x^{n-1} = 0 \Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} 2n c_n x^{n-1} = 0$$

<u>power of x</u>	<u>coefficient equation</u>	
x^0	$2(1)c_2 + 2(1)c_1 = 0$	$\Rightarrow c_2 = -c_1$
x^1	$3(2)c_3 + 2(2)c_2 = 0$	$\Rightarrow c_3 = -\frac{2}{3}c_2 = \frac{2}{3}c_1$
x^2	$4(3)c_4 + 2(3)c_3 = 0$	$\Rightarrow c_4 = -\frac{1}{2}c_3 = -\frac{1}{3}c_1$
x^3	$5(4)c_5 + 2(4)c_4 = 0$	$\Rightarrow c_5 = -\frac{2}{5}c_4 = \frac{2}{15}c_1$
x^4	$6(5)c_6 + 2(5)c_5 = 0$	$\Rightarrow c_6 = -\frac{1}{3}c_5 = -\frac{2}{45}c_1$
$\vdots$	$\vdots$	$\vdots$
x^n	$(n+2)(n+1)c_{n+2} + 2(n+1)c_{n+1} = 0$	$\Rightarrow c_{n+2} = -\frac{2}{n+2}c_{n+1}$

or $c_n = -\frac{2}{n}c_{n-1} = \left(-\frac{2}{n}\right)\left(-\frac{2}{n-1}c_{n-2}\right) = \left(-\frac{2}{n}\right)\left(-\frac{2}{n-1}\right)\left(-\frac{2}{n-2}c_{n-3}\right) = \frac{(-2)^{n-1}}{n!}c_1$, $n \geq 2$. Thus

$$y = c_0 + c_1 x - c_1 x^2 + \frac{2}{3}c_1 x^3 - \frac{1}{3}c_1 x^4 + \frac{2}{15}c_1 x^5 - \frac{2}{45}c_1 x^6 + \dots = c_0 + c_1 \left(x - x^2 + \frac{2}{3}x^3 - \frac{1}{3}x^4 + \frac{2}{15}x^5 - \frac{2}{45}x^6 + \dots\right)$$

$$\text{or } y = c_0 + \frac{c_1}{2} - \frac{c_1}{2} + \frac{c_1}{2}(2x) - \frac{c_1}{2}\frac{(2x)^2}{2!} + \frac{c_1}{2}\frac{(2x)^3}{3!} - \frac{c_1}{2}\frac{(2x)^4}{4!} + \frac{c_1}{2}\frac{(2x)^5}{5!} - \frac{c_1}{2}\frac{(2x)^6}{6!} + \dots$$

$$y = \left(c_0 + \frac{c_1}{2}\right) - \frac{c_1}{2} \left(1 - (2x) + \frac{(2x)^2}{2!} - \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} - \frac{(2x)^5}{5!} + \frac{(2x)^6}{6!} + \dots\right) = \left(c_0 + \frac{c_1}{2}\right) - \frac{c_1}{2} \sum_{n=0}^{\infty} \frac{(-2x)^n}{n!}$$

$$= \left(c_0 + \frac{c_1}{2}\right) - \frac{c_1}{2} e^{-2x} = a + b e^{-2x}, \text{ where } a = c_0 + \frac{c_1}{2} \text{ and } b = -\frac{c_1}{2}$$

$$2. \quad y'' + 2y' + y = 0 \Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 2 \sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} 2n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

<u>power of x</u>	<u>coefficient equation</u>	
x^0	$2(1)c_2 + 2(1)c_1 + c_0 = 0$	$\Rightarrow c_2 = -c_1 - \frac{1}{2}c_0$
x^1	$3(2)c_3 + 2(2)c_2 + c_1 = 0$	$\Rightarrow c_3 = -\frac{2}{3}c_2 - \frac{1}{6}c_1 = \frac{1}{2}c_1 + \frac{1}{3}c_0$
x^2	$4(3)c_4 + 2(3)c_3 + c_2 = 0$	$\Rightarrow c_4 = -\frac{1}{2}c_3 - \frac{1}{12}c_2 = -\frac{1}{6}c_1 - \frac{1}{8}c_0$
x^3	$5(4)c_5 + 2(4)c_4 + c_3 = 0$	$\Rightarrow c_5 = -\frac{2}{5}c_4 - \frac{1}{20}c_3 = \frac{1}{24}c_1 + \frac{1}{30}c_0$
x^4	$6(5)c_6 + 2(5)c_5 + c_4 = 0$	$\Rightarrow c_6 = -\frac{1}{3}c_5 - \frac{1}{30}c_4 = -\frac{1}{120}c_1 - \frac{1}{144}c_0$
$\vdots$	$\vdots$	$\vdots$
x^n	$(n+2)(n+1)c_{n+2} + 2(n+1)c_{n+1} + c_n = 0$	$\Rightarrow c_{n+2} = -\frac{2}{n+2}c_{n+1} - \frac{1}{(n+2)(n+1)}c_n$

$$y = c_0 + c_1 x + \left(-c_1 - \frac{1}{2}c_0\right)x^2 + \left(\frac{1}{2}c_1 + \frac{1}{3}c_0\right)x^3 + \left(-\frac{1}{6}c_1 - \frac{1}{8}c_0\right)x^4 + \left(\frac{1}{24}c_1 + \frac{1}{30}c_0\right)x^5 + \left(-\frac{1}{120}c_1 - \frac{1}{144}c_0\right)x^6 + \dots$$

$$= c_0 - \frac{1}{2}c_0 x^2 + \frac{1}{3}c_0 x^3 - \frac{1}{8}c_0 x^4 + \frac{1}{30}c_0 x^5 - \frac{1}{144}c_0 x^6 + \dots + c_1 x - c_1 x^2 + \frac{1}{2}c_1 x^3 - \frac{1}{6}c_1 x^4 + \frac{1}{24}c_1 x^5 - \frac{1}{120}c_1 x^6 + \dots$$

$$= c_0 \left(1 - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{8}x^4 + \frac{1}{30}x^5 - \frac{1}{144}x^6 + \dots\right) + c_1 \left(x - x^2 + \frac{1}{2}x^3 - \frac{1}{6}x^4 + \frac{1}{24}x^5 - \frac{1}{120}x^6 + \dots\right)$$

Note that in each coefficient equation, the coefficient of c_1 can be given by $\frac{(-1)^{n-1}}{(n-1)!}$. To find the coefficient of c_0 , note that:

$$-\frac{1}{2} = -\left(1 - \frac{1}{2}\right) = -\left(\frac{1}{1!} - \frac{1}{2!}\right), \frac{1}{3} = \frac{1}{2} - \frac{1}{6} = \frac{1}{2!} - \frac{1}{3!}, -\frac{1}{8} = -\left(\frac{1}{6} - \frac{1}{24}\right) = -\left(\frac{1}{3!} - \frac{1}{4!}\right), \frac{1}{30} = \frac{1}{24} - \frac{1}{120} = \frac{1}{4!} - \frac{1}{5!}$$

so the coefficient of c_0 can be given by $(-1)^{n-1} \left[\frac{1}{(n-1)!} - \frac{1}{n!} \right] = \frac{(-1)^{n-1}}{(n-1)!} + \frac{(-1)^n}{n!}$. Thus $c_n = \left[\frac{(-1)^{n-1}}{(n-1)!} + \frac{(-1)^n}{n!} \right] c_0 + \frac{(-1)^{n-1}}{(n-1)!} c_1$

$$\text{or } c_n = \frac{(-1)^n}{n!} c_0 + \frac{(-1)^{n-1}}{(n-1)!} (c_0 + c_1) \text{ for } n \geq 2. \text{ Thus } y = c_0 + c_1 x + \sum_{n=2}^{\infty} \left(\frac{(-1)^n}{n!} c_0 + \frac{(-1)^{n-1}}{(n-1)!} (c_0 + c_1) \right) x^n$$

$$= c_0 + c_1 x + c_0 \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} x^n + c_0 \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(n-1)!} x^n + c_1 \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(n-1)!} x^n$$

$$= c_0 - c_0 x + c_0 \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} x^n + c_0 x + c_0 \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(n-1)!} x^n + c_1 x + c_1 \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(n-1)!} x^n$$

$$= c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n + c_0 x \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!} x^{n-1} + c_1 x \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!} x^{n-1} = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n + (c_0 + c_1) x \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!} x^{n-1}$$

$$= c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n + (c_0 + c_1) x \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k = c_0 e^{-x} + (c_0 + c_1) x e^{-x} = a e^{-x} + b x e^{-x}, \text{ where } a = c_0 \text{ and } b = c_0 + c_1$$

$$3. \quad y'' + 4y = 0 \Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 4 \sum_{n=0}^{\infty} c_n x^n = 0 \Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} 4c_n x^n = 0$$

<u>power of x</u>	<u>coefficient equation</u>	
x^0	$2(1)c_2 + 4c_0 = 0$	$\Rightarrow c_2 = -2c_0$
x^1	$3(2)c_3 + 4c_1 = 0$	$\Rightarrow c_3 = -\frac{2}{3}c_1$
x^2	$4(3)c_4 + 4c_2 = 0$	$\Rightarrow c_4 = -\frac{1}{3}c_2 = \frac{2}{3}c_0$
x^3	$5(4)c_5 + 4c_3 = 0$	$\Rightarrow c_5 = -\frac{1}{5}c_3 = \frac{2}{15}c_1$
x^4	$6(5)c_6 + 4c_4 = 0$	$\Rightarrow c_6 = -\frac{2}{15}c_4 = -\frac{4}{45}c_0$
$\vdots$	$\vdots$	$\vdots$
x^n	$(n+2)(n+1)c_{n+2} + 4c_n = 0$	$\Rightarrow c_{n+2} = -\frac{4}{(n+2)(n+1)}c_n$

$$\begin{aligned} y &= c_0 + c_1 x - 2c_0 x^2 - \frac{2}{3}c_1 x^3 + \frac{2}{3}c_0 x^4 + \frac{2}{15}c_1 x^5 - \frac{4}{45}c_0 x^6 + \dots \\ &= c_0 - 2c_0 x^2 + \frac{2}{3}c_0 x^4 - \frac{4}{45}c_0 x^6 + \dots + c_1 x - \frac{2}{3}c_1 x^3 + \frac{2}{15}c_1 x^5 + \dots \\ &= c_0 \left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \dots \right) + \frac{c_1}{2} \left(2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + \dots \right) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (2x)^{2n} + \frac{c_1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (2x)^{2n+1} \\ &= c_0 \cos 2x + \frac{c_1}{2} \sin 2x = a \cos 2x + b \sin 2x, \text{ where } a = c_0 \text{ and } b = \frac{c_1}{2} \end{aligned}$$

$$4. \quad y'' - 3y' + 2y = 0 \Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 3 \sum_{n=1}^{\infty} n c_n x^{n-1} + 2 \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} 3n c_n x^{n-1} + \sum_{n=0}^{\infty} 2c_n x^n = 0$$

<u>power of x</u>	<u>coefficient equation</u>	
x^0	$2(1)c_2 - 3(1)c_1 + 2c_0 = 0$	$\Rightarrow c_2 = \frac{3}{2}c_1 - c_0$
x^1	$3(2)c_3 - 3(2)c_2 + 2c_1 = 0$	$\Rightarrow c_3 = c_2 - \frac{1}{3}c_1 = \frac{7}{6}c_1 - c_0$
x^2	$4(3)c_4 - 3(3)c_3 + 2c_2 = 0$	$\Rightarrow c_4 = \frac{3}{4}c_3 - \frac{1}{6}c_2 = \frac{5}{8}c_1 - \frac{7}{12}c_0$
x^3	$5(4)c_5 - 3(4)c_4 + 2c_3 = 0$	$\Rightarrow c_5 = \frac{3}{5}c_4 - \frac{1}{10}c_3 = \frac{31}{120}c_1 - \frac{1}{4}c_0$
x^4	$6(5)c_6 - 3(5)c_5 + 2c_4 = 0$	$\Rightarrow c_6 = \frac{1}{2}c_5 - \frac{1}{15}c_4 = \frac{7}{80}c_1 - \frac{31}{360}c_0$
$\vdots$	$\vdots$	$\vdots$
x^n	$(n+2)(n+1)c_{n+2} - 3(n+1)c_{n+1} + 2c_n = 0$	$\Rightarrow c_{n+2} = \frac{3}{n+2}c_{n+1} - \frac{2}{(n+2)(n+1)}c_n$

$$\begin{aligned} y &= c_0 + c_1 x + \left(\frac{3}{2}c_1 - c_0\right)x^2 + \left(\frac{7}{6}c_1 - c_0\right)x^3 + \left(\frac{5}{8}c_1 - \frac{7}{12}c_0\right)x^4 + \left(\frac{31}{120}c_1 - \frac{1}{4}c_0\right)x^5 + \left(\frac{7}{80}c_1 - \frac{31}{360}c_0\right)x^6 + \dots \\ &= c_0 - c_0 x^2 - c_0 x^3 - \frac{7}{12}c_0 x^4 - \frac{1}{4}c_0 x^5 - \frac{31}{360}c_0 x^6 + \dots + c_1 x + \frac{3}{2}c_1 x^2 + \frac{7}{6}c_1 x^3 + \frac{5}{8}c_1 x^4 + \frac{31}{120}c_1 x^5 + \frac{7}{80}c_1 x^6 + \dots \\ &= c_0 \left(1 - x^2 - \frac{1}{3}x^3 - \frac{7}{12}x^4 - \frac{1}{4}x^5 - \frac{31}{360}x^6 - \dots \right) + c_1 \left(x + \frac{3}{2}x^2 + \frac{7}{6}x^3 + \frac{5}{8}x^4 + \frac{31}{120}x^5 + \frac{7}{80}x^6 + \dots \right) \end{aligned}$$

Note that if we use the techniques of Section 17.2, our solution is $y = a e^x + b e^{2x} = a \sum_{n=0}^{\infty} \frac{x^n}{n!} + b \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$. Thus

$$(a+b) + (a+2b)x + \dots = c_0 + c_1 x + \left(\frac{3}{2}c_1 - c_0\right)x^2 + \left(\frac{7}{6}c_1 - c_0\right)x^3 + \left(\frac{5}{8}c_1 - \frac{7}{12}c_0\right)x^4 + \left(\frac{31}{120}c_1 - \frac{1}{4}c_0\right)x^5 + \dots$$

If the series are equivalent, then take $c_0 = a+b$ and $c_1 = a+2b$ and substitute in our first series for c_0 and c_1 to obtain

$$\begin{aligned} y &= (a+b) + (a+2b)x + \left(\frac{3}{2}(a+2b) - (a+b)\right)x^2 + \left(\frac{7}{6}(a+2b) - (a+b)\right)x^3 + \left(\frac{5}{8}(a+2b) - \frac{7}{12}(a+b)\right)x^4 \\ &\quad + \left(\frac{31}{120}(a+2b) - \frac{1}{4}(a+b)\right)x^5 + \left(\frac{7}{80}(a+2b) - \frac{31}{360}(a+b)\right)x^6 + \dots \\ &= (a+b) + (a+2b)x + \left(\frac{1}{2}a + 2b\right)x^2 + \left(\frac{1}{6}a + \frac{4}{3}b\right)x^3 + \left(\frac{1}{24}a + \frac{2}{3}b\right)x^4 + \left(\frac{1}{120}a + \frac{4}{15}b\right)x^5 + \left(\frac{1}{720}a + \frac{4}{45}b\right)x^6 + \dots \\ &= a + ax + \frac{1}{2}ax^2 + \frac{1}{6}ax^3 + \frac{1}{24}ax^4 + \frac{1}{120}ax^5 + \frac{1}{720}ax^6 + \dots + b + 2bx + 2bx^2 + \frac{4}{3}bx^3 + \frac{2}{3}bx^4 + \frac{4}{15}bx^5 + \frac{4}{45}bx^6 + \dots \\ &= a \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \dots \right) + b \left(1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \frac{(2x)^5}{5!} + \frac{(2x)^6}{6!} + \dots \right) \\ &= a \sum_{n=0}^{\infty} \frac{x^n}{n!} + b \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = a e^x + b e^{2x} \end{aligned}$$

$$5. \quad x^2 y'' - 2xy' + 2y = 0 \Rightarrow x^2 \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 2x \sum_{n=1}^{\infty} n c_n x^{n-1} + 2 \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^n - \sum_{n=1}^{\infty} 2n c_n x^n + \sum_{n=0}^{\infty} 2c_n x^n = 0$$

<u>power of x</u>	<u>coefficient equation</u>	
x^0	$2c_0 = 0$	$\Rightarrow c_0 = 0$
x^1	$-2(1)c_1 + 2c_1 = 0$	$\Rightarrow 0 = 0$
x^2	$2(1)c_2 - 2(2)c_2 + 2c_2 = 0$	$\Rightarrow 0 = 0$
x^3	$3(2)c_3 - 2(3)c_3 + 2c_3 = 0$	$\Rightarrow c_3 = 0$
x^4	$4(3)c_4 - 2(4)c_4 + 2c_4 = 0$	$\Rightarrow c_4 = 0$
$\vdots$	$\vdots$	$\vdots$
x^n	$n(n-1)c_n - 2n c_n + 2c_n = 0$	$\Rightarrow c_n = 0, n \geq 3$

$$y = c_1 x + c_2 x^2$$

$$6. \quad y'' - xy' + y = 0 \Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - x \sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

<u>power of x</u>	<u>coefficient equation</u>	
x^0	$2(1)c_2 + c_0 = 0$	$\Rightarrow c_2 = -\frac{1}{2}c_0$
x^1	$3(2)c_3 - (1)c_1 + c_1 = 0$	$\Rightarrow c_3 = 0$
x^2	$4(3)c_4 - (2)c_2 + c_2 = 0$	$\Rightarrow c_4 = \frac{1}{12}c_2 = -\frac{1}{24}c_0$
x^3	$5(4)c_5 - (3)c_3 + c_3 = 0$	$\Rightarrow c_5 = \frac{1}{10}c_3 = 0$
x^4	$6(5)c_6 - (4)c_4 + c_4 = 0$	$\Rightarrow c_6 = \frac{2}{15}c_4 = -\frac{1}{180}c_0$
$\vdots$	$\vdots$	$\vdots$
x^n	$(n+2)(n+1)c_{n+2} - n c_n + c_n = 0$	$\Rightarrow c_{n+2} = -\frac{n-1}{(n+2)(n+1)}c_n$

$$y = c_0 + c_1 x - \frac{1}{2}c_0 x^2 - \frac{1}{24}c_0 x^4 - \frac{1}{180}c_0 x^6 + \dots = c_0 \left(1 - \frac{1}{2}x^2 - \frac{1}{24}x^4 - \frac{1}{180}x^6 + \dots \right) + c_1 x$$

$$7. \quad (1+x)y'' - y = 0 \Rightarrow (1+x) \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

<u>power of x</u>	<u>coefficient equation</u>	
x^0	$2(1)c_2 - c_0 = 0$	$\Rightarrow c_2 = \frac{1}{2}c_0$
x^1	$3(2)c_3 + 2(1)c_2 - c_1 = 0$	$\Rightarrow c_3 = -\frac{1}{3}c_2 + \frac{1}{6}c_1 = \frac{1}{6}c_1 - \frac{1}{6}c_0$
x^2	$4(3)c_4 + 3(2)c_3 - c_2 = 0$	$\Rightarrow c_4 = -\frac{1}{2}c_3 + \frac{1}{12}c_2 = -\frac{1}{12}c_1 + \frac{1}{8}c_0$
x^3	$5(4)c_5 + 4(3)c_4 - c_3 = 0$	$\Rightarrow c_5 = -\frac{3}{5}c_4 + \frac{1}{12}c_3 = \frac{7}{120}c_1 - \frac{1}{12}c_0$
x^4	$6(5)c_6 + 5(4)c_5 - c_4 = 0$	$\Rightarrow c_6 = -\frac{2}{3}c_5 + \frac{1}{30}c_4 = -\frac{1}{24}c_1 + \frac{43}{720}c_0$
$\vdots$	$\vdots$	$\vdots$
x^n	$(n+2)(n+1)c_{n+2} + n(n+1)c_{n+1} - c_n = 0$	$\Rightarrow c_{n+2} = -\frac{n}{(n+2)}c_{n+1} + \frac{1}{(n+2)(n+1)}c_n$

$$y = c_0 + c_1 x + \frac{1}{2}c_0 x^2 + \left(\frac{1}{6}c_1 - \frac{1}{6}c_0 \right) x^3 + \left(-\frac{1}{12}c_1 + \frac{1}{8}c_0 \right) x^4 + \left(\frac{7}{120}c_1 - \frac{1}{12}c_0 \right) x^5 + \left(-\frac{1}{24}c_1 + \frac{43}{720}c_0 \right) x^6 + \dots$$

$$= c_0 \left(1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{8}x^4 - \frac{1}{12}x^5 + \frac{43}{720}x^6 + \dots \right) + c_1 \left(x + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{7}{120}x^5 - \frac{1}{24}x^6 + \dots \right)$$

$$8. (1 - x^2)y'' - 4xy' + 6y = 0 \Rightarrow (1 - x^2) \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 4x \sum_{n=1}^{\infty} n c_n x^{n-1} + 6 \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)c_n x^n - \sum_{n=1}^{\infty} 4n c_n x^n + \sum_{n=0}^{\infty} 6c_n x^n = 0$$

power of xcoefficient equation

x^0	$2(1)c_2 + 6c_0 = 0$	$\Rightarrow c_2 = -3c_0$
x^1	$3(2)c_3 - 4(1)c_1 + 6c_1 = 0$	$\Rightarrow c_3 = -\frac{1}{3}c_1$
x^2	$4(3)c_4 - 2(1)c_2 - 4(2)c_2 + 6c_2 = 0$	$\Rightarrow c_4 = \frac{1}{3}c_2 = -c_0$
x^3	$5(4)c_5 - 3(2)c_3 - 4(3)c_3 + 6c_3 = 0$	$\Rightarrow c_5 = \frac{3}{5}c_3 = -\frac{1}{5}c_1$
x^4	$6(5)c_6 - 4(3)c_4 - 4(4)c_4 + 6c_4 = 0$	$\Rightarrow c_6 = \frac{11}{15}c_4 = -\frac{11}{15}c_0$
$\vdots$	$\vdots$	$\vdots$
x^n	$(n+2)(n+1)c_{n+2} - n(n-1)c_n - 4nc_n + 6c_n = 0$	$\Rightarrow c_{n+2} = \frac{n^2+3n-6}{(n+2)(n+1)}c_n$

$$y = c_0 + c_1x - 3c_0x^2 - \frac{1}{3}c_1x^3 - c_0x^4 - \frac{1}{5}c_1x^5 - \frac{11}{15}c_0x^6 - \dots$$

$$= c_0(1 - 3x^2 - x^4 - \frac{11}{15}x^6 - \dots) + c_1(x - \frac{1}{3}x^3 - \frac{1}{5}x^5 - \dots)$$

$$9. (x^2 - 1)y'' + 2xy' - 2y = 0 \Rightarrow (x^2 - 1) \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 2x \sum_{n=1}^{\infty} n c_n x^{n-1} - 2 \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^n - \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} 2n c_n x^n - \sum_{n=0}^{\infty} 2c_n x^n = 0$$

power of xcoefficient equation

x^0	$-2(1)c_2 - 2c_0 = 0$	$\Rightarrow c_2 = -c_0$
x^1	$-3(2)c_3 + 2(1)c_1 - 2c_1 = 0$	$\Rightarrow c_3 = 0$
x^2	$2(1)c_2 - 4(3)c_4 + 2(2)c_2 - 2c_2 = 0$	$\Rightarrow c_4 = \frac{1}{3}c_2 = -\frac{1}{3}c_0$
x^3	$3(2)c_3 - 5(4)c_5 + 2(3)c_3 - 2c_3 = 0$	$\Rightarrow c_5 = \frac{1}{2}c_3 = 0$
x^4	$4(3)c_4 - 6(5)c_6 + 2(4)c_4 - 2c_4 = 0$	$\Rightarrow c_6 = \frac{3}{5}c_4 = -\frac{1}{5}c_0$
$\vdots$	$\vdots$	$\vdots$
x^n	$n(n-1)c_n - (n+2)(n+1)c_{n+2} + 2nc_n - 2c_n = 0$	$\Rightarrow c_{n+2} = \frac{n-1}{(n+1)}c_n$

$$y = c_0 + c_1x - c_0x^2 - \frac{1}{3}c_0x^4 - \frac{1}{5}c_0x^6 - \dots = c_0(1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - \dots) + c_1x$$

$$10. y'' + y' - x^2y = 0 \Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^{n-1} - x^2 \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^{n+2} = 0$$

power of xcoefficient equation

x^0	$2(1)c_2 + (1)c_1 = 0$	$\Rightarrow c_2 = -\frac{1}{2}c_1$
x^1	$3(2)c_3 + (2)c_2 = 0$	$\Rightarrow c_3 = -\frac{1}{3}c_2 = \frac{1}{6}c_1$
x^2	$4(3)c_4 + (3)c_3 - c_0 = 0$	$\Rightarrow c_4 = -\frac{1}{4}c_3 + \frac{1}{12}c_0 = -\frac{1}{24}c_1 + \frac{1}{12}c_0$
x^3	$5(4)c_5 + (4)c_4 - c_1 = 0$	$\Rightarrow c_5 = -\frac{1}{5}c_4 + \frac{1}{20}c_1 = \frac{1}{120}c_1 + \frac{1}{30}c_0$
x^4	$6(5)c_6 + (5)c_5 - c_2 = 0$	$\Rightarrow c_6 = -\frac{1}{6}c_5 + \frac{1}{30}c_2 = -\frac{13}{720}c_1 - \frac{1}{180}c_0$
$\vdots$	$\vdots$	$\vdots$
x^n	$(n+2)(n+1)c_{n+2} + (n+1)c_{n+1} - c_{n-2} = 0$	$\Rightarrow c_{n+2} = -\frac{1}{(n+2)}c_{n+1} + \frac{1}{(n+2)(n+1)}c_{n-2}$

$$y = c_0 + c_1x - \frac{1}{2}c_0x^2 + \frac{1}{6}c_1x^3 + \left(-\frac{1}{24}c_1 + \frac{1}{12}c_0\right)x^4 + \left(\frac{1}{120}c_1 + \frac{1}{30}c_0\right)x^5 + \left(-\frac{13}{720}c_1 - \frac{1}{180}c_0\right)x^6 + \dots$$

$$= c_0\left(1 - \frac{1}{2}x^2 + \frac{1}{12}x^4 - \frac{1}{180}x^6 + \dots\right) + c_1\left(x + \frac{1}{6}x^3 - \frac{1}{24}x^4 + \frac{1}{120}x^5 - \frac{13}{720}x^6 + \dots\right)$$

$$11. (x^2 - 1)y'' - 6y = 0 \Rightarrow (x^2 - 1) \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 6 \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^n - \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=0}^{\infty} 6c_n x^n = 0$$

power of xcoefficient equation

x^0	$-2(1)c_2 - 6c_0 = 0$	$\Rightarrow c_2 = -3c_0$
x^1	$-3(2)c_3 - 6c_1 = 0$	$\Rightarrow c_3 = -c_1$
x^2	$2(1)c_2 - 4(3)c_4 - 6c_2 = 0$	$\Rightarrow c_4 = -\frac{1}{3}c_2 = c_0$
x^3	$3(2)c_3 - 5(4)c_5 - 6c_3 = 0$	$\Rightarrow c_5 = 0$
x^4	$4(3)c_4 - 6(5)c_6 - 6c_4 = 0$	$\Rightarrow c_6 = -\frac{1}{5}c_4 = -\frac{1}{5}c_0$
$\vdots$	$\vdots$	$\vdots$
x^n	$n(n-1)c_n - (n+2)(n+1)c_{n+2} - 6c_n = 0$	$\Rightarrow c_{n+2} = -\frac{n-3}{(n+1)}c_n$

$$y = c_0 + c_1x - 3c_0x^2 - c_1x^3 + c_0x^4 - \frac{1}{5}c_0x^6 - \dots = c_0(1 - 3x^2 + x^4 - \frac{1}{5}x^6 + \dots) + c_1(x - x^3)$$

$$12. xy'' - (x+2)y' + 2y = 0 \Rightarrow x \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - (x+2) \sum_{n=1}^{\infty} n c_n x^{n-1} + 2 \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-1} - \sum_{n=1}^{\infty} n c_n x^n - \sum_{n=1}^{\infty} 2n c_n x^{n-1} + \sum_{n=0}^{\infty} 2c_n x^n = 0$$

power of xcoefficient equation

x^0	$-2(1)c_1 + 2c_0 = 0$	$\Rightarrow c_1 = c_0$
x^1	$2(1)c_2 - (1)c_1 - 2(2)c_2 + 2c_1 = 0$	$\Rightarrow c_2 = \frac{1}{2}c_1 = \frac{1}{2}c_0$
x^2	$3(2)c_3 - (2)c_2 - 2(3)c_3 + 2c_2 = 0$	$\Rightarrow 0 = 0$
x^3	$4(3)c_4 - (3)c_3 - 2(4)c_4 + 2c_3 = 0$	$\Rightarrow c_4 = \frac{1}{4}c_3$
x^4	$5(4)c_5 - (4)c_4 - 2(5)c_5 + 2c_4 = 0$	$\Rightarrow c_5 = \frac{1}{5}c_4 = \frac{1}{20}c_3$
$\vdots$	$\vdots$	$\vdots$
x^n	$(n+1)n c_{n+1} - n c_n - 2(n+1)c_{n+1} + 2c_n = 0$	$\Rightarrow c_{n+1} = \frac{1}{(n+1)}c_n$

$$y = c_0 + c_0x + \frac{1}{2}c_0x^2 + c_3x^3 + \frac{1}{4}c_3x^4 + \frac{1}{20}c_3x^5 + \dots = c_0(1 + x + \frac{1}{2}x^2) + c_3(x^3 + \frac{1}{4}x^4 + \frac{1}{20}x^5 + \dots)$$

$$13. (x^2 - 1)y'' + 4xy' + 2y = 0 \Rightarrow (x^2 - 1) \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 4x \sum_{n=1}^{\infty} n c_n x^{n-1} + 2 \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^n - \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} 4n c_n x^n + \sum_{n=0}^{\infty} 2c_n x^n = 0$$

power of xcoefficient equation

x^0	$-2(1)c_2 + 2c_0 = 0$	$\Rightarrow c_2 = c_0$
x^1	$-3(2)c_3 + 4(1)c_1 + 2c_1 = 0$	$\Rightarrow c_3 = c_1$
x^2	$2(1)c_2 - 4(3)c_4 + 4(2)c_2 + 2c_2 = 0$	$\Rightarrow c_4 = c_2 = c_0$
x^3	$3(2)c_3 - 5(4)c_5 + 4(3)c_3 + 2c_3 = 0$	$\Rightarrow c_5 = c_3 = c_1$
x^4	$4(3)c_4 - 6(5)c_6 + 4(4)c_4 + 2c_4 = 0$	$\Rightarrow c_6 = c_4 = c_0$
$\vdots$	$\vdots$	$\vdots$
x^n	$n(n-1)c_n - (n+2)(n+1)c_{n+2} + 4nc_n + 2c_n = 0$	$\Rightarrow c_{n+2} = c_n$

$$y = c_0 + c_1x + c_0x^2 + c_3x^3 + c_1x^4 + c_0x^5 + c_0x^6 - \dots = c_0(1 + x^2 + x^4 + x^6 + \dots) + c_1(x + x^3 + x^5 + \dots)$$

$$14. y'' - 2xy' + 4y = 0 \Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 2x \sum_{n=1}^{\infty} n c_n x^{n-1} + 4 \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} 2n c_n x^n + \sum_{n=0}^{\infty} 4c_n x^n = 0$$

<u>power of x</u>	<u>coefficient equation</u>	
x^0	$2(1)c_2 + 4c_0 = 0$	$\Rightarrow c_2 = -2c_0$
x^1	$3(2)c_3 - 2(1)c_1 + 4c_1 = 0$	$\Rightarrow c_3 = -\frac{1}{3}c_1$
x^2	$4(3)c_4 - 2(2)c_2 + 4c_2 = 0$	$\Rightarrow c_4 = 0$
x^3	$5(4)c_5 - 2(3)c_3 + 4c_3 = 0$	$\Rightarrow c_5 = \frac{1}{10}c_3 = -\frac{1}{30}c_1$
x^4	$6(5)c_6 - 2(4)c_4 + 4c_4 = 0$	$\Rightarrow c_6 = \frac{2}{15}c_4 = 0$
$\vdots$	$\vdots$	$\vdots$
x^n	$(n+2)(n+1)c_{n+2} - 2n c_n + 4c_n = 0$	$\Rightarrow c_{n+2} = \frac{2(n-1)}{(n+2)(n+1)}c_n$

$$y = c_0 + c_1 x - 2c_0 x^2 - \frac{1}{3}c_1 x^3 - \frac{1}{30}c_1 x^5 - \dots = c_0(1 - 2x^2) + c_1(x - \frac{1}{3}x^3 - \frac{1}{30}x^5 - \dots)$$

$$15. y'' - 2xy' + 3y = 0 \Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 2x \sum_{n=1}^{\infty} n c_n x^{n-1} + 3 \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} 2n c_n x^n + \sum_{n=0}^{\infty} 3c_n x^n = 0$$

<u>power of x</u>	<u>coefficient equation</u>	
x^0	$2(1)c_2 + 3c_0 = 0$	$\Rightarrow c_2 = -\frac{3}{2}c_0$
x^1	$3(2)c_3 - 2(1)c_1 + 3c_1 = 0$	$\Rightarrow c_3 = -\frac{1}{6}c_1$
x^2	$4(3)c_4 - 2(2)c_2 + 3c_2 = 0$	$\Rightarrow c_4 = \frac{1}{12}c_2 = -\frac{1}{8}c_0$
x^3	$5(4)c_5 - 2(3)c_3 + 3c_3 = 0$	$\Rightarrow c_5 = \frac{3}{20}c_3 = -\frac{1}{40}c_1$
x^4	$6(5)c_6 - 2(4)c_4 + 3c_4 = 0$	$\Rightarrow c_6 = \frac{1}{6}c_4 = -\frac{1}{48}c_0$
$\vdots$	$\vdots$	$\vdots$
x^n	$(n+2)(n+1)c_{n+2} - 2n c_n + 3c_n = 0$	$\Rightarrow c_{n+2} = \frac{2n-3}{(n+2)(n+1)}c_n$

$$y = c_0 + c_1 x - \frac{3}{2}c_0 x^2 - \frac{1}{6}c_1 x^3 - \frac{1}{8}c_0 x^4 - \frac{1}{40}c_1 x^5 - \frac{1}{48}c_0 x^6 - \dots$$

$$= c_0(1 - \frac{3}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{48}x^6 - \dots) + c_1(x - \frac{1}{6}x^3 - \frac{1}{40}x^5 - \dots)$$

$$16. (1-x^2)y'' - xy' + 4y = 0 \Rightarrow (1-x^2) \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - x \sum_{n=1}^{\infty} n c_n x^{n-1} + 4 \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)c_n x^n - \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} 4c_n x^n = 0$$

<u>power of x</u>	<u>coefficient equation</u>	
x^0	$2(1)c_2 + 4c_0 = 0$	$\Rightarrow c_2 = -2c_0$
x^1	$3(2)c_3 - (1)c_1 + 4c_1 = 0$	$\Rightarrow c_3 = -\frac{1}{2}c_1$
x^2	$4(3)c_4 - 2(1)c_2 - (2)c_2 + 4c_2 = 0$	$\Rightarrow c_4 = 0$
x^3	$5(4)c_5 - 3(2)c_3 - (3)c_3 + 4c_3 = 0$	$\Rightarrow c_5 = \frac{1}{4}c_3 = -\frac{1}{8}c_1$
x^4	$6(5)c_6 - 4(3)c_4 - (4)c_4 + 4c_4 = 0$	$\Rightarrow c_6 = \frac{2}{5}c_4 = 0$
$\vdots$	$\vdots$	$\vdots$
x^n	$(n+2)(n+1)c_{n+2} - n(n-1)c_n - n c_n + 4c_n = 0$	$\Rightarrow c_{n+2} = \frac{n-2}{(n+1)}c_n$

$$y = c_0 + c_1 x - 2c_0 x^2 - \frac{1}{2}c_1 x^3 - \frac{1}{8}c_1 x^5 - \dots = c_0(1 - x^2) + c_1(x - \frac{1}{2}x^3 - \frac{1}{8}x^5 - \dots)$$

$$17. \quad y'' - xy' + 3y = 0 \Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - x \sum_{n=1}^{\infty} n c_n x^{n-1} + 3 \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} 3c_n x^n = 0$$

power of xcoefficient equation x^0

$2(1)c_2 + 3c_0 = 0$

$\Rightarrow c_2 = -\frac{3}{2}c_0$

 x^1

$3(2)c_3 - (1)c_1 + 3c_1 = 0$

$\Rightarrow c_3 = -\frac{1}{3}c_1$

 x^2

$4(3)c_4 - (2)c_2 + 3c_2 = 0$

$\Rightarrow c_4 = -\frac{1}{12}c_2 = \frac{1}{8}c_0$

 x^3

$5(4)c_5 - (3)c_3 + 3c_3 = 0$

$\Rightarrow c_5 = 0$

 x^4

$6(5)c_6 - (4)c_4 + 3c_4 = 0$

$\Rightarrow c_6 = \frac{1}{30}c_4 = \frac{1}{240}c_0$

 $\vdots$ $\vdots$ $\vdots$ x^n

$(n+2)(n+1)c_{n+2} - n c_n + 3c_n = 0$

$\Rightarrow c_{n+2} = \frac{n-3}{(n+2)(n+1)}c_n$

$$y = c_0 + c_1 x - \frac{3}{2}c_0 x^2 - \frac{1}{6}c_1 x^3 + \frac{1}{8}c_0 x^4 + \frac{1}{240}c_0 x^6 - \dots = c_0 \left(1 - \frac{3}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{240}x^6 + \dots\right) + c_1 \left(x - \frac{1}{3}x^3\right)$$

$$18. \quad x^2 y'' - 4xy' + 6y = 0 \Rightarrow x^2 \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 4x \sum_{n=1}^{\infty} n c_n x^{n-1} + 6 \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^n - \sum_{n=1}^{\infty} 4n c_n x^n + \sum_{n=0}^{\infty} 6c_n x^n = 0$$

power of xcoefficient equation x^0

$6c_0 = 0$

$\Rightarrow c_0 = 0$

 x^1

$-4(1)c_1 + 6c_1 = 0$

$\Rightarrow c_1 = 0$

 x^2

$2(1)c_2 - 4(2)c_2 + 6c_2 = 0$

$\Rightarrow 0 = 0$

 x^3

$3(2)c_3 - 4(3)c_3 + 6c_3 = 0$

$\Rightarrow 0 = 0$

 x^4

$4(3)c_4 - 4(4)c_4 + 6c_4 = 0$

$\Rightarrow c_4 = 0$

 $\vdots$ $\vdots$ $\vdots$ x^n

$n(n-1)c_n - 4n c_n + 6c_n = 0$

$\Rightarrow c_n = 0$

$$y = c_2 x^2 + c_3 x^3$$