Chapter 2
Introduction to Optimization & Linear Programming

1. If an LP model has more than one optimal solution it has an ***infinite*** number of alternate optimal solutions. In Figure 2.8, the two extreme points at (122, 78) and (174, 0) are alternate optimal solutions, but there are an infinite number of alternate optimal solutions along the edge connecting these extreme points. This is true of all LP models with alternate optimal solutions.

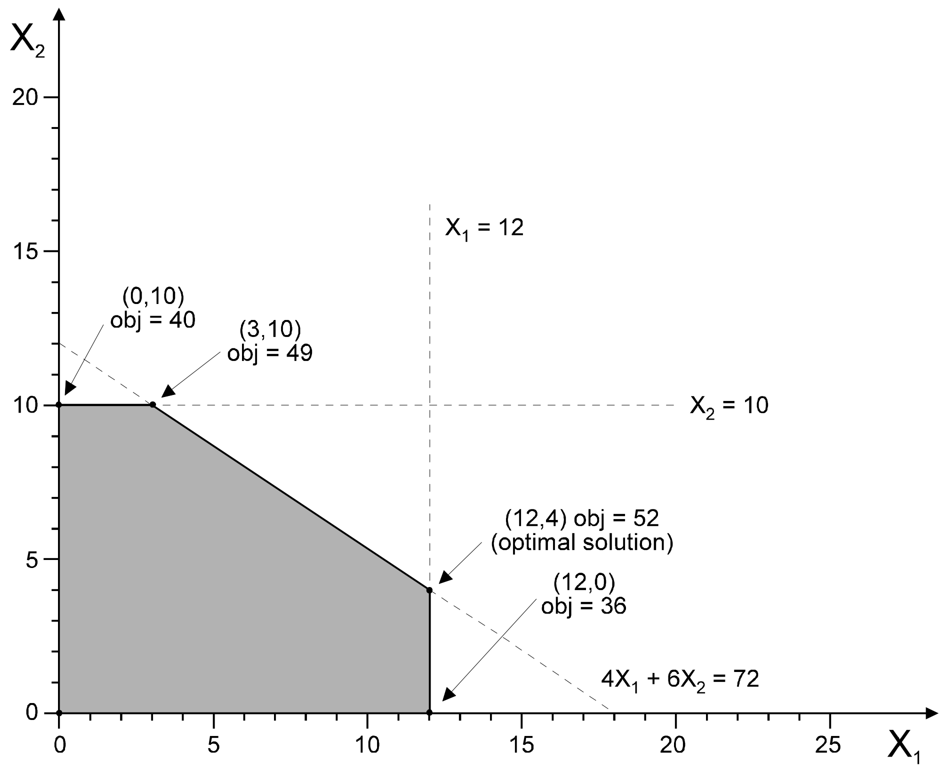
2. There is no guarantee that the optimal solution to an LP problem will occur at an integer-valued extreme point of the feasible region. (An exception to this general rule is discussed in Chapter 5 on networks).

3. We can graph an inequality as if they were an equality because the condition imposed by the equality corresponds to the boundary line (or most extreme case) of the inequality.

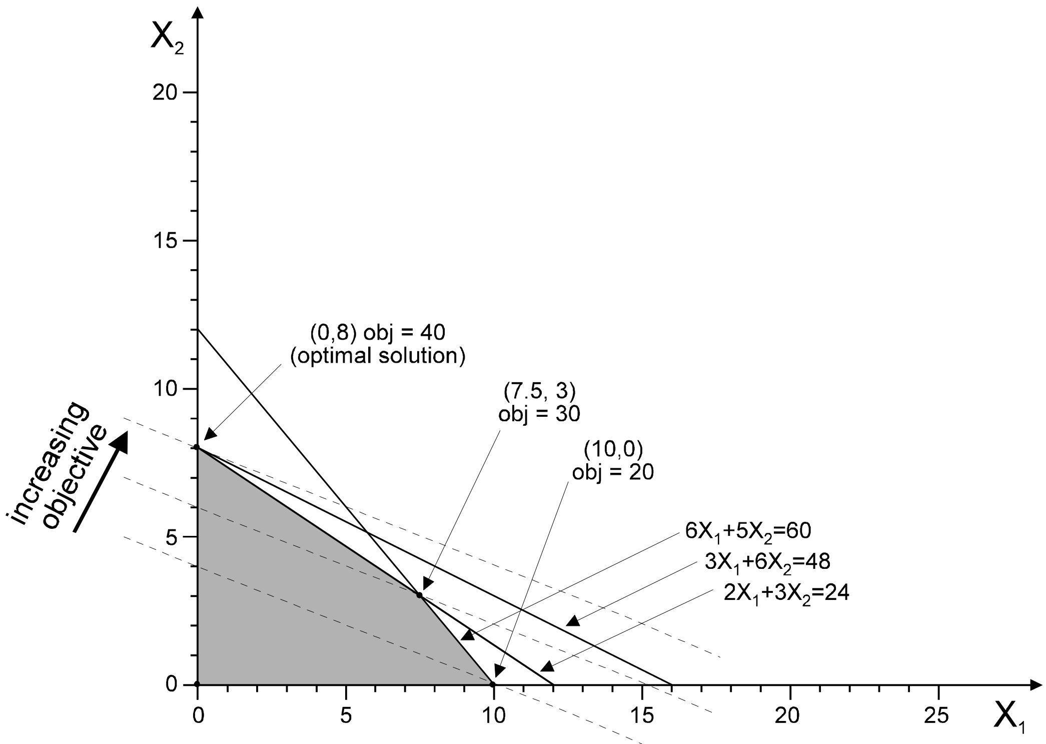
4. The objectives are equivalent. For any values of X_1 and X_2, the absolute value of the objectives are the same. Thus, maximizing the value of the first objective is equivalent to minimizing the value of the second objective.

5. a. linear
   b. nonlinear
   c. linear, can be re-written as: $4\ X_1 - .3333\ X_2 = 75$
   d. linear, can be re-written as: $2.1\ X_1 + 1.1\ X_2\ - 3.9\ X_3 \leq 0$
   e. nonlinear

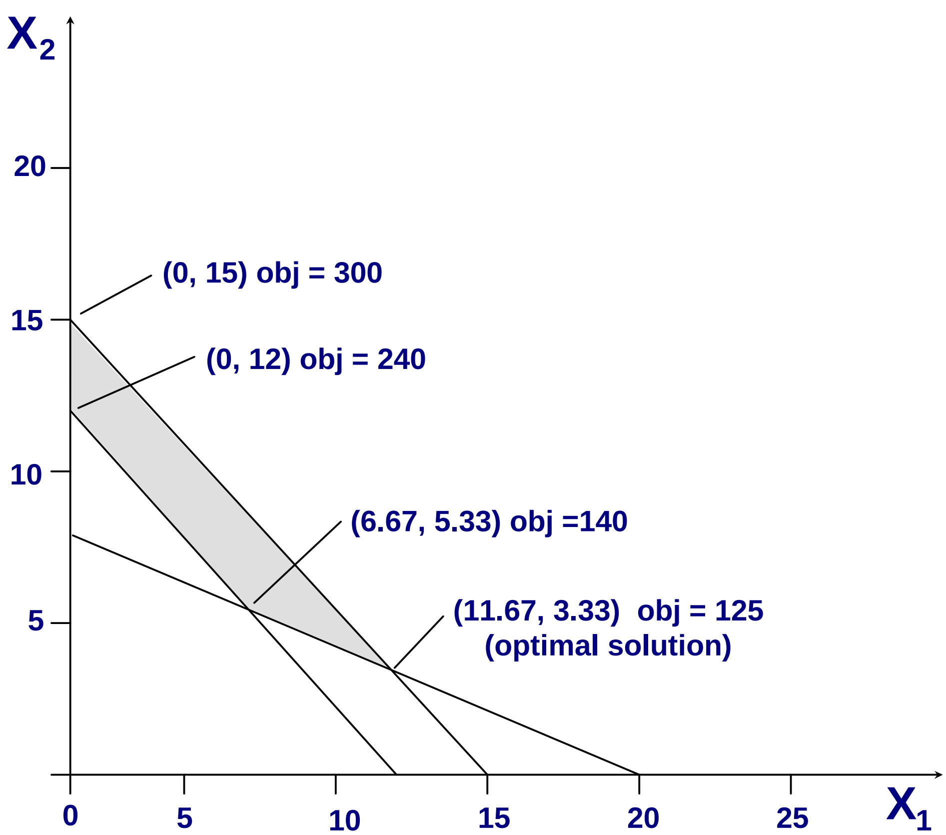
6.



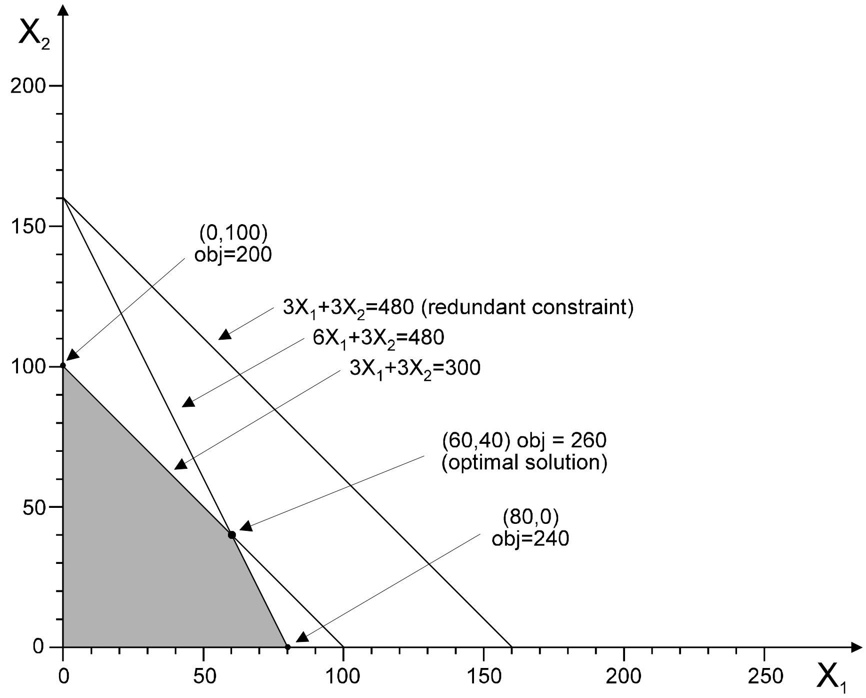
7.


X2
20
15
10
5
0
0
5
10
15
20
25
X1
(0,8) obj = 40
(optimal solution)
(7.5, 3)
obj = 30
(10,0)
obj = 20
increasing
objective
6X1+5X2=60
3X1+6X2=48
2X1+3X2=24

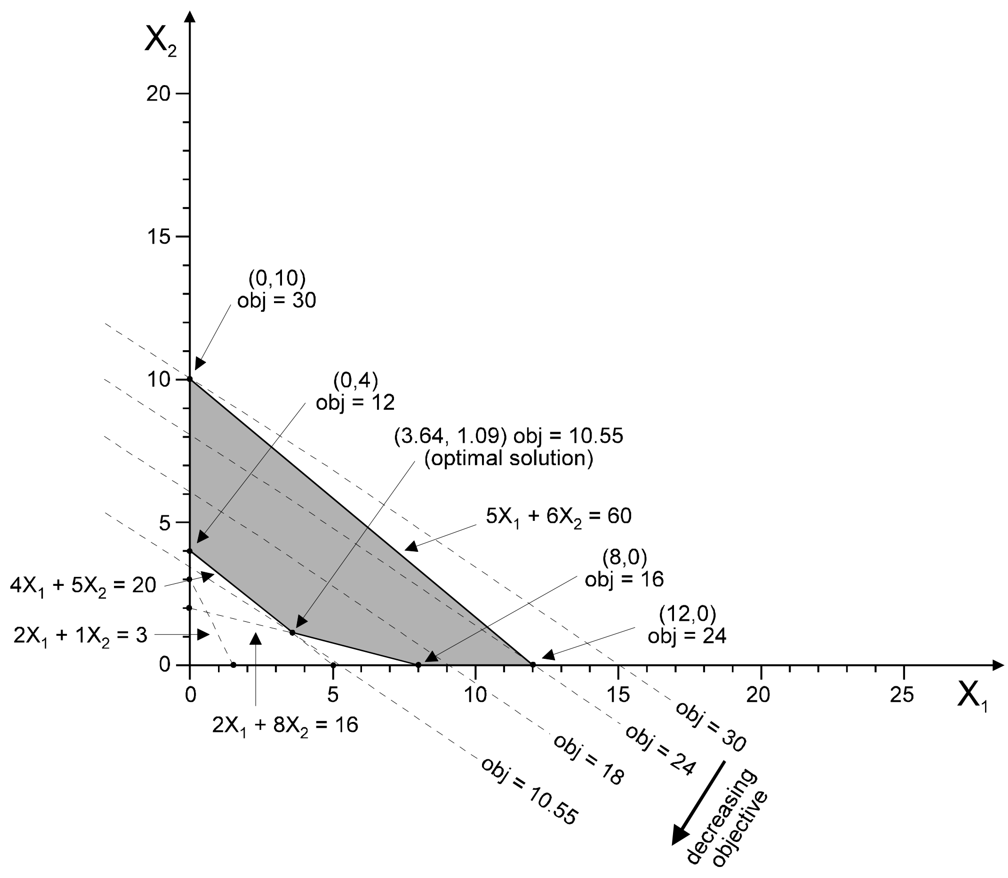
8.


X2
20
15
10
5
0
5
10
15
20
25
X1
(0, 15) obj = 300
(0, 12) obj = 240
(6.67, 5.33) obj =140
(11.67, 3.33) obj = 125
(optimal solution)

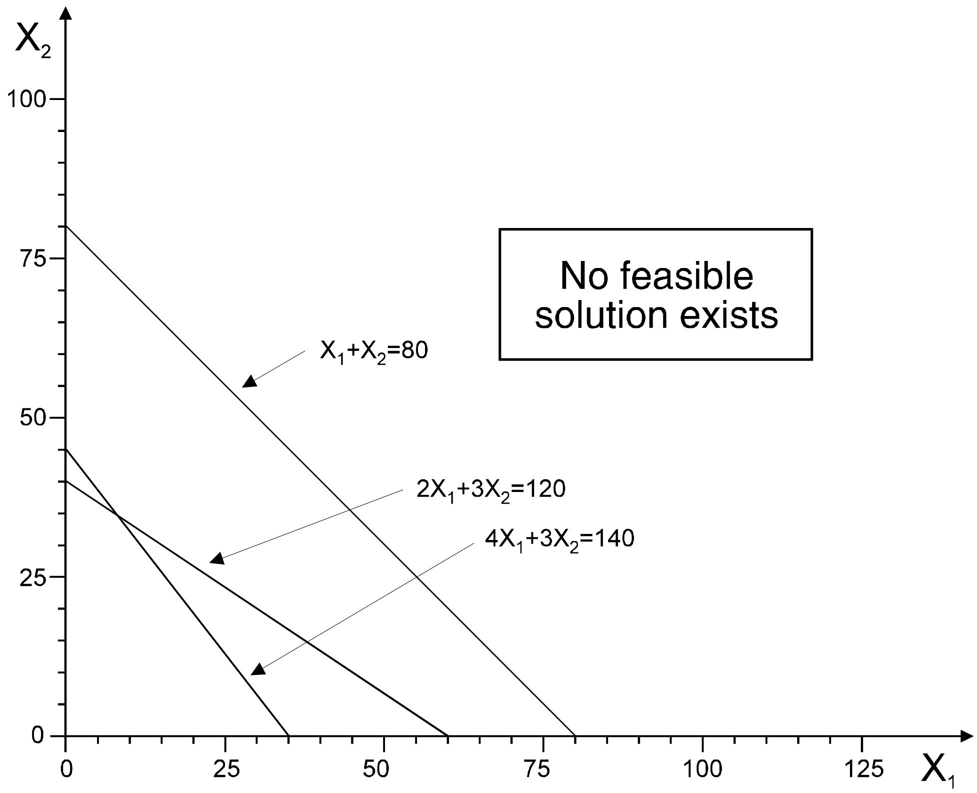
9.


X2
200
150
100
50
0
(0,100)
obj=200
3X1+3X2=480 (redundant constraint)
6X1+3X2=480
3X1+3X2=300
(60,40) obj = 260
(optimal solution)
(80,0)
obj=240
0
50
100
150
200
250
X1

10.

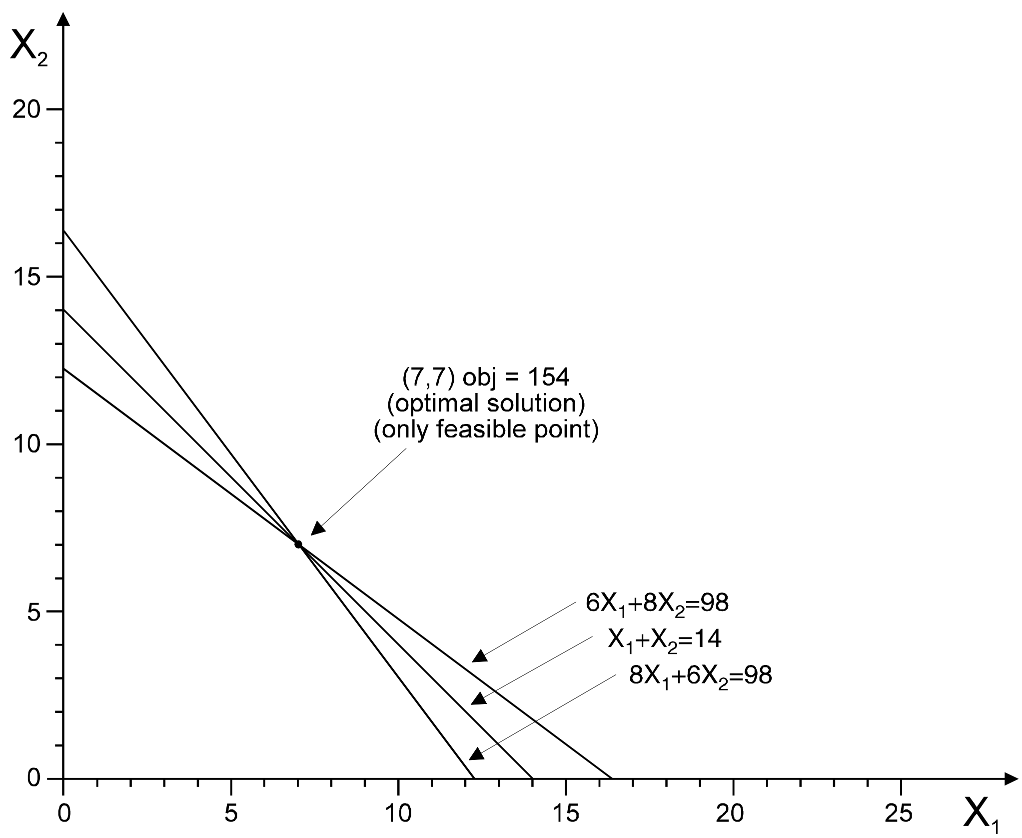

X2
20
15
10
5
0
(0,10)
obj = 30
(0,4)
obj = 12
(3.64, 1.09) obj = 10.55
(optimal solution)
5X1 + 6X2 = 60
(8,0)
obj = 16
(12,0)
obj = 24
4X1 + 5X2 = 20
2X1 + 1X2 = 3
2X1 + 8X2 = 16
0
5
10
15
20
25
X1
obj = 30
obj = 24
obj = 18
obj = 10.55
decreasing
objective

11.



X_2
100
75
50
25
0
No feasible solution exists
$X_1+X_2=80$
$2X_1+3X_2=120$
$4X_1+3X_2=140$
0
25
50
75
100
125
X_1

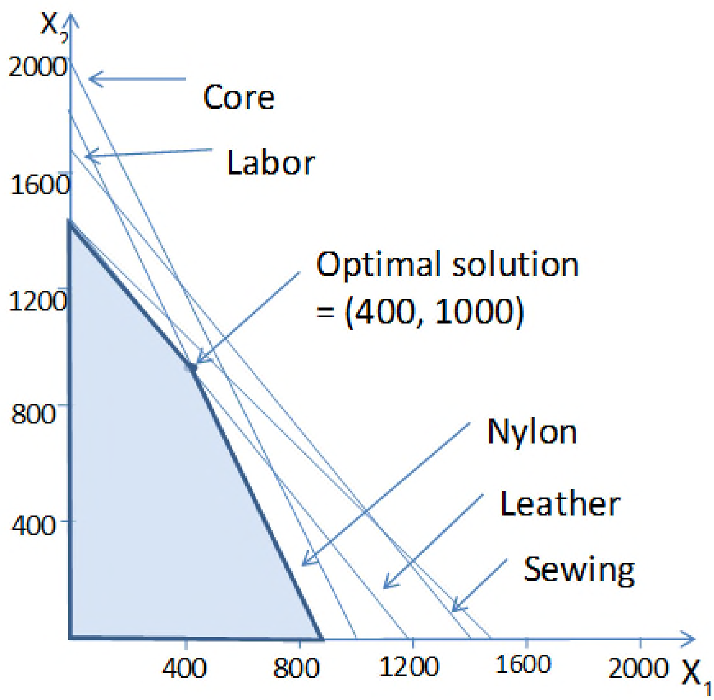
12.



X_2
20
15
10
5
0
(7,7) obj = 154
(optimal solution)
(only feasible point)
$6X_1+8X_2=98$
$X_1+X_2=14$
$8X_1+6X_2=98$
0
5
10
15
20
25
X_1

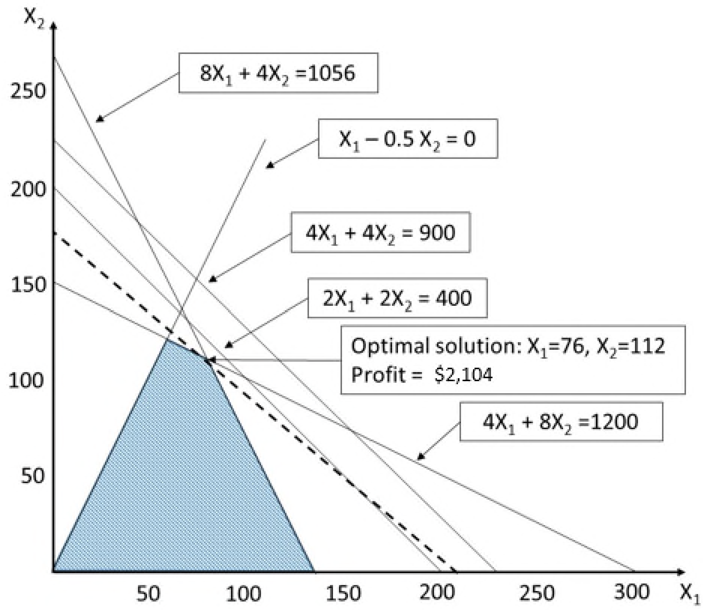
13. X_1 = number of softballs to produce, X_2 = number of baseballs to produce

MAX $6 X_1 + 4.5 X_2$
ST $5X_1 + 4 X_2 \le 6000$
$6 X_1 + 3 X_2 \le 5400$
$4 X_1 + 2 X_2 \le 4000$
$2.5 X_1 + 2 X_2 \le 3500$
$1 X_1 + 1 X_2 \le 1500$
$X_1, X_2 \ge 0$



14. X_1 = number of His chairs to produce, X_2 = number of Hers chairs to produce

MAX $10 X_1 + 12 X_2$
ST $4 X_1 + 8 X_2 \le 1200$
$8 X_1 + 4 X_2 \le 1056$
$2 X_1 + 2 X_2 \le 400$
$4 X_1 + 4 X_2 \le 900$
$1 X_1 - 0.5 X_2 \ge 0$
$X_1, X_2 \ge 0$



15. X_1 = number of propane grills to produce, X_2 = number of electric grills to produce

MAX $100X_1 + 80X_2$

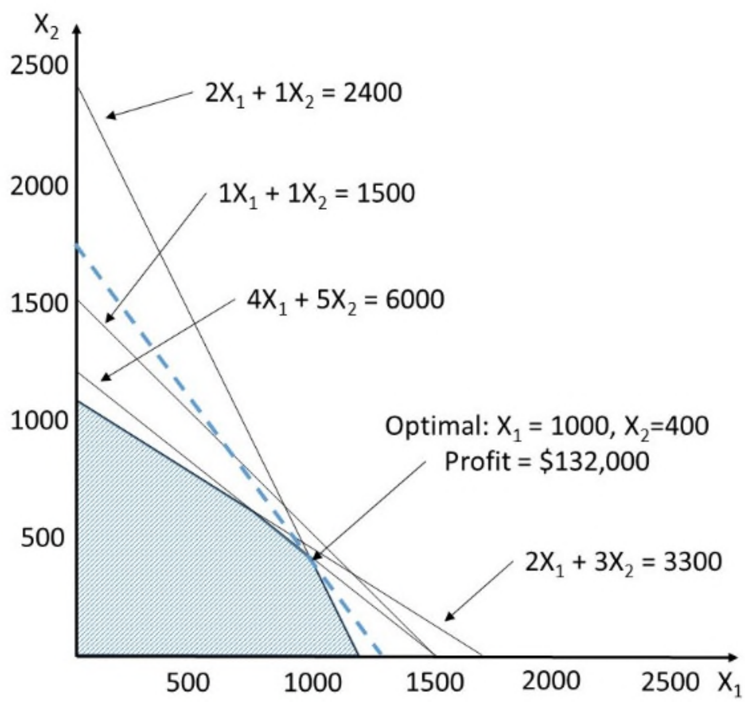
ST $2X_1 + 1X_2 \le 2400$

$4X_1 + 5X_2 \le 6000$

$2X_1 + 3X_2 \le 3300$

$1X_1 + 1X_2 \le 1500$

$X_1, X_2 \ge 0$



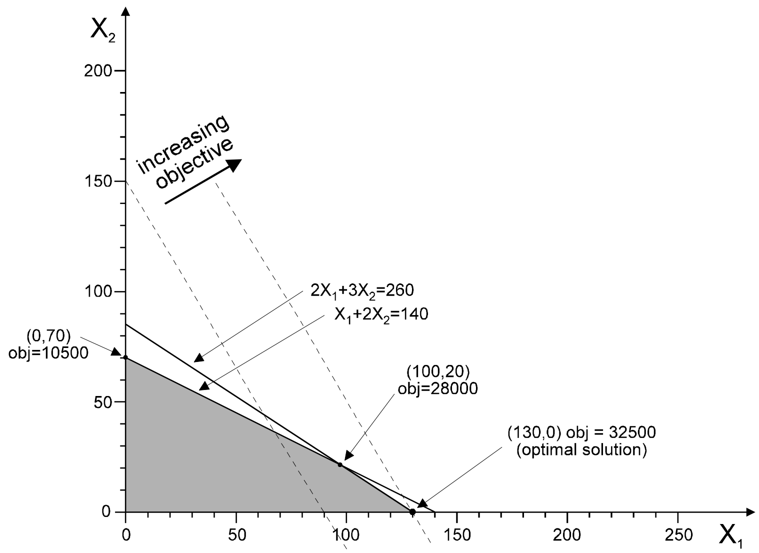
16. X_1 = number of generators, X_2 = number of alternators

MAX $250 X_1 + 150 X_2$

ST $2 X_1 + 3 X_2 \le 260$

$1 X_1 + 2 X_2 \le 140$

$X_1, X_2 \ge 0$



17. X_1 = number of generators, X_2 = number of alternators

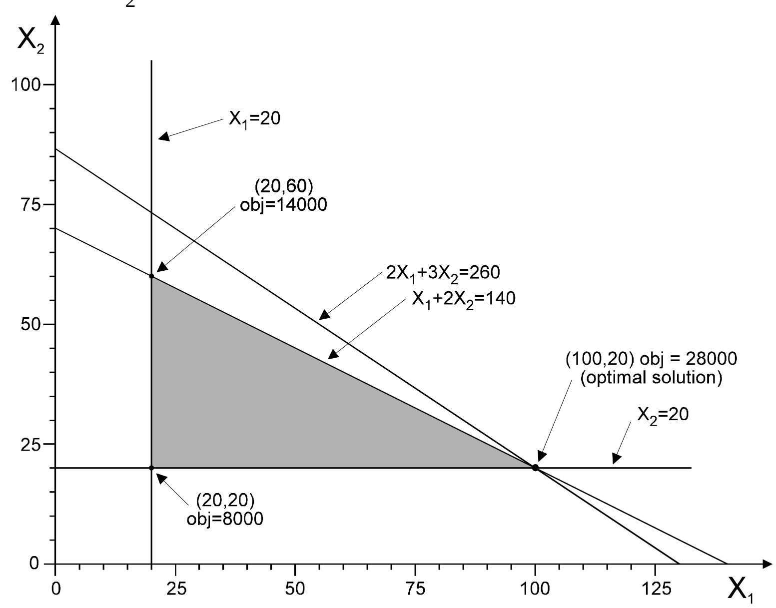
MAX $250 X_1 + 150 X_2$

ST $2 X_1 + 3 X_2 \le 260$

$1 X_1 + 2 X_2 \le 140$

$X_1 \ge 20$

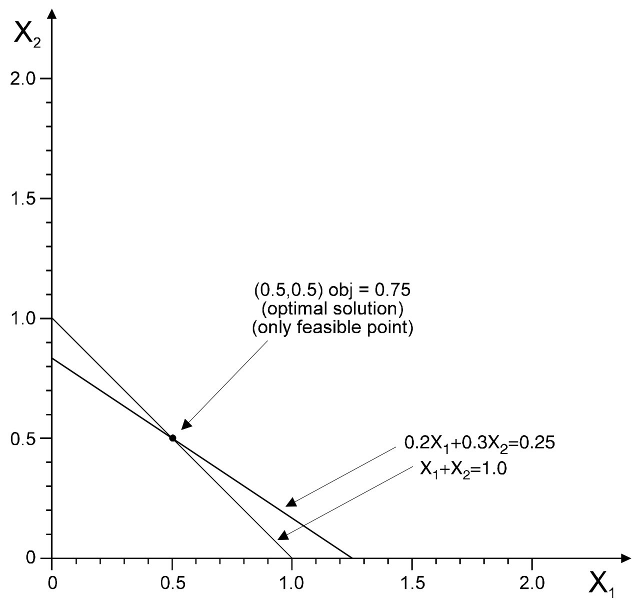
$X_2 \ge 20$



d. No, the feasible region would not increase so the solution would not change -- you'd just have extra (unused) wiring capacity.

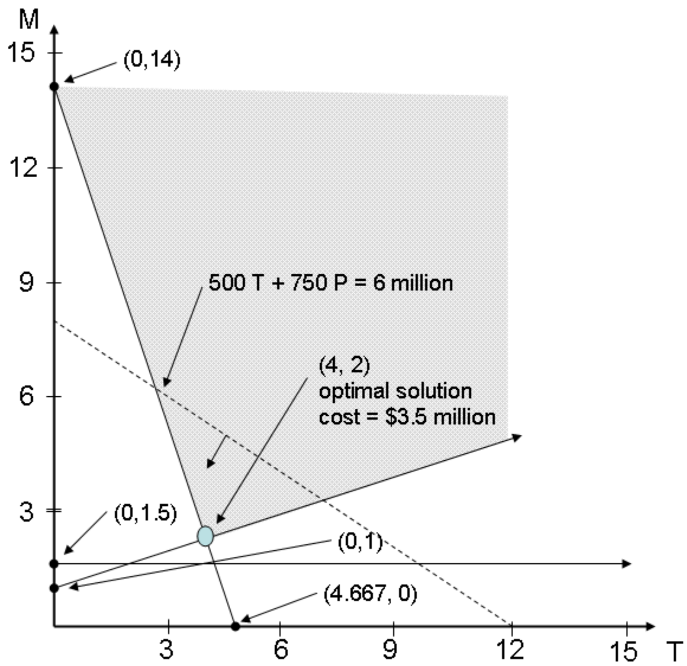
18. X_1 = proportion of beef in the mix, X_2 = proportion of pork in the mix

MIN $.85\ X_1 + .65\ X_2$
ST $1X_1 + 1\ X_2 = 1$
$0.2\ X_1 + 0.3\ X_2 \le 0.25$
$X_1, X_2 \ge 0$



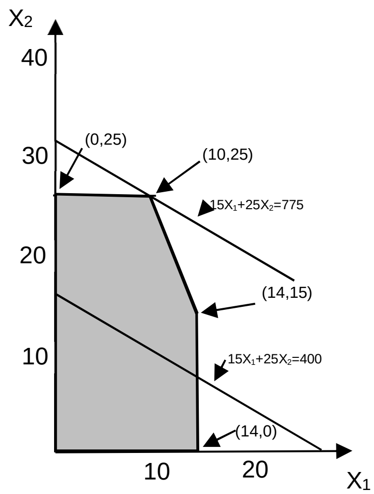
19. T= number of TV ads to run, M = number of magazine ads to run

MIN $500\ T + 750\ P$
ST $3T + 1P \ge 14$
$-1T + 4P \ge 4$
$0T + 2P \ge 3$
$T, P \ge 0$



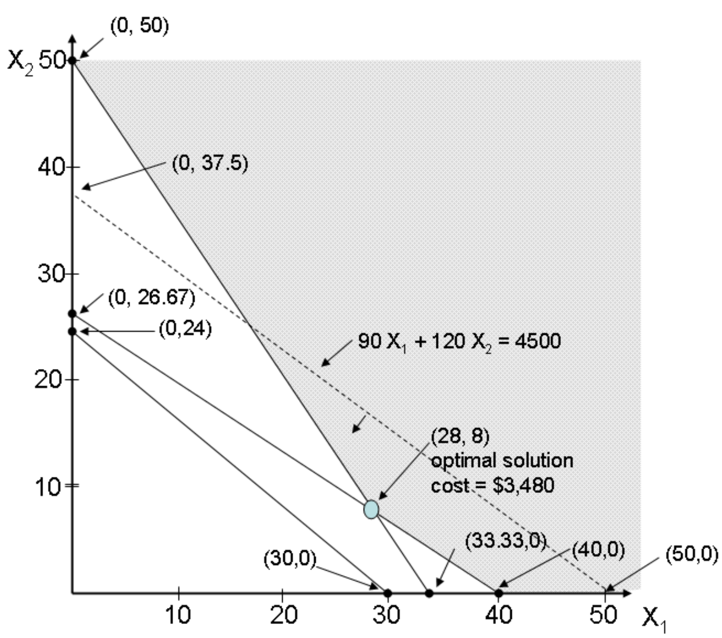
20. X_1 = # of TV spots, X_2 = # of magazine ads

MAX	$15\ X_1 + 25\ X_2$	(profit)
ST	$5\ X_1 + 2\ X_2 \le 100$	(ad budget)
	$5\ X_1 + 0\ X_2 \le 70$	(TV limit)
	$0\ X_1 + 2\ X_2 \le 50$	(magazine limit)
	$X_1, X_2 \ge 0$	



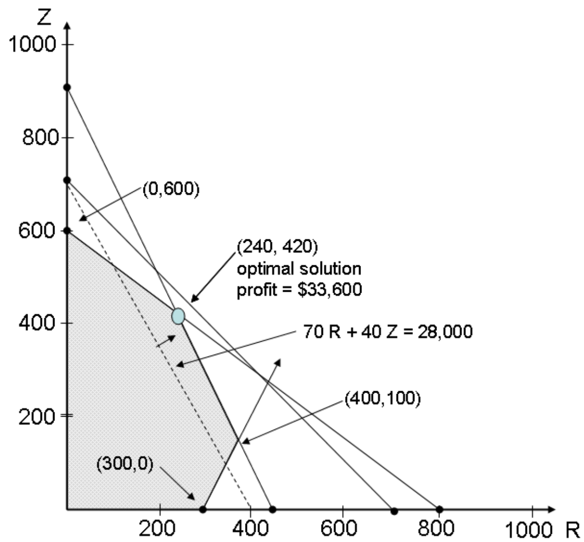
21. X_1 = tons of ore purchased from mine 1, X_2 = tons of ore purchased from mine 2

MIN	$90\ X_1 + 120\ X_2$	(cost)
ST	$0.2\ X_1 + 0.3\ X_2 \ge 8$	(copper)
	$0.2\ X_1 + 0.25\ X_2 \ge 6$	(zinc)
	$0.15\ X_1 + 0.1\ X_2 \ge 5$	(magnesium)
	$X_1, X_2 \ge 0$	



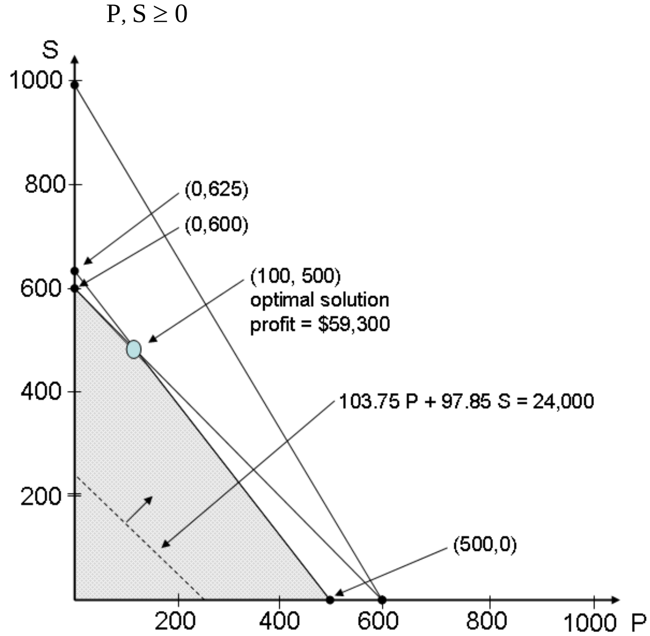
22. R = number of Razors produced, Z = number of Zoomers produced

MAX 70 R + 40 Z
ST R + Z ≤ 700
R – Z ≤ 300
2 R + 1 Z ≤ 900
3 R + 4 Z ≤ 2400
R, Z ≥ 0



23. P = number of Presidential desks produced, S = number of Senator desks produced

MAX 103.75 P + 97.85 S
ST 30 P + 24 S ≤ 15,000
1 P + 1 S ≤ 600
5 P + 3 S ≤ 3000
P, S ≥ 0



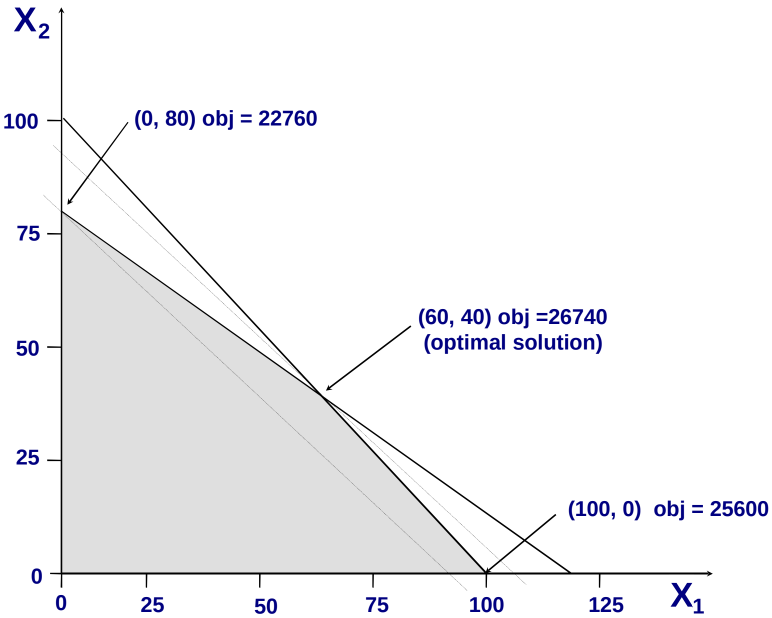
24. X_1 = acres planted in watermelons, X_2 = acres planted in cantaloupes

MAX $256 X_1 + 284.5 X_2$

ST $50 X_1 + 75 X_2 \le 6000$

$X_1 + X_2 \le 100$

$X_1, X_2 \ge 0$



25. D = number of doors produced, W = number of windows produced

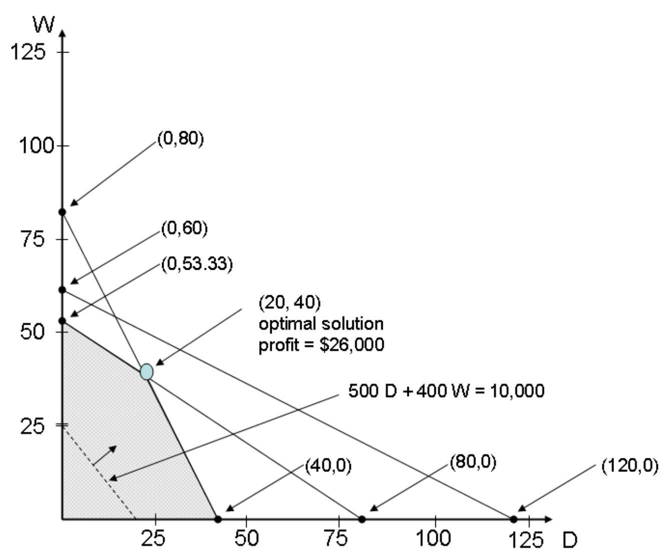
MAX $500 D + 400 W$

ST $1 D + 0.5 W \le 40$

$0.5 D + 0.75 W \le 40$

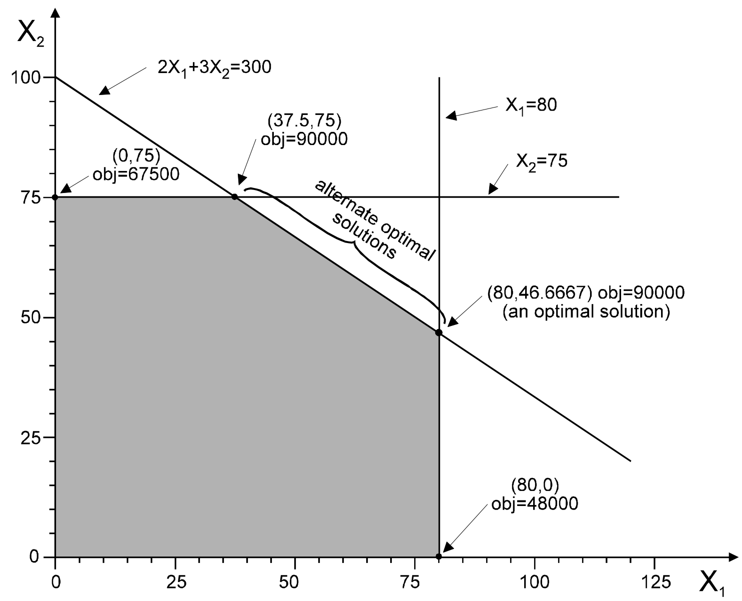
$0.5 D + 1 W \le 60$

$D, W \ge 0$



26. X_1 = number of desktop computers, X_2 = number of laptop computers

MAX $600 X_1 + 900 X_2$
ST $2 X_1 + 3 X_2 \leq 300$
$X_1 \leq 80$
$X_2 \leq 75$
$X_1, X_2 \geq 0$



Case 2-1: For The Lines They Are A-Changin'

1. 200 pumps, 1566 labor hours, 2712 feet of tubing.

2. Pumps are a binding constraint and should be increased to 207, if possible. This would increase profits by $1,400 to $67,500.

3. Labor is a binding constraint and should be increased to 1800, if possible. This would increase profits by $3,900 to $70,000.

4. Tubing is a non-binding constraint. They've already got more than they can use and don't need any more.

5. 9 to 8: profit increases by $3,050
8 to 7: profit increases by $850
7 to 6: profit increases by $0

6. 6 to 5: profit increases by $975
5 to 4: profit increases by $585
4 to 3: profit increases by $390

7. 12 to 13: profit changes by $0
13 to 14: profit decreases by $760
14 to 15: profit decreases by $1,440

8. 16 to 17: profit changes by $0
   17 to 18: profit changes by $0
   18 to 19: profit decreases by $400

9. The profit on Aqua-Spas can vary between $300 and $450 without changing the optimal solution.

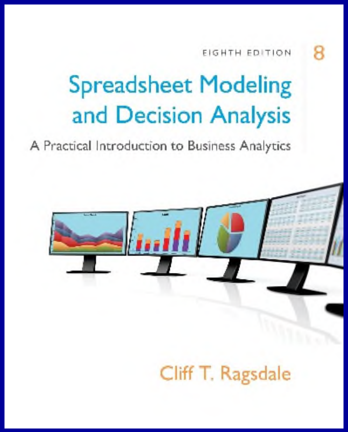
10. The profit on Hydro-Luxes can vary between $233.33 and $350 without changing the optimal solution.

Spreadsheet Modeling & Decision Analysis

A Practical Introduction to Business Analytics

8th edition

Cliff T. Ragsdale



Chapter 2

Introduction to Optimization and Linear Programming

Introduction

- We all face decision about how to use limited resources such as:
  - Oil in the earth
  - Land for dumps
  - Time
  - Money
  - Workers

Mathematical Programming...

- MP is a field of management science that finds the optimal, or most efficient, way of using limited resources to achieve the objectives of an individual of a business.
- *a.k.a.* Optimization

Applications of Optimization

- Determining Product Mix
- Manufacturing
- Routing and Logistics
- Financial Planning

Characteristics of Optimization Problems

- Decisions
- Constraints
- Objectives

General Form of an Optimization Problem

MAX (or MIN): $f_0(X_1, X_2, \ldots, X_n)$

Subject to: $f_1(X_1, X_2, \ldots, X_n) <= b_1$

$$\vdots$$

$$f_k(X_1, X_2, \ldots, X_n) >= b_k$$

$$\vdots$$

$$f_m(X_1, X_2, \ldots, X_n) = b_m$$

Note: If all the functions in an optimization are linear, the problem is a Linear Programming (LP) problem

Linear Programming (LP) Problems

MAX (or MIN): $c_1X_1 + c_2X_2 + \ldots + c_nX_n$

Subject to: $a_{11}X_1 + a_{12}X_2 + \ldots + a_{1n}X_n <= b_1$

$$\vdots$$

$$a_{k1}X_1 + a_{k2}X_2 + \ldots + a_{kn}X_n >= b_k$$

$$\vdots$$

$$a_{m1}X_1 + a_{m2}X_2 + \ldots + a_{mn}X_n = b_m$$

An Example LP Problem

Blue Ridge Hot Tubs produces two types of hot tubs: Aqua-Spas & Hydro-Luxes.

	Aqua-Spa	Hydro-Lux
Pumps	1	1
Labor	9 hours	6 hours
Tubing	12 feet	16 feet
Unit Profit	$350	$300

There are 200 pumps, 1566 hours of labor, and 2880 feet of tubing available.

5 Steps In Formulating LP Models:

1. Understand the problem.
2. Identify the decision variables.

   X_1=number of Aqua-Spas to produce

   X_2=number of Hydro-Luxes to produce
3. State the objective function as a linear combination of the decision variables.

   MAX: $350X_1 + 300X_2$

5 Steps In Formulating LP Models
(continued)

4. State the constraints as linear combinations of the decision variables.

$1X_1 + 1X_2 <= 200$ } pumps

$9X_1 + 6X_2 <= 1566$ } labor

$12X_1 + 16X_2 <= 2880$ } tubing

5. Identify any upper or lower bounds on the decision variables.

$X_1 >= 0$

$X_2 >= 0$

LP Model for Blue Ridge Hot Tubs

MAX: $350X_1 + 300X_2$

S.T.: $1X_1 + 1X_2 <= 200$

$9X_1 + 6X_2 <= 1566$

$12X_1 + 16X_2 <= 2880$

$X_1 >= 0$

$X_2 >= 0$

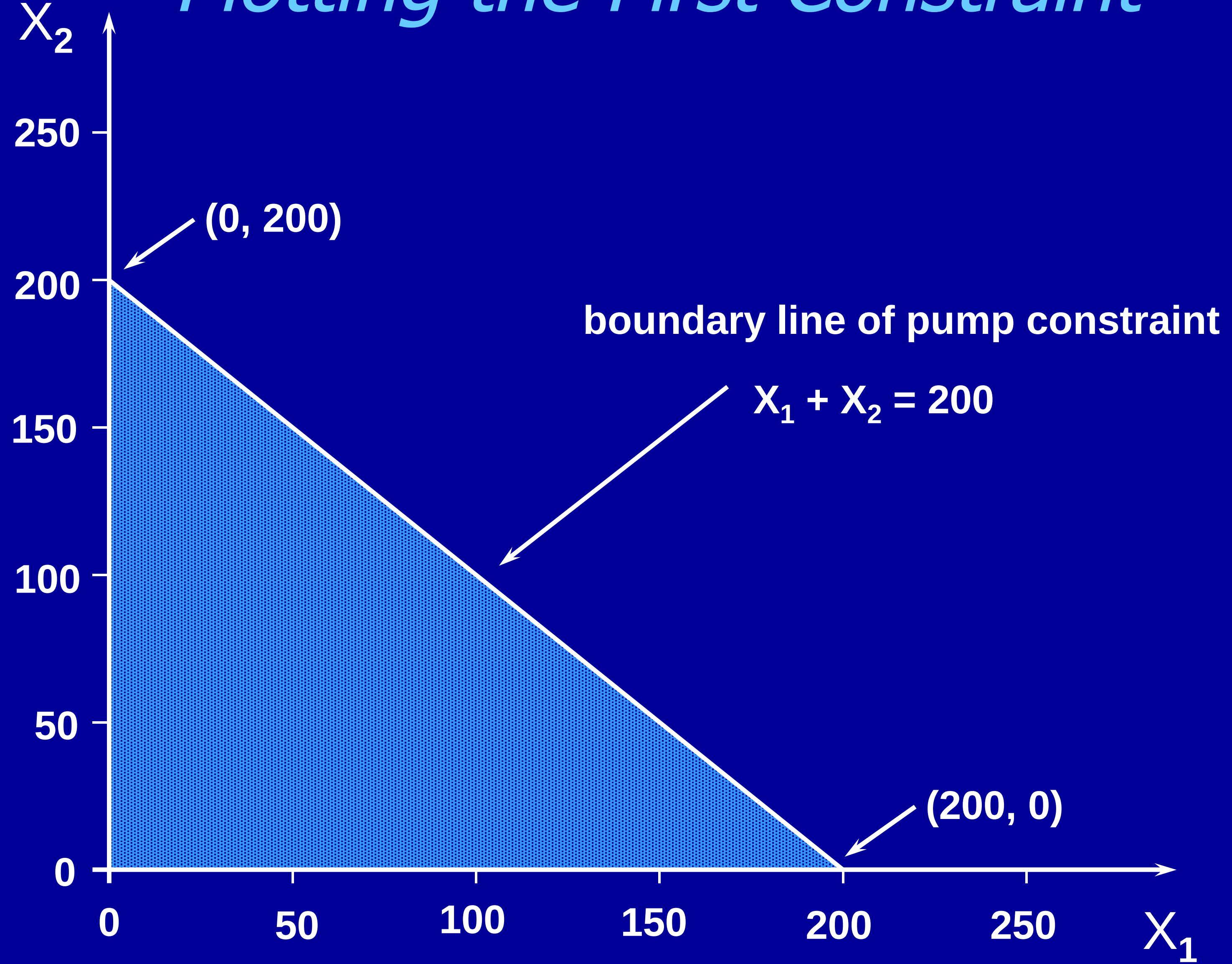
Solving LP Problems: An Intuitive Approach

- Idea: Each Aqua-Spa (X_1) generates the highest unit profit ($350), so let's make as many of them as possible!
- How many would that be?
  - Let $X_2 = 0$
    - 1st constraint: $1X_1 <= 200$
    - 2nd constraint: $9X_1 <= 1566$ or $X_1 <= 174$
    - 3rd constraint: $12X_1 <= 2880$ or $X_1 <= 240$
- If $X_2=0$, the maximum value of X_1 is 174 and the total profit is $350*174 + $300*0 = $60,900
- This solution is *feasible*, but is it *optimal?*
- ***No!***

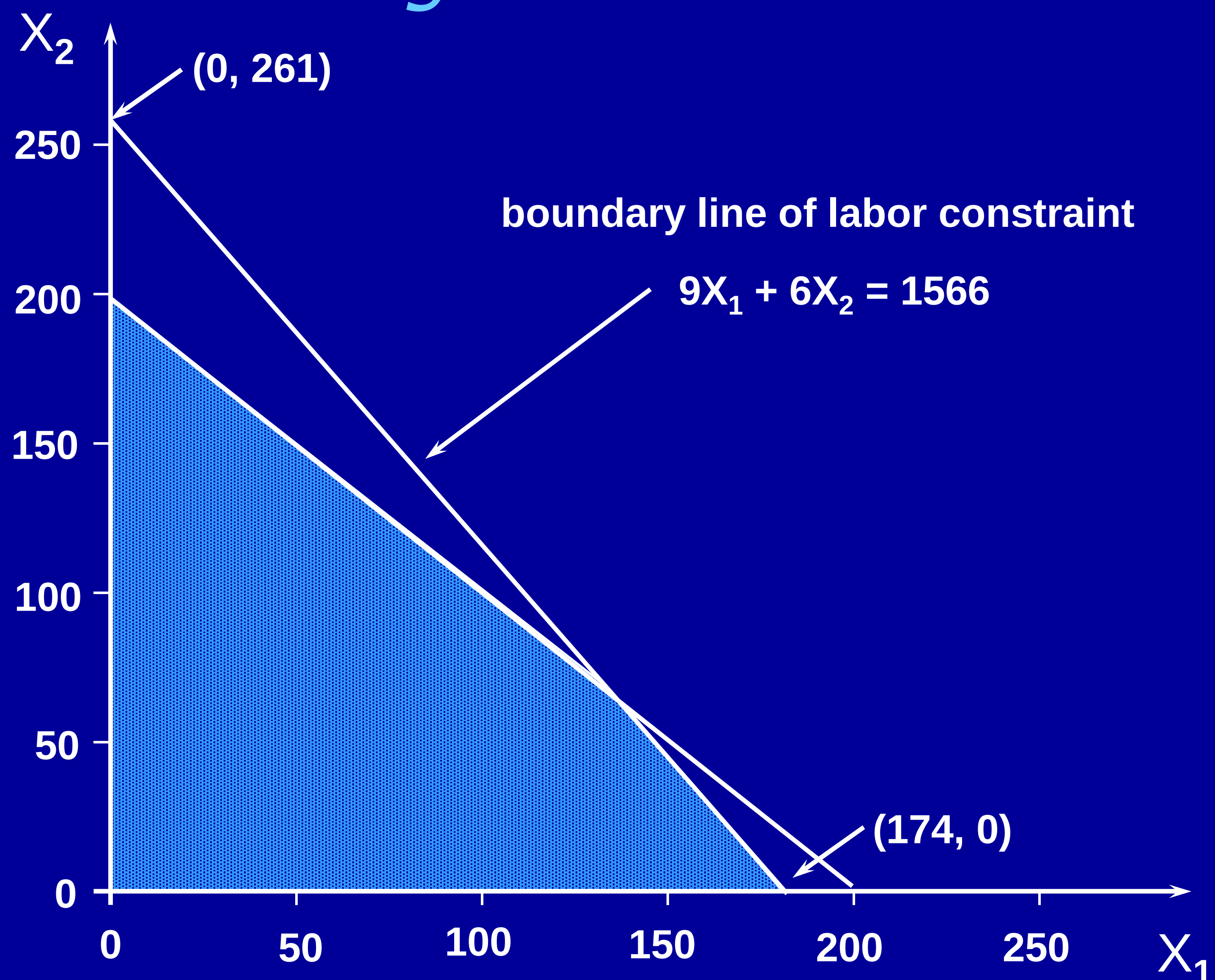
Solving LP Problems: A Graphical Approach

- The constraints of an LP problem defines its feasible region.
- The best point in the feasible region is the optimal solution to the problem.
- For LP problems with 2 variables, it is easy to plot the feasible region and find the optimal solution.

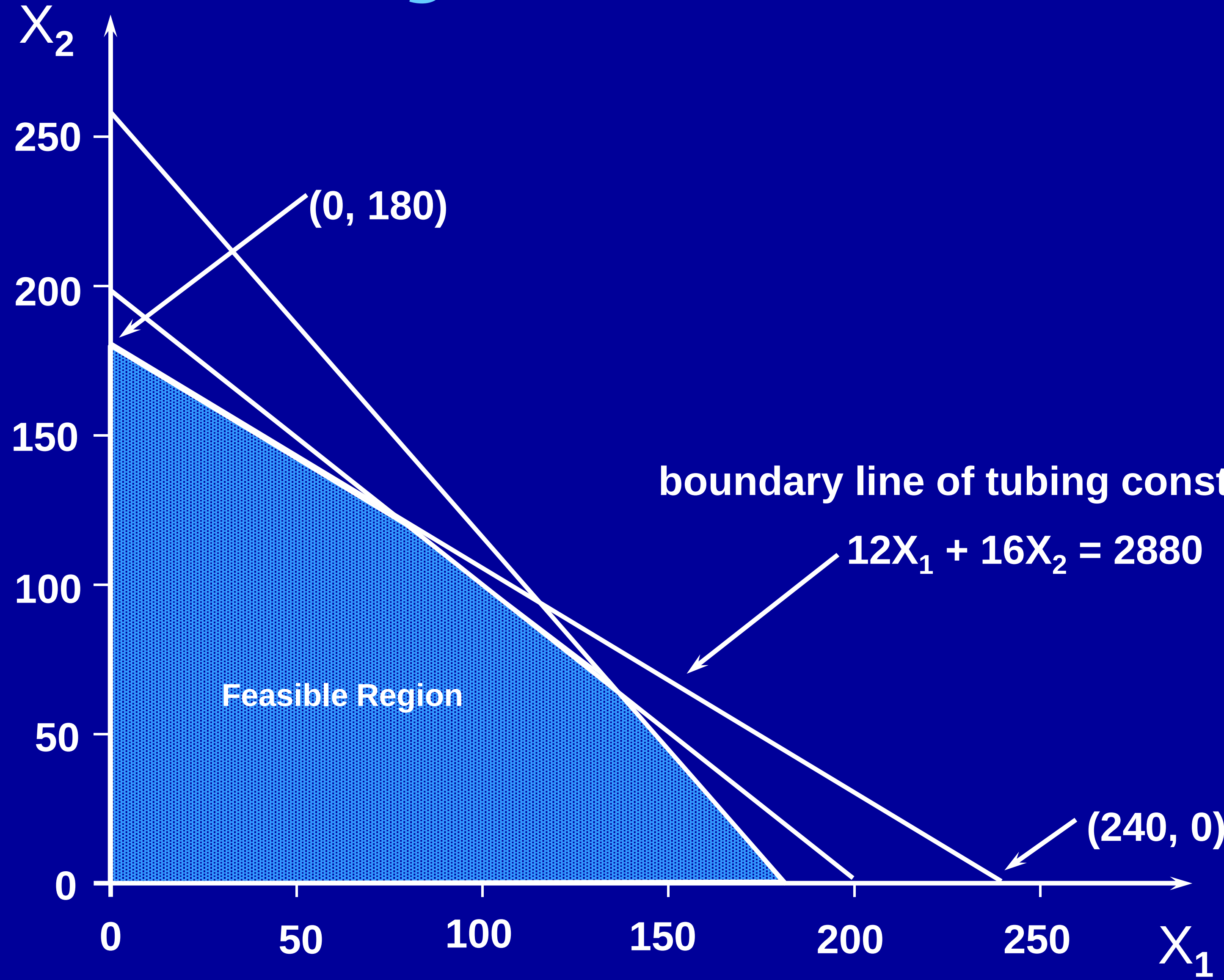
Plotting the First Constraint



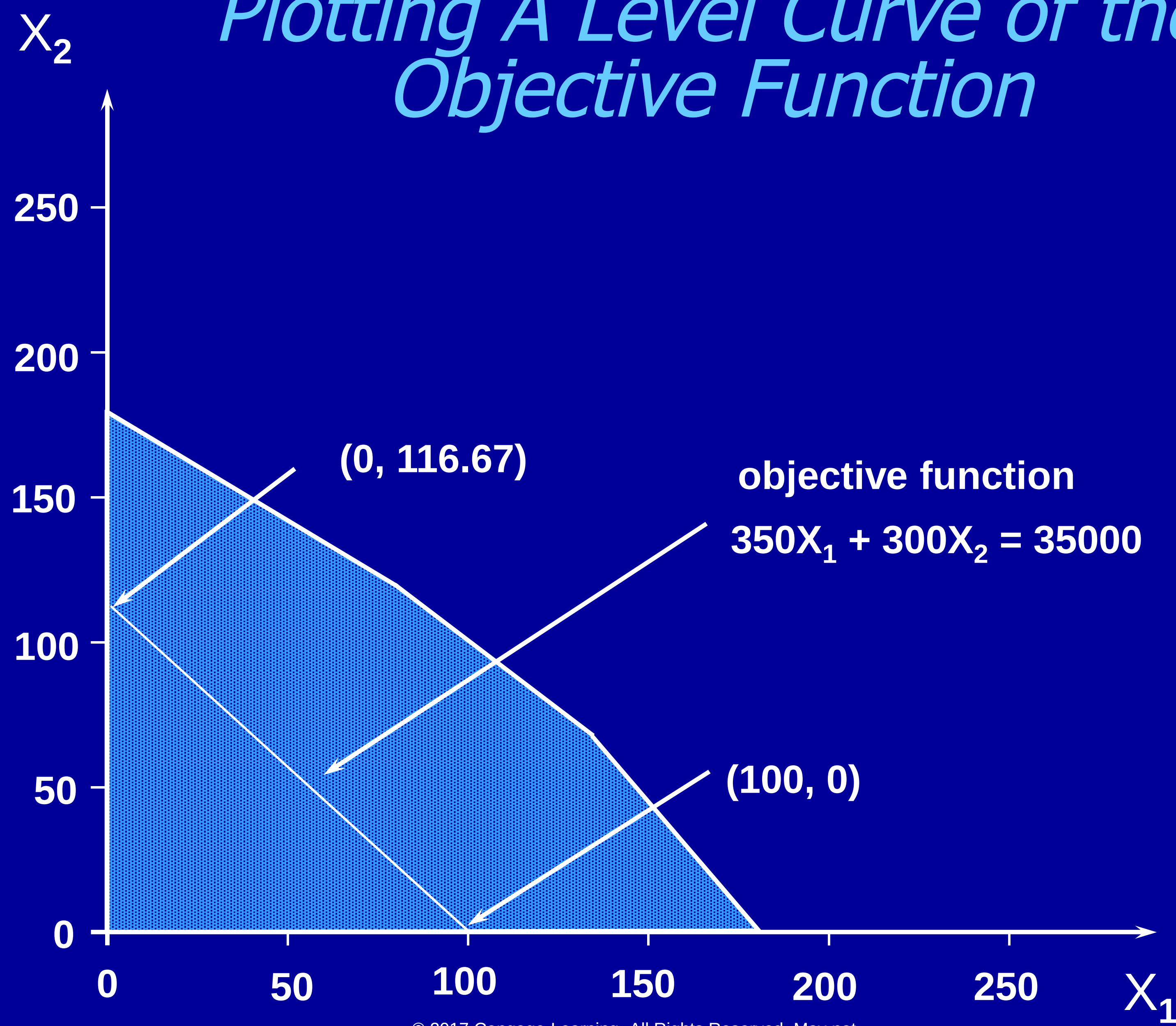
Plotting the Second Constraint



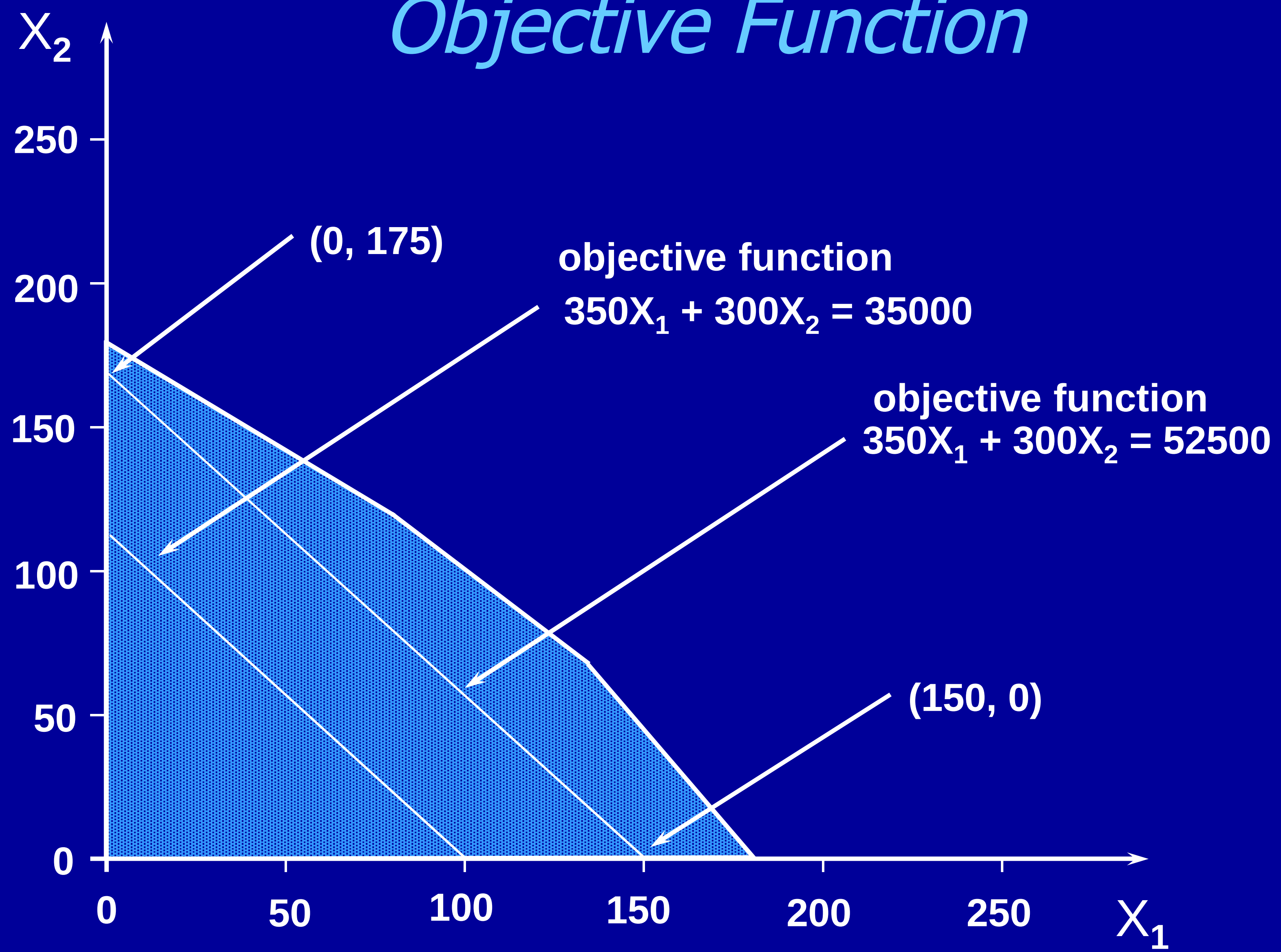
Plotting the Third Constraint



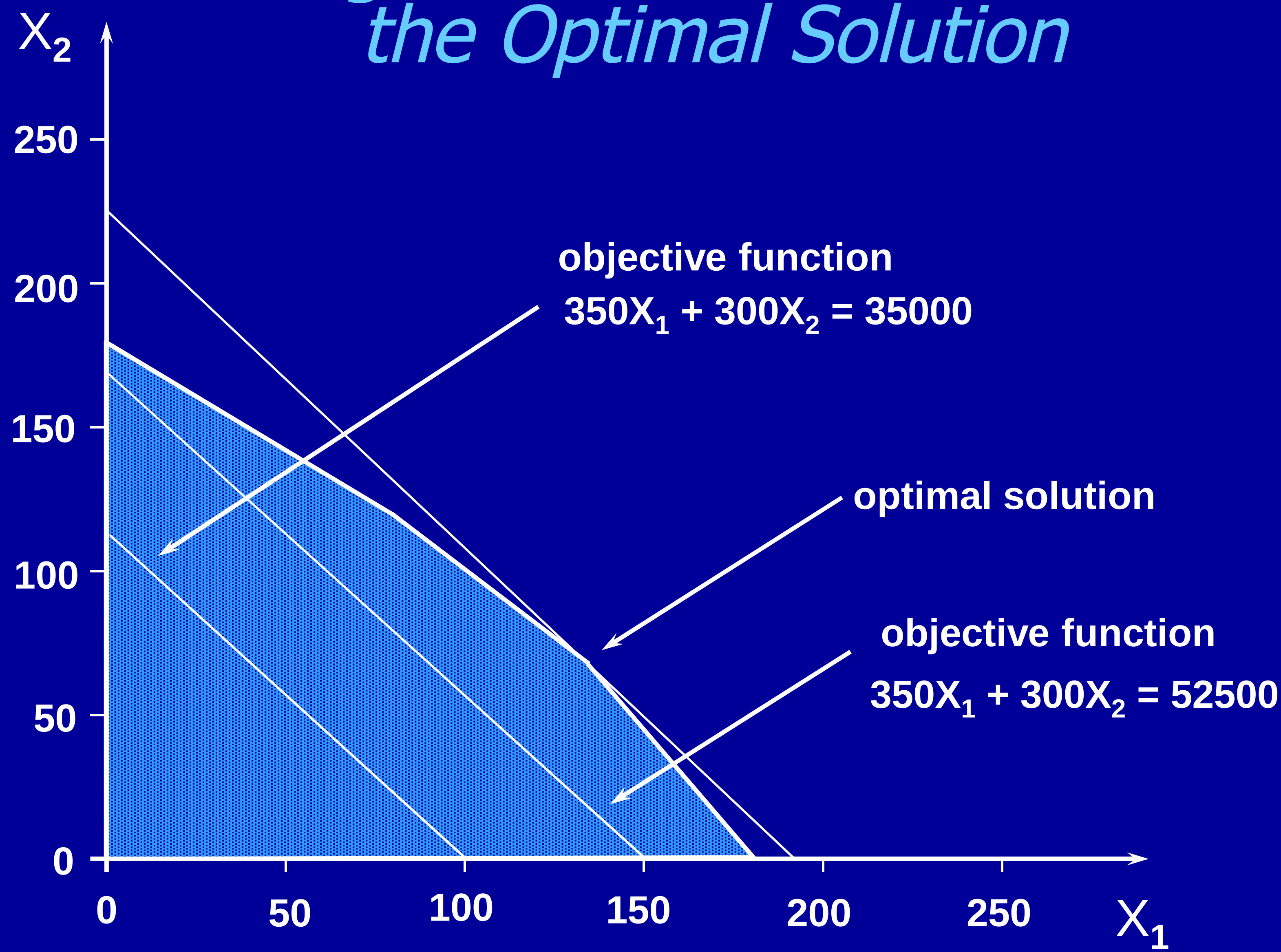
Plotting A Level Curve of the Objective Function



A Second Level Curve of the Objective Function



Using A Level Curve to Locate the Optimal Solution



Calculating the Optimal Solution

- The optimal solution occurs where the “pumps” and “labor” constraints intersect.
- This occurs where:

$$X_1 + X_2 = 200 \qquad (1)$$

$$\text{and} \quad 9X_1 + 6X_2 = 1566 \qquad (2)$$

- From (1) we have, $X_2 = 200 - X_1$ (3)
- Substituting (3) for X_2 in (2) we have,

$$9X_1 + 6\,(200 - X_1) = 1566$$

which reduces to $X_1 = 122$

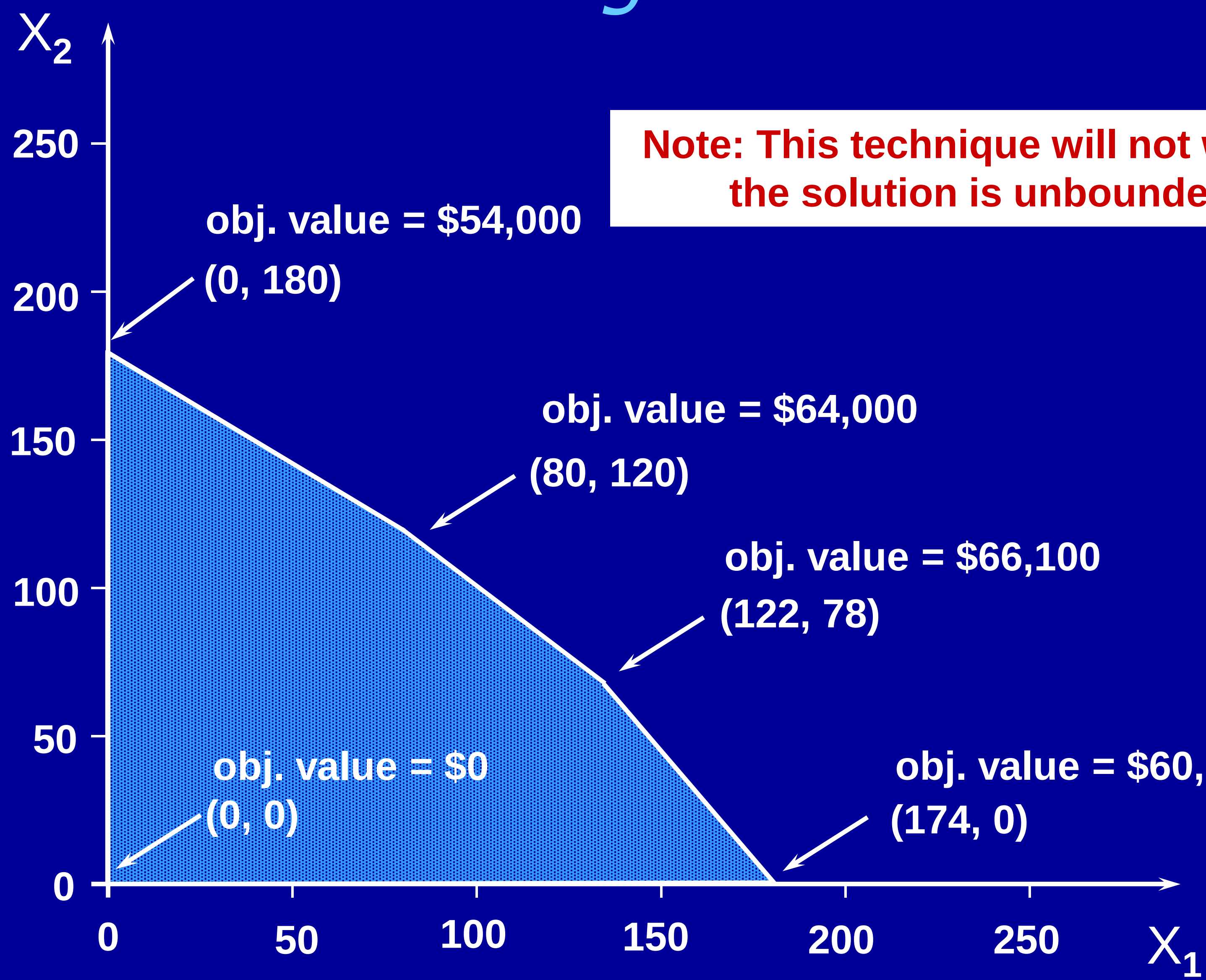
- So the optimal solution is,

$$X_1 = 122,\ X_2 = 200 - X_1 = 78$$

Total Profit = $350*122 + $300*78 = $66,100

Enumerating The Corner Points

Note: This technique will not work if the solution is unbounded.



Summary of Graphical Solution to LP Problems

1. Plot the boundary line of each constraint
2. Identify the feasible region
3. Locate the optimal solution by either:
   a. Plotting level curves
   b. Enumerating the extreme points

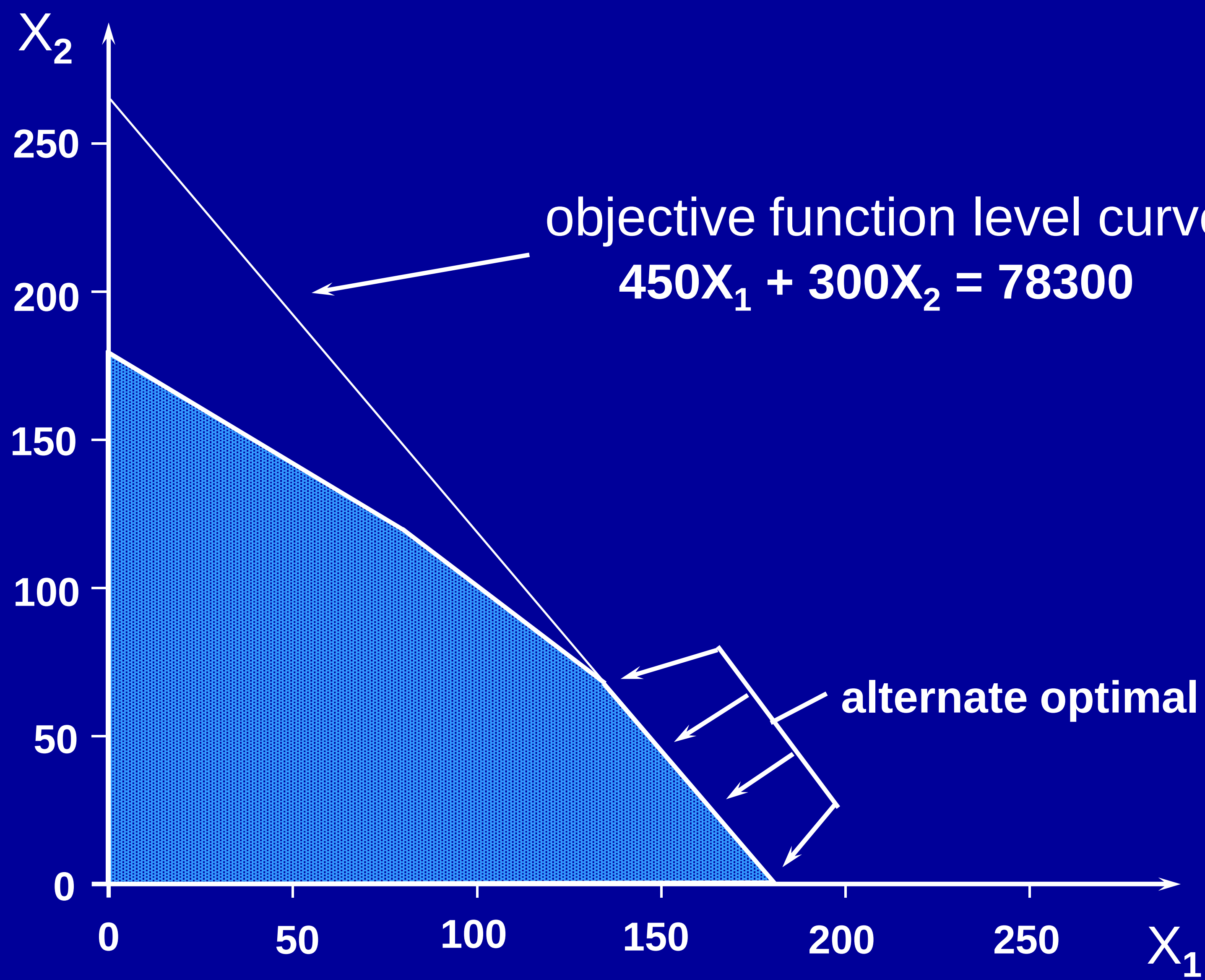
Understanding How Things Change

See file Fig2-8.xlsm

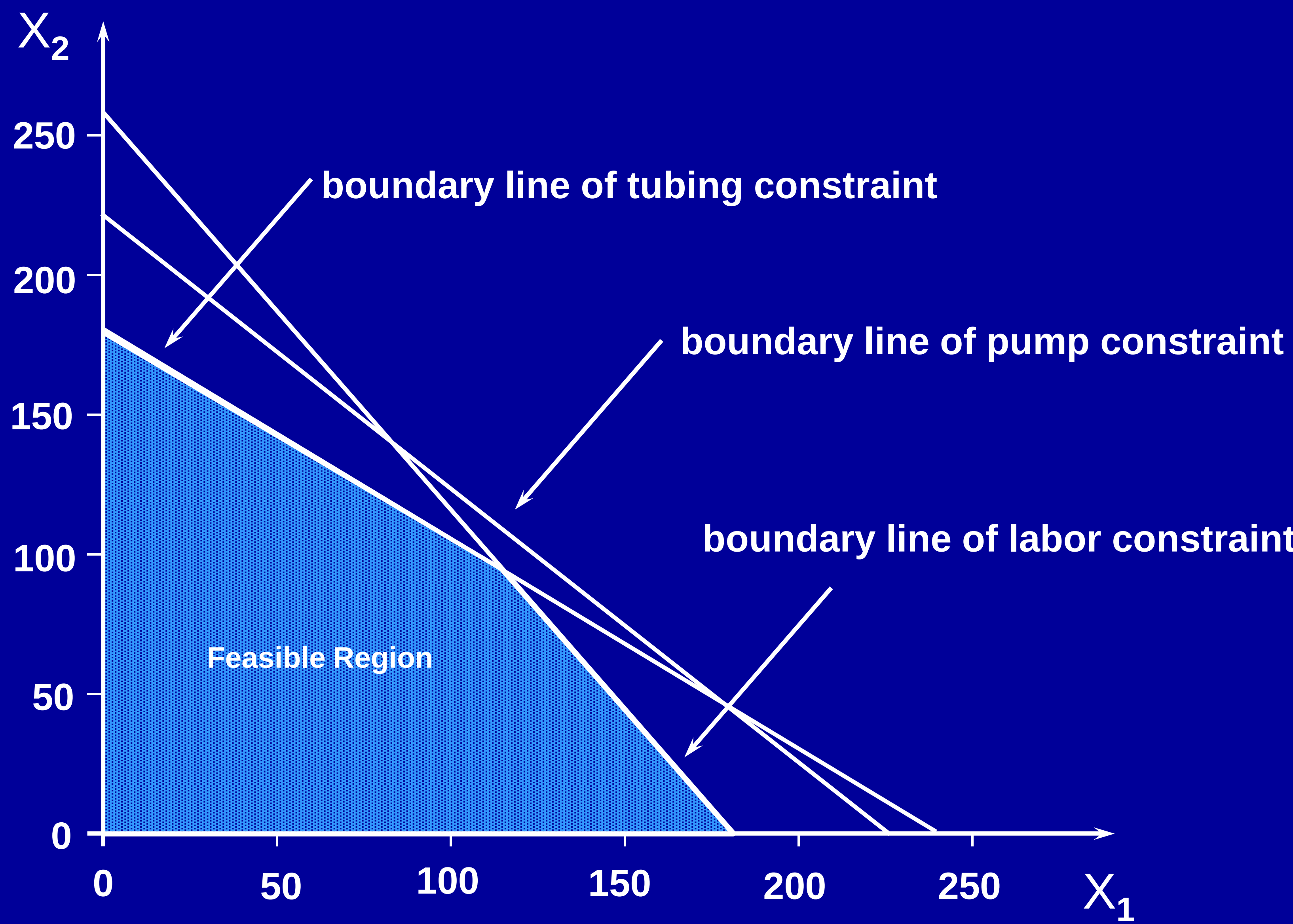
Special Conditions in LP Models

- A number of anomalies can occur in LP problems:
  - – Alternate Optimal Solutions
  - – Redundant Constraints
  - – Unbounded Solutions
  - – Infeasibility

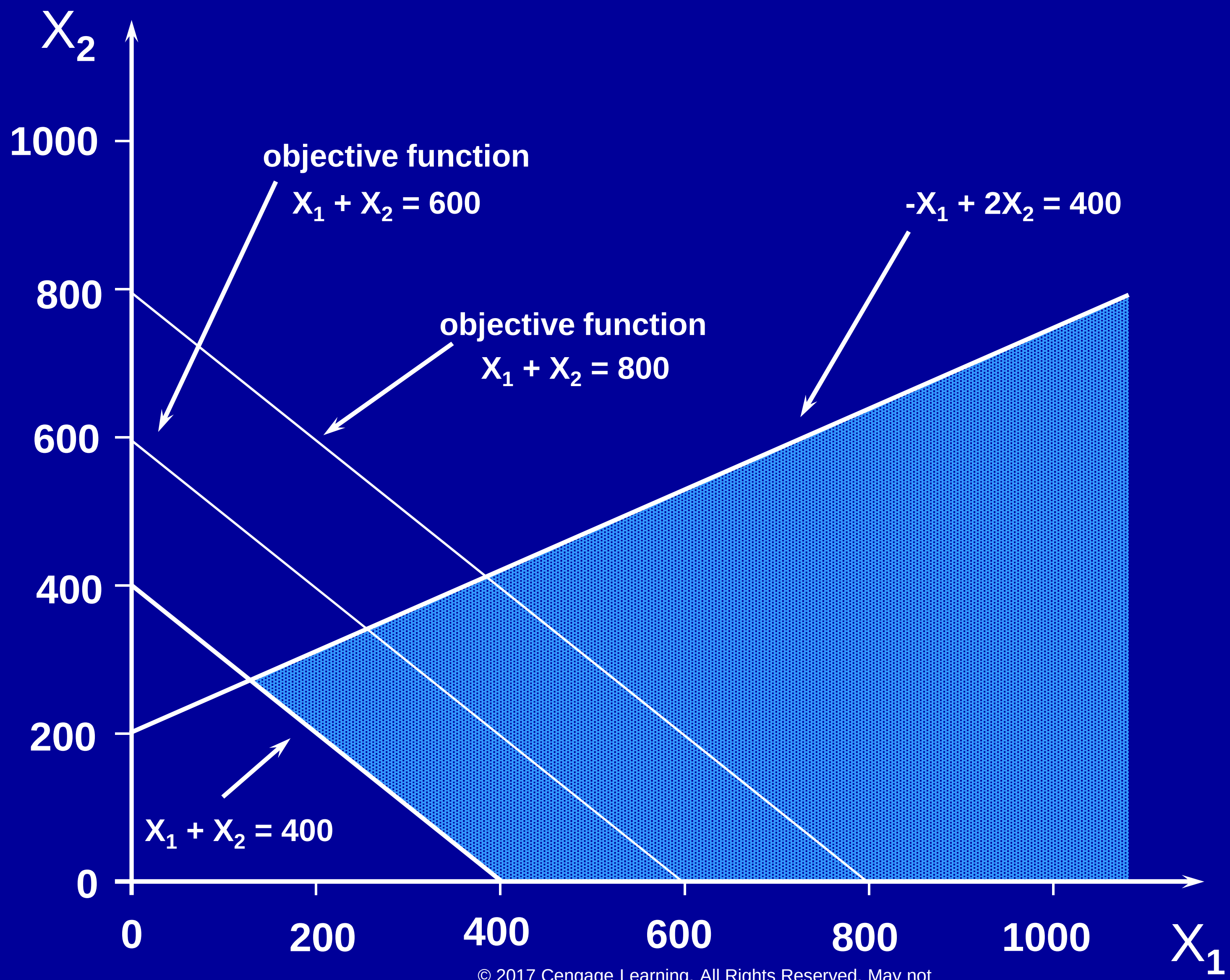
Example of Alternate Optimal Solutions



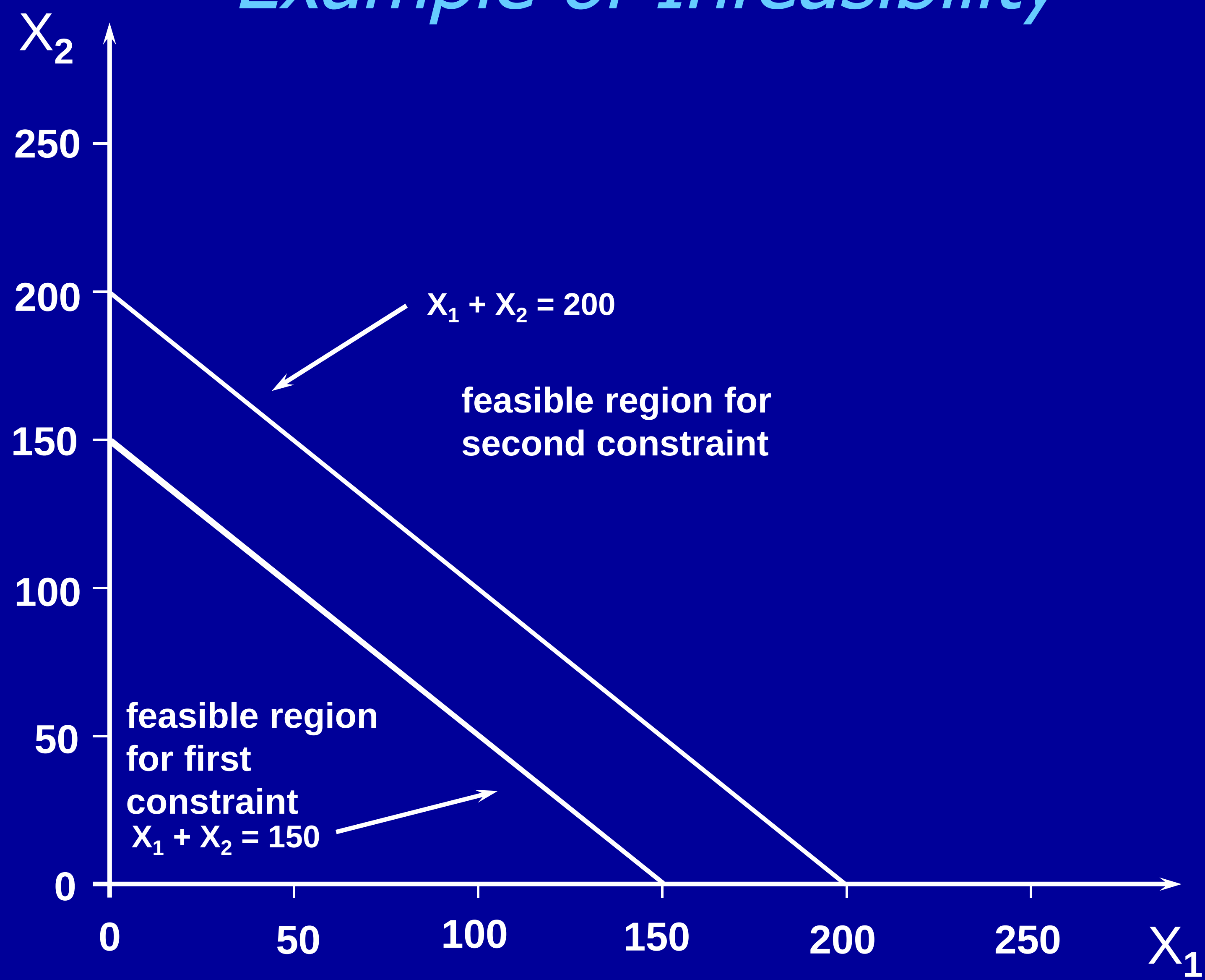
Example of a Redundant Constraint



Example of an Unbounded Solution



Example of Infeasibility



End of Chapter 2

The Analytic Solver Platform software featured in this book is provided by Frontline Systems.

http://www.solver.com