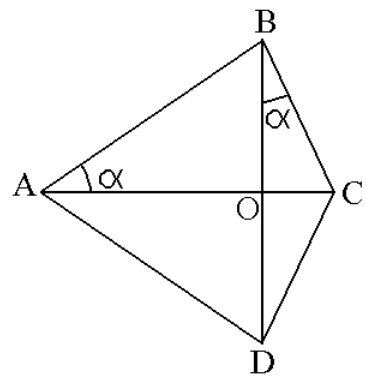
Solution to Chapter 2 Problems.

Chapter Problems

Problem 13. The four corners of a quadrilateral in the xy plane are labeled in clockwise direction as $ABCD$. It is given that $AC = 1$, ABC and ADC are right angles, and BD is perpendicular to AC. The intersection of AC and BD is labeled O and the angle BAC called α. Show that $OC = \sin^2\alpha$ and $BD = \sin 2\alpha$.



Solution.

i) From the right triangle ABC we have $BC = \sin\alpha$.
ii) The angles $B\hat{A}C$ and $O\hat{B}C$ are equal to each other, therefore, $O\hat{B}C = \alpha$.
iii) From the right triangle BOC we obtain
$OC = BC\sin\alpha = \sin^2\alpha$,
$OB = BC\cos\alpha = \sin\alpha\cos\alpha$,
$BD = 2OB = 2\sin\alpha\cos\alpha = \sin 2\alpha$.

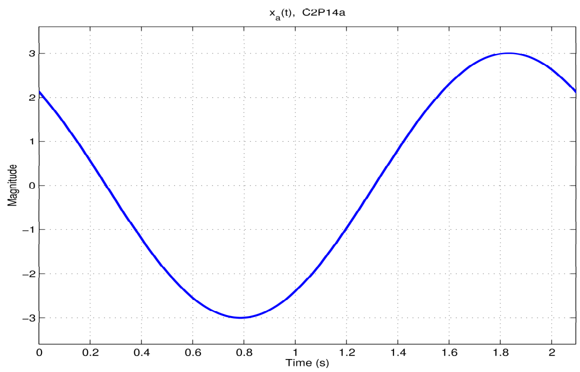
Problem 14. The expression $A\cos(2\pi ft + \theta)$, where $A > 0$ is the magnitude, f is the frequency (in Hz), and $-\pi < \theta < \pi$ is the phase in radians, represents a sinusoidal signal in cosine form. Determine the amplitude, frequency, and phase of the following signals when represented in the above form.

a) $3\cos(3t + 45^o)$ b) $2\sin(\sqrt{2}t - 120^o)$ c) $-4\cos 5t$ d) $2\cos\pi(t - 0.5)$

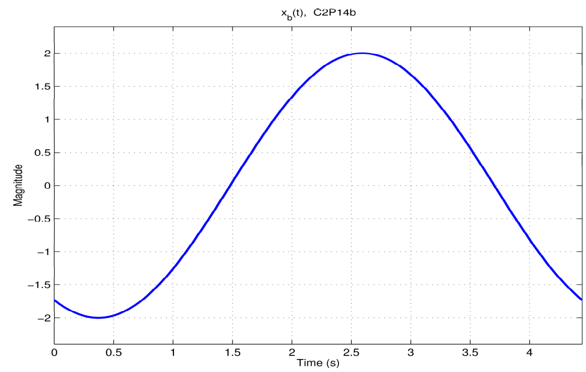
e) $2.5\cos(\pi t + 10^o)$ f) $-\cos(2\pi t + \pi/3)$ g) $-3\sin(6.28t - 2\pi/3)$ h) $2\cos 0.2(t - 1)$

Solution.	signal	A	f	θ
	$a)$ $3\cos(3t + 45^o)$	3	$\frac{3}{2\pi} = 0.4775$	45^o
	$b)$ $2\sin(\sqrt{2}t - 120^o) = 2\cos(\sqrt{2}t + 150^o)$	2	$\frac{\sqrt{2}}{2\pi} = 0.2251$	150^o
	$c)$ $-4\cos 5t = 4\cos(5t \pm 180^o)$	4	$\frac{5}{2\pi} = 0.7958$	$\pm 180^o$
	$d)$ $2\cos\pi(t - 0.5) = 2\cos(\pi t - 90^o)$	2	0.5	-90^o
	$e)$ $2.5\cos(\pi t + 10^o)$	2.5	0.5	10^o
	$f)$ $-\cos(2\pi t + \pi/3) = \cos(2\pi t - 2\pi/3)$	1	1	-120^o
	$g)$ $-3\sin(6.28t - 2\pi/3) = 3\cos(6.28t - \pi/6)$	3	$\frac{3.14}{\pi} = 0.9995$	-30^o
	$h)$ $2\cos 0.2(t - 1) = 2\cos(0.2t - 11.46^o)$	2	$\frac{1}{10\pi} = 0.0318$	-11.46^o

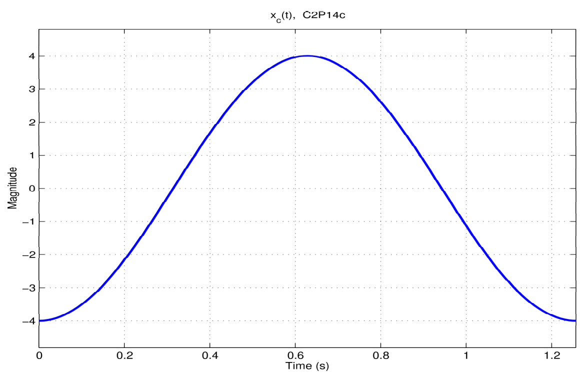
Problem 15. Sketch and label the signals given in Problem 14.



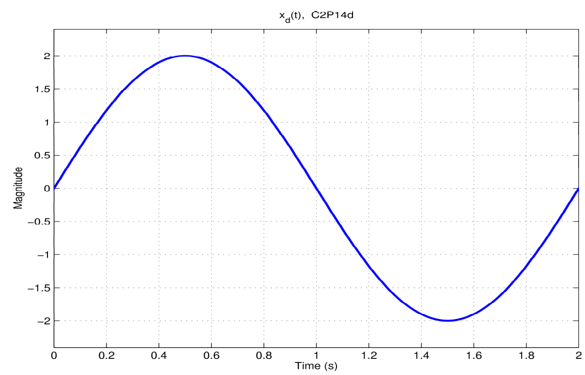
$x_a(t)$



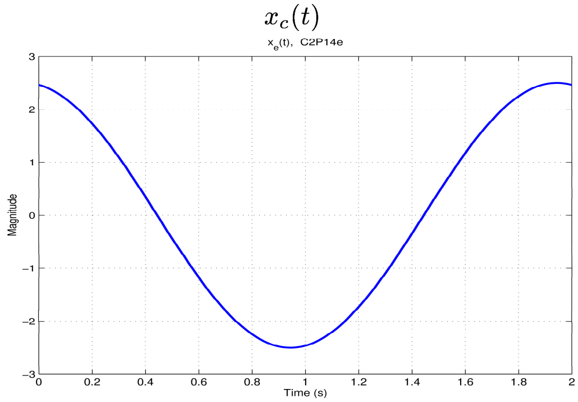
$x_b(t)$



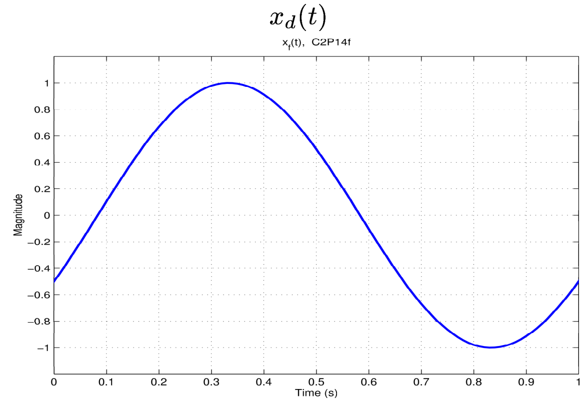
$x_c(t)$



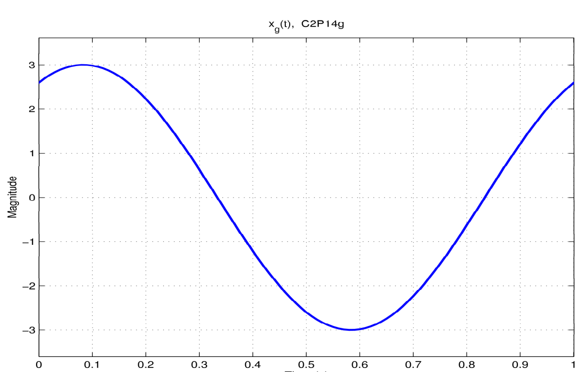
$x_d(t)$



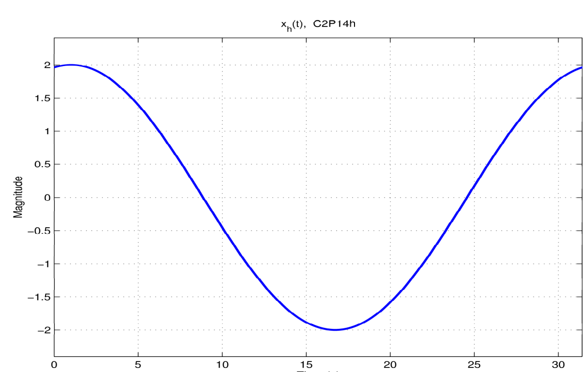
$x_e(t)$



$x_f(t)$



$x_g(t)$



$x_h(t)$

Problems 16-21. Identify the correct frequency of the given sinusoids.

Problem 16. The frequency of $\sin 10^6 \pi t$ is
a) 2 MHz
b) 2 MHz
* c) 500 kHz
d) 250 kHz
e) none of the above.

Problem 17. The frequency of $\cos 10^6 t$ is most nearly
a) 318 kHz
* b) 159 kHz
c) 159 Hz
d) 16 Hz
e) none of the above.

Problem 18. The frequency of $\sin(\pi t/6 + 3)$ is
a) 1/6 Hz
* b) 1/12 Hz
c) 1/3 Hz
d) $2/\pi$ Hz
e) none of the above.

Problem 19. The frequency of $\sin(t/6) + 3$ is
a) $\pi/3$ Hz
* b) $1/(12\pi)$ Hz
c) 1/12 Hz
d) $\pi/3$ Hz
e) none of the above.

Problem 20. The frequency of $\cos(3000\pi t + \theta)$ is
a) 6 kHz
b) 3 kHz
* c) 1.5 kHz
d) 300 Hz
e) none of the above.

Problem 21. The frequency of $\cos 10^3 t$ is most nearly
a) 159 kHz
* b) 159 Hz
c) 80 Hz
d) 50 Hz
e) none of the above.

Problems 22-31. Identify the correct period of the given signals.

Problem 22. The period of $\cos 10^6 \pi t$ is
a) 1 ms
* b) 2 μs
c) 1 μs
d) 1 ns
e) none of the above.

Problem 23. The period of $3 \sin 10^6 t$ is most nearly
a) 6 ms
* b) 6 μs
c) 1 μs
d) 6 ns
e) 1 ns .

Problem 24. The period of $2 + \cos(2 \times 10^5 \pi t)$ is
a) 2 ms
b) 20 μs
* c) 10 μs
d) 2 ns
e) none of the above.

Problem 25. The period of $\sin 20\pi t + 4 \cos 2\pi t$ is
* a) 1 s.
b) 500 ms
c) 50 ms
d) 500 μs
e) none of the above.

Problem 26. The period of $\sin 5\pi t + \cos 3\pi t$ is
a) 5 s
b) 2.5 s
* c) 2 s
d) 1.2 s
e) none of the above.

Problem 27. The period of $\sin t + \cos 2t$ is
a) $3\pi/2$ s
b) π s
c) 1 s
d) 1 ms
* e) none of the above.

Problem 28. The period of $\sin 5t + \cos 3t$ is
a) 5π s
* b) 2π s
c) $(1/5 + 1/3)2\pi$ s
d) $3\pi/5$ s
e) none of the above.

Problem 29. The period of $\sin 4t + 3 \cos 2t$ is
a) 2π s
* b) π s
c) $\pi/2$ s
d) $\pi/3$ s
e) none of the above.

Problem 30. The period of $\sin 200t + \sin 201t$ is
a) 2.1π s
* b) 2π s
c) 1.99π s
d) π s
e) none of the above.

Problem 31. The period of $\sin 200\pi t + \sin 201\pi t$ is
a) 2.2 s
b) 2.1 s
* c) 2 s
d) 1.9 s
e) none of the above.

Problems 32-37. Identify the correct phase θ of the following sinusoids when expressed in the form of $\cos(\omega t+\theta)$.

Problem 32. The phase of $\cos(\pi t + 30^o)$ is
* a) 30 degrees
b) 3π radians
c) 3 radians
d) 60 degrees
e) none of the above.

Problem 33. The phase of $\sin(2\pi t - 30^o)$ is
a) -30 degrees
b) 90 degrees
* c) -120 degrees
d) 120 degrees
e) none of the above.

Problem 34. The phase of $\cos(\pi t + 1)$ is
a) 30 degrees
b) 3π radians
* c) 1 radian
d) 60 degrees
e) none of the above.

Problem 35. The phase of $\sin(\pi t + 1)$ is most nearly
a) -1 radians
b) -23 degrees
c) 1 radians
d) 23 radians
* e) -33 degrees

Problem 36. The phase of $\cos\pi(t - 0.5)$ is
a) 90 degrees
* b) $-\pi/2$ radians
c) $-\pi$ radians
d) -5 radians
e) none of the above.

Problem 37. The phase of $\cos 500\pi(t + 10^{-3})$ is
a) 30 degrees
b) 0.5π radians
* c) 0.5 radians
d) 2 radians
e) none of the above.

Problems 38-45. Determine the correct phase relationship between the sinusoids given below.

Problem 38. $x_1 = \cos(t + 10^o)$, $x_2 = \cos(t - 30^o)$
a) x_1 leads x_2 by 10 degrees
b) x_1 lags x_2 by 130 degrees
* c) x_1 leads x_2 by 40 degrees
d) x_2 lags x_1 by 130 degrees
e) none of the above.

Problem 39. $x_1 = \cos(t + 30^o)$, $x_2 = \cos(t + 55^o)$
a) x_1 lags x_2 by 65 degrees
* b) x_1 lags x_2 by 25 degrees
c) x_2 lags x_1 by 55 degrees
d) x_1 leads x_2 by 85 degrees
e) none of the above.

Problem 40. $x_1 = \cos(t + 10^o)$, $x_2 = \sin(t - 30^o)$
a) x_1 leads x_2 by 10 degrees
b) x_1 leads x_2 by 40 degrees
* c) x_2 lags x_1 by 130 degrees
d) x_1 lags x_1 by 130 degrees
e) none of the above.

Problem 41. $x_1 = \cos(t + 30^o)$, $x_2 = \sin(t + 55^o)$
* a) x_1 leads x_2 by 65 degrees
b) x_1 lags x_2 by 85 degrees
c) x_1 lags x_2 by 25 degrees
d) x_2 leads x_1 by 35 degrees
e) none of the above.

Problem 42. $x_1 = \cos t$, $x_2 = \cos(t - 5\ ms)$
a) x_1 leads x_2 by 1 degree
* b) x_1 leads x_2 by $9/(10\pi)$ degrees
c) x_1 lags x_2 by 1 radian
d) x_1 and x_2 are almost in phase
e) none of the above.

Problem 43. $x_1 = \cos t$, $x_2 = \cos(t - 1)$
a) x_1 leads x_2 by 55 degrees
* b) x_1 leads x_2 by 1 radian
c) x_1 leads x_2 by 1 degrees
d) x_1 and x_2 are almost in phase
e) none of the above.

Problem 44. $x_1 = \cos 10^4 t$, $x_2 = \cos 10^4(t - 10\mu s)$

a) x_1 lags x_2 by 10 degrees
b) x_1 and x_2 are almost in phase
c) x_1 leads x_2 by 5 degrees
d) x_2 lags x_1 by 10 degrees
* e) x_2 lags x_1 by 0.1 radian.

Problem 45. $x_1 = \cos 10^5 t$, $x_2 = \cos 10^6(t + 1\mu s)$

a) x_1 leads x_2 by 1 degrees
b) x_1 leads x_2 by 1 radian
c) x_1 and x_2 are almost in phase
* d) x_1 lags x_2 by 1 radian
e) none of the above.

Problem 46. Use phasors to convert each time function given below into the form $A\cos(2\pi f t + \theta)$.

a) $3\cos(3t + 45^o) + 2\sin(3t - 120^o)$

b) $-4\cos(5t) + 2\cos 5(t - 0.5)$

c) $2.5\cos(\pi t + 10^o) - \cos(\pi t + \pi/3)$

d) $-3\sin(6.28t - 2\pi/3) + 2\cos 6.28(t - 0.5)$

Solution. The answers are

a) $3.1455\cos(3t + 82.89^o)$

b) $5.7287\cos(5t - 167.93^o)$

c) $2.0090\cos(\pi t - 12.41^o)$

d) $1.6178\cos(6.28t - 68.26^o)$

The following Matlab code was executed to obtain the answers.

```
r1=3;  p1=45*pi/180;         v1=r1*exp(i*p1);
r2=2;  p2=-120*pi/180-pi/2;  v2=r2*exp(i*p2);
A1=abs(v1+v2);
P1=180*angle(v1+v2)/pi;
%
r1=-4;  p1=0;         v1=r1*exp(i*p1);
r2=2;   p2=-2.5;     v2=r2*exp(i*p2);
A2=abs(v1+v2);
P2=180*angle(v1+v2)/pi;
%
r1=2.5;  p1=10*pi/180;         v1=r1*exp(i*p1);
r2=-1;   p2=pi/3;     v2=r2*exp(i*p2);
A3=abs(v1+v2);
P3=180*angle(v1+v2)/pi;
%
r1=-3;  p1=-2*pi/3-pi/2;         v1=r1*exp(i*p1);
r2=2;   p2=-3.14;     v2=r2*exp(i*p2);
A4=abs(v1+v2);
P4=180*angle(v1+v2)/pi;
%
A=[A1 A2 A3 A4]; P=[P1 P2 P3 P4]; V=[A;P]

3.1455    5.7287    2.0090    1.6178
82.8912 -167.9399  -12.4147  -68.3037
```

Problem 47. For each time function given below determine if it is periodic or aperiodic, and specify its period if periodic.

a) $\cos 5t + \cos \pi t$

b) $\cos(5t) + \cos 3.1416t$

c) $\cos(\pi t + 10^o) - \cos(2\pi t + \pi/3)$

d) $\sin(6.28t - 2\pi/3) + \cos 0.2(t-1)$

e) $\sin 5t + \cos(3t + \theta)$

f) $\sin t + \sin 2\pi t$

g) $\sin \pi t + \sin 3.141592t$

h) $\cos(1.14t + \theta) + \sin 3.141592t$

Solution. If $\dfrac{T_1}{T_2} = \dfrac{k_2}{k_1}$, where k_1 and k_2 are integers, then the sum is periodic with $T = k_1 T_1 = k_2 T_2$. Equivalently,

$$\frac{k_1}{k_2} = \frac{\omega_1}{\omega_2}, \quad T = k_1 \frac{2\pi}{\omega_1} = k_2 \frac{2\pi}{\omega_2}$$

	ω_1	ω_2	$\dfrac{\omega_1}{\omega_2} \stackrel{?}{=} \dfrac{k_1}{k_2}$	Periodic?	$T = k_1 \dfrac{2\pi}{\omega_1} = k_2 \dfrac{2\pi}{\omega_2}$
a)	5,	π,	$\dfrac{5}{\pi} \neq \dfrac{k_1}{k_2}$,	no	
b)	5,	3.1416,	$\dfrac{5}{3.1416} = \dfrac{6250}{3927} = \dfrac{k_1}{k_2}$,	yes	$T = 6250\dfrac{2\pi}{5} = 3927\dfrac{2\pi}{3.1416} = 2500\pi$
c)	π,	2π,	$\dfrac{\pi}{2\pi} = \dfrac{1}{2} = \dfrac{k_1}{k_2}$,	yes	$T = \dfrac{2\pi}{\pi} = 2\dfrac{2\pi}{2\pi} = 2$
d)	6.28,	0.2,	$\dfrac{6.28}{0.2} = \dfrac{157}{5} = \dfrac{k_1}{k_2}$,	yes	$T = 157\dfrac{2\pi}{6.28} = 5\dfrac{2\pi}{0.2} = 50\pi$
e)	5,	3,	$\dfrac{5}{3} = \dfrac{k_1}{k_2}$,	yes,	$T = 5\dfrac{2\pi}{5} = 3\dfrac{2\pi}{3} = 2\pi$
f)	1,	2π,	$\dfrac{1}{2\pi} \neq \dfrac{k_1}{k_2}$,	no	
g)	π,	3.141592,	$\dfrac{\pi}{3.141592} \neq \dfrac{k_1}{k_2}$,	no	
h)	1.14,	3.141592,	$\dfrac{1.14}{3.141592} = \dfrac{142500}{392699} = \dfrac{k_1}{k_2}$,	yes	$T = 142500\dfrac{2\pi}{1.14} = 392699\dfrac{2\pi}{3.141592} = 250,000\pi$

Problem 48. For each time function given below determine if it is periodic or aperiodic, and specify its period if periodic.

a) $\cos 2\pi t + \cos 6.28t$

b) $\cos 2\pi t + \cos 6.2816t$

c) $\cos 2\pi t + \cos 6.28159t$

d) $\cos 6.2816t + \cos 6.28159t$

e) $\cos \sqrt{2}t + \cos 1.41t$

f) $\cos 1.4142t + \cos 1.41t$

	ω_1	ω_2	$\frac{\omega_1}{\omega_2} \stackrel{?}{=} \frac{k_1}{k_2}$	Periodic?	$T = k_1\frac{2\pi}{\omega_1} = k_2\frac{2\pi}{\omega_2}$
a)	2π,	6.28,	$\frac{2\pi}{6.28} \neq \frac{k_1}{k_2}$,	no	
b)	2π,	6.2816,	$\frac{2\pi}{6.2816} \neq \frac{k_1}{k_2}$,	no	
c)	2π,	6.28159,	$\frac{2\pi}{6.28159} \neq \frac{k_1}{k_2}$,	no	
d)	6.2816,	6.28159,	$\frac{6.2816}{6.28159} = \frac{628160}{628159} = \frac{k_1}{k_2}$,	yes	$T = 628160\frac{2\pi}{6.2816} = 628159\frac{2\pi}{6.28159} = 2 \times 10^5\pi$
e)	$\sqrt{2}$,	1.41,	$\frac{\sqrt{2}}{1.41} \neq \frac{k_1}{k_2}$,	no	
f)	1.4142,	1.41,	$\frac{1.4142}{1.41} = \frac{2357}{2350} = \frac{k_1}{k_2}$,	yes	$T = 2357\frac{2\pi}{1.4142} = 2350\frac{2\pi}{1.41} = \frac{10^4\pi}{3}$

Problem 49. For each time function given below determine if it is a) periodic, b) aperiodic, or c) whether more information is needed. Specify its period if periodic.

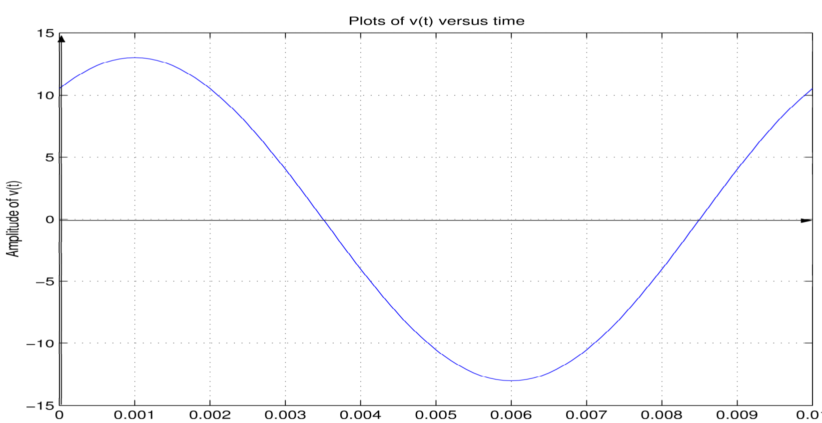
a) $\cos 3.14t + \sin 2\pi t$ b) $\cos 3.14t + \sin 3.1416t$

c) $\cos \pi t - \sin(100\pi t + \pi/3)$ d) $\sin(6.28t - \pi/3) + \cos(t-1)$

	ω_1	ω_2	$\frac{\omega_1}{\omega_2} \stackrel{?}{=} \frac{k_1}{k_2}$	Periodic?	$T = k_1\frac{2\pi}{\omega_1} = k_2\frac{2\pi}{\omega_2}$
a)	3.14,	2π,	$\frac{3.14}{2\pi} \neq \frac{k_1}{k_2}$,	no	
b)	3.14,	3.1416,	$\frac{3.14}{3.1416} = \frac{3925}{3927} = \frac{k_1}{k_2}$,	yes	$T = 3925\frac{2\pi}{3.14} = 3927\frac{2\pi}{3.1416} = 2500\pi$
c)	π,	100π,	$\frac{\pi}{100\pi} = \frac{1}{100} = \frac{k_1}{k_2}$,	yes	$T = 1\frac{2\pi}{\pi} = 100\frac{2\pi}{100\pi} = 2$
d)	6.28,	1,	$\frac{6.28}{1} = \frac{157}{25} = \frac{k_1}{k_2}$,	yes	$T = 157\frac{2\pi}{6.28} = 25\frac{2\pi}{1} = 50\pi$

Problem 50. A sinusoidal voltage $v(t)$ has a frequency of 100 Hz, a zero DC value, and a peak value of 13 V which it reaches at t=1 ms. Write its equation as a function of time in cosine form and plot it for $0 < t < 10$ msec.

Solution. $v(t) = 13\cos[200\pi(t - 0.001)] = 13\cos(200\pi t - \pi/5)$.



$v(t) = 13\cos(200\pi t - \pi/5)$

Problems 51. For the following cases determine the phase lag of $x_2(t)$ with reference to $x_1(t)$.

a) $x_1(t) = \cos(t + 30^o)$ and $x_2(t) = \cos(t - 10^o)$ b) $x_1(t) = \sin(t + 30^o)$ and $x_2(t) = \cos(t - 10^o)$

c) $x_1(t) = \cos(t + 30^o)$ and $x_2(t) = \sin(t - 10^o)$ d) $x_1(t) = \sin(t + 30^o)$ and $x_2(t) = \sin(t - 10^o)$

Solution. a) 40^o, b) -50^o (or 50^o lead), c) 130^o, d) 40^o.

Problem 52. Consider the sum of two sinusoids having the same frequency but different phases $x(t) = \cos \omega t + \cos(\omega t + \theta)$, where $0 \leq \theta \leq 2\pi$ is the phase. Problem 4 expressed it as $x(t) = A\cos(\omega t + \phi)$ and obtained the A and ϕ values for $\theta = k\pi/4, 0 \leq k \leq 8$, k an integer. Write a Matlab program to plot $x(t)$ for $\omega = 2\pi$ and the above 9 values of θ. As in Problem 4 consider the following two conditions for A and ϕ:

a) $A \geq 0$, and $-\pi \leq \phi \leq \pi$.

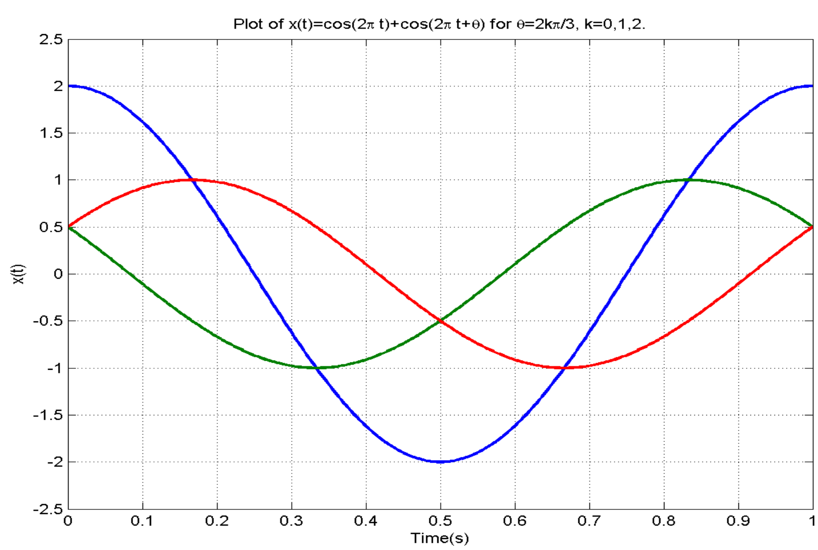
b) A any value, and $0 \leq \phi \leq \pi$.

Solution. The Matlab program of Problem 5 is modified and used. It plots

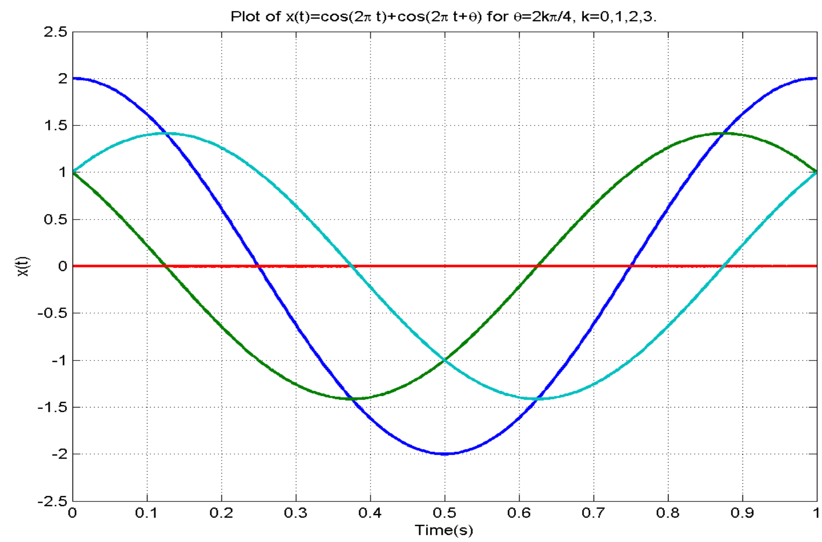
$$x(t) = \cos 2\pi t + \cos(2\pi t + \theta), \ \theta = \frac{2k\pi}{N}, \ 0 \leq k \leq (N-1), \ k \text{ an integer}$$

Representations suggested in (a) and (b) appear analytically different from each other but result in identical numerical values. The program shown below doesn't use models of (a) or (b). It generates numbers and plots them. Plots are shown below for $N = 3$, 4, and 8.

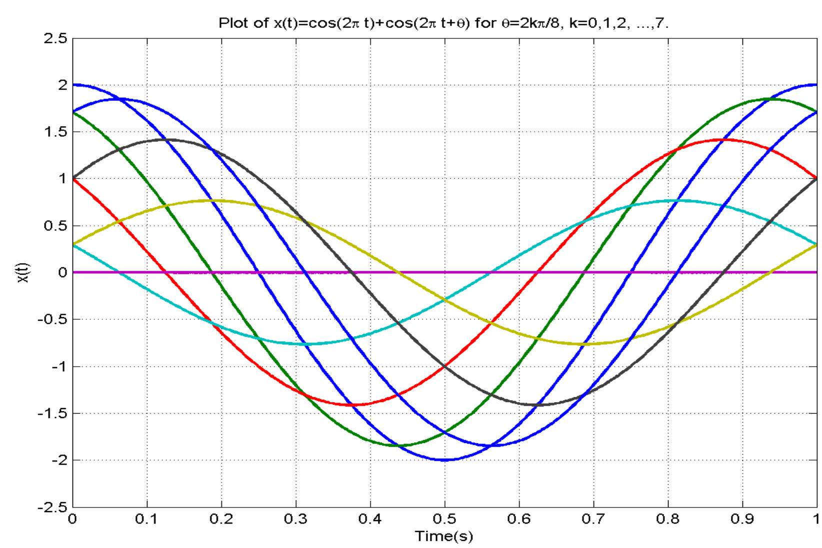
```
clear
N=3;
k=0:N-1;
theta=2*pi*k/N;
t=linspace(0,1,500);
for j=1:N;
   for i=1:500;
       x(i,j)=cos(2*pi*t(i))+cos(2*pi*t(i)+theta(j));
   end
end
for j=1:N;
    plot(t,x,'LineWidth',2)
    axis([0 1 -2.5 2.5]);
 end
```



Plots for problem 52 with $N = 3$



Plots for problem 52 with $N = 4$



Plots for problem 52 with $N = 8$

Problem 53. a) The following three measurements are made off a sinusoidal signal $x(t) = X_0 \cos(2\pi f t + \theta)$, where the frequency is assumed to be 1 MHz.

t	$t_0 = 0$	$t_1 = 0.1\ \mu$s	$t_2 = 0.2\ \mu$s
$x(t)$	$x_0 = 2.1213$	$x_1 = 0.4693$	$x_2 = -1.3620$

a) Verify the above frequency assumption. b) Find the amplitude and phase of $x(t)$. c) Knowing the frequency, what is the minimum number of samples from which the amplitude may be computed? Show how this minimum sample number is obtained.

Solution. a) Let $x(t) = A\cos(2\pi f_0 t + \theta)$. Then

$$\begin{aligned}
&(1)\quad A\cos\theta = x_0\\
&(2)\quad A\cos(\alpha+\theta) = A[\cos\alpha\cos\theta - \sin\alpha\sin\theta] = x_1, \text{ where } \alpha = 2\times 10^{-7}\pi f_0\\
&(3)\quad A\cos(2\alpha+\theta) = A[\cos 2\alpha\cos\theta - \sin 2\alpha\sin\theta] = x_2
\end{aligned}$$

From (1) one obtains $\quad (4)\quad \cos\theta = \dfrac{x_0}{A}$

From (2) one obtains $\quad (5)\quad \sin\theta = \dfrac{x_0\cos\alpha - x_1}{A\sin\alpha}$

From (4) and (5) one obtains $\quad (6)\quad \tan\theta = \dfrac{1}{\sin\alpha}\left(\cos\alpha - \dfrac{x_1}{x_0}\right)$

From (1) and (3) one obtains $\quad (7)\quad \tan\theta = \dfrac{1}{\sin 2\alpha}\left(\cos 2\alpha - \dfrac{x_2}{x_0}\right)$

Equate (6) and (7) to get $\quad (8)\quad \dfrac{1}{\sin\alpha}\left(\cos\alpha - \dfrac{x_1}{x_0}\right) = \dfrac{1}{\sin 2\alpha}\left(\cos 2\alpha - \dfrac{x_2}{x_0}\right)$

The result is $\quad (9)\quad \alpha = \cos^{-1}\left(\dfrac{x_0+x_2}{2x_1}\right), \quad f = 10^6 \text{ Hz}$

b) From (1) and (2) we get

$$\begin{aligned}
A\cos(\alpha+\theta) &= A[\cos\alpha\cos\theta - \sin\alpha\sin\theta] = x_1\\
A\sin\theta &= x_0\cot\alpha - \frac{x_1}{\sin\alpha},\\
\tan\theta &= \frac{\sin\theta}{\cos\theta} = \cot\alpha - \frac{x_1}{x_0\sin\alpha}
\end{aligned}$$

In the present case

$\alpha = 0.2\pi, \quad x_0 = 2.1213, \quad x_1 = 0.4693$ which results in $\tan\theta = \cot(0.2\pi) - \dfrac{0.4693}{2.1213\sin 0.2\pi} = 1, \quad \theta = 45^o, \quad A = 3$

The answer is $x(t) = 3\cos(2\times 10^6\pi t + 45^o)$.

The same answer will be obtained from (1) and (3)

$$\begin{aligned}
A\sin(\theta) &= 2.1213\cot(0.4\pi) + 1.3620/\sin(0.4\pi) = 2.1213,\\
\tan\theta &= \left(\frac{2.1213}{2.1213}\right) = 1, \quad \theta = 45^o, \quad A = 3
\end{aligned}$$

c) Knowing the frequency, two samples that produce two independent equations are needed to determine A and θ. (Samples that are $k\pi$ degrees apart do not give enough information).

Problem 54. For each set of measurements given in Table 4, model the signal by $x(t) = A\cos(2\pi ft + \theta)$ (i.e., find the amplitude, frequency, and phase of the sinusoid).

Problem 55. Write the mathematical expression $A\cos(2\pi ft + \theta)$ for the sinusoidal signals whose measurements are recorded in Table 5 and then sketch them.

Table 4. (Problem 54)

	t	0	0.1 μs	0.2 μs
a)	$x_1(t)$	1.7321	1.8437	1.9263
b)	$x_2(t)$	1.6209	0.7615	−0.1725
c)	$x_3(t)$	0.1732	0.1889	0.1979
d)	$x_4(t)$	1.4494	2.0148	2.5306
e)	$x_5(t)$	1.4330	1.3831	1.3201
f)	$x_6(t)$	4.2092	4.9875	5.0184
g)	$x_7(t)$	−0.2499	−0.3516	−0.4500

Table 5. (Problem 55)

	signal	period	$x_{\max}$	$x_{\min}$	$x(0)$	slope at $t = 0$
a)	$x_1(t)$	10 ms	3	−1	2.7321	+
b)	$x_2(t)$	6,67 μs	3.5	0.9	2.85	−
c)	$x_3(t)$	8,33 μs	4	0	−0.3823	−
d)	$x_4(t)$	12.5 ms	−1	−4	−1.0511	+
e)	$x_5(t)$	10 ms	2.5	0.5	2.366	+
f)	$x_6(t)$	1,334 μs	−2.5	−7.5	−3.1823	−
g)	$x_7(t)$	4 ms	1	−5	−1.0729	+

Solution for Problem 54. Using the three sample values and the formulation developed in Problem 53 we compute A, f, and θ in $x(t) = A\cos(2\pi ft + \theta)$. The following Matlab program is a sample program.

```
fprintf('Part a');
t=linspace(0, 2*10^(-7),3);
x=[1.7321   1.8437 1.9263]
x0=1.7321;    x1=1.8437; x2=1.9263;
f=(10^(7)/(2*pi))*acos((x0+x2)/(2*x1))
alpha=2*10^(-7)*pi*f;
theta=atan(cot(alpha)-x1/(x0*sin(alpha)))
A=x0/cos(theta)
x=A*cos(2*pi*f*t+theta)
```

The results are given below.

```
Part a
x= 1.7321    1.8437     1.9263
f= 1.9974e+005
theta = -0.5238
A = 2.0003
%
Part b
x= 1.6209    0.7615    -0.1725
f= 5.0020e+005
theta= 0.9998
A = 2.9990
%
Part c
x = 0.1732    0.1889    0.1979
f = 3.0018e+005
theta = -0.5241
A = 0.2001
```

```
%
Part d
x = 1.4494    2.0148    2.5306
f = 2.4997e+005
theta = -1.2001
A = 4.0005
%
Part e
x = 1.4330    1.3831    1.3201
f = 1.5495e+005
theta = 0.3002
A = 1.5001
%
Part f
x =  4.2092    4.9875    5.0184
f =  6.2002e+005
theta = -0.6000
A =  5.1000
%
Part g
x = -0.2499   -0.3516  -0.4500
f = 1.5425e+005
theta = -1.3399
A = -1.0921
```

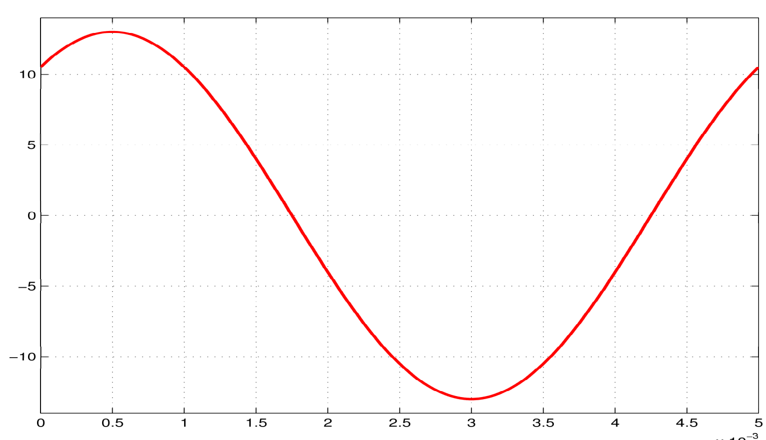
The answers are rounded to one decimal points and given below. The rounding error is, however, noticeable.

Answers to Problem 54.
a) $x_1 = 2\cos(4 \times 10^5 \pi t - \pi/6)$
b) $x_2 = 3\cos(10^6 \pi t + 1)$
c) $x_3 = 0.2\cos(6 \times 10^5 \pi t - \pi/6)$
d) $x_4 = 4\cos(5 \times 10^5 \pi t - 1.2)$
e) $x_5 = 1.5\cos(3.1 \times 10^5 \pi t - 0.3)$
f) $x_6 = 5.1\cos(1.24 \times 10^6 \pi t - 0.6)$
g) $x_7 = 1.1\cos(3.06 \times 10^5 \pi t + 1.8)$

Answers to Problem 55.
a) $x_1 = 2\cos(200\pi t - \pi/6) + 1$
b) $x_2 = 1.3\cos(300\pi t + \pi/3) + 2.2$
c) $x_3 = 2\cos(240\pi t + \pi/5) - 2$
d) $x_4 = 1.5\cos(160\pi t - \pi/12) - 2.5$
e) $x_5 = \cos(200\pi t - \pi/6) + 1.5$
f) $x_6 = 2.5\cos(1500\pi t + \pi/4) - 5$
g) $x_7 = 3\cos(500\pi t - 2\pi/5) - 2$

Problem 56. A sinusoidal voltage $v(t)$ has a frequency of 200 Hz, a zero DC value, and a peak value of 13 V which it reaches at t=0.5 ms. Write its equation as a function of time in a cosine form.

Ans. See below.



$$v(t) = 13\cos\left[400\pi(t - 0.0005)\right] = 13\cos(400\pi t - \pi/5)$$

Problem 57. A low-frequency periodic signal $s(t)$ is modeled by a DC value added to the sum of the first N harmonics of a fundamental frequency f_s (in Hz) as given below:

$$s(t) = C_0 + \frac{1}{2}\sum_{n=1}^{N} C_n \cos(2\pi n f_s t)$$

The highest frequency in $s(t)$ is $f_0 = Nf_s$. The signal $s(t)$ modulates a sinusoidal carrier $\cos(2\pi f_c t),\ f_c > f_0$. The modulated waveform is $x(t) = s(t)\cos(2\pi f_c t)$. Let $f_s = 100$ kHz, $f_0 = 1$ MHz, and $f_c = 100$ MHz. The modulated waveform $x(t)$ is passed through an ideal bandpass filter with unity amplitude gain within the band of 99 MHz to 101 MHz and zero gain outside it. The phase lag introduced by the filter in a sinusoidal input at frequency f within the above band is $\theta = 0.2\pi\left(1 + 10^{-8} f\right)$ radians. Find the output of the filter $y(t)$ and discuss possible distortions.

Solution. The input to the filter is

$$x(t) = s(t)\cos(2\pi f_c t) = C_0\cos(2\pi f_c t) + \frac{1}{2}\sum_{n=1}^{N} C_n \cos(2\pi n f_s t)\cos(2\pi f_c t) = C_0\cos(2\pi f_c t) + \frac{1}{2}\sum_{n=1}^{N} C_n x_n(t)$$

where $\quad x_n(t) = \cos(2\pi n f_s t)\cos(2\pi f_c t)$

But $\quad \cos a \cos b = \frac{1}{2}\left[\cos(a+b) + \cos(a-b)\right]$

Therefore, $\quad x_n(t) = \frac{1}{2}\left[\cos 2\pi(f_c + nf_s)t + \cos 2\pi(f_c - nf_s)t\right]$

In passing through the filter the magnitude of the above component remains the same but its phase changes and lags by $\theta = 0.2\pi\left(1 + 10^{-8} f\right) = 0.4\pi + 2\pi \times 10^{-9}(f - f_c)$ radians, where $f_c = 10^8$.

Frequency		Phase lag
f	$\Longrightarrow$	$\theta = 0.4\pi + 2\pi \times 10^{-9}(f - f_c)$
$f_c + nf_s$	$\Longrightarrow$	$\theta = 0.4\pi + 2\pi \times 10^{-9} n f_s$
$f_c - nf_s$	$\Longrightarrow$	$\theta = 0.4\pi - 2\pi \times 10^{-9} n f_s$
Input component	$\Longrightarrow$	**Output component**
$\cos 2\pi(f_c + nf_s)t$	$\Longrightarrow$	$\cos[2\pi(f_c + nf_s)t - 0.4\pi - 2\pi \times 10^{-9} n f_s]$ $= \cos\left[2\pi f_c(t - \tau_c) + 2\pi n f_s(t - \tau_s)\right]$ where $\tau_c = \frac{1}{5f_c} = 2 \times 10^{-9}$ and $\tau_s = 10^{-9}$
$\cos 2\pi(f_c - nf_s)t$	$\Longrightarrow$	$\cos[2\pi(f_c - nf_s)t - 0.4\pi + 2\pi \times 10^{-9} n f_s]$ $= \cos\left[2\pi f_c(t - \tau_c) - 2\pi n f_s(t - \tau_s)\right]$

The nth input-output pair to the filter are

$$x_n(t) = \frac{1}{2}\left[\cos 2\pi(f_c + nf_s)t + \cos 2\pi(f_c - nf_s)t\right]$$

$$y_n(t) = \frac{1}{2}\left\{\cos\left[2\pi f_c(t - \tau_c) + 2\pi n f_s(t - \tau_s)\right] + \cos\left[2\pi f_c(t - \tau_c) - 2\pi n f_s(t - \tau_s)\right]\right\}$$

But

$$\cos(a+b) + \cos(a-b) = 2\cos a \cos b$$

Therefore,

$$y_n(t) = \cos 2\pi f_c(t - \tau_c)\cos 2\pi n f_s(t - \tau_s), \quad \text{where } \tau_c = 2 \times 10^{-9} \text{ and } \tau_s = 10^{-9}$$

$$y(t) = C_0 \cos 2\pi f_c(t - \tau_c) + \frac{1}{2}\sum_{n=1}^{N} C_n \cos 2\pi n f_s(t - \tau_s)\cos 2\pi f_c(t - \tau_c)$$

$$= s(t - \tau_s)\cos 2\pi f_c(t - \tau_c)$$

In summary,

$$s(t)\cos 2\pi f_c t \implies s(t-\tau_s)\cos 2\pi f_c(t-\tau_c)$$

All components of the modulating signal $s(t)$ undergo an equal delay of $\tau_s = 0.1$ μs, resulting in no distortion. The sinuspoidal carrier undergoes a delay of $\tau_s = 0.2$ μs. No other changes occur in $x(t)$.

Problem 58. Use the phasor notation to show that $V_1\cos(\omega t+\theta_1) + V_2\cos(\omega t+\theta_2) = V\cos(\omega t+\theta)$, where

$$\theta = \tan^{-1}\left\{\frac{V_1\sin\theta_1 + V_2\sin\theta_2}{V_1\cos\theta_1 + V_2\cos\theta_2}\right\} \text{ and } V = \sqrt{V_1^2+V_2^2+2V_1V_2\cos(\theta_1-\theta_2)}$$

Solution.

$$\begin{aligned}
&V_1\cos(\omega t+\theta_1) = \mathcal{RE}\left\{V_1 e^{j\theta_1}e^{j\omega t}\right\}\\
&V_2\cos(\omega t+\theta_2) = \mathcal{RE}\left\{V_2 e^{j\theta_2}e^{j\omega t}\right\}\\
&V_1\cos(\omega t+\theta_1) + V_2\cos(\omega t+\theta_2) = \mathcal{RE}\left\{(V_1 e^{j\theta_1} + V_2 e^{j\theta_2})e^{j\omega t}\right\} = \mathcal{RE}\left\{V e^{j\theta}e^{j\omega t}\right\}
\end{aligned}$$

$$V e^{j\theta} = V_1 e^{j\theta_1} + V_2 e^{j\theta_2} \implies \begin{cases} V\cos\theta = V_1\cos\theta_1 + V_2\cos\theta_2 & \text{(Eq-1)}\\ V\sin\theta = V_1\sin\theta_1 + V_2\sin\theta_2 & \text{(Eq-2)}\end{cases}$$

To find the phase divide Eq-2 and Eq-1.

$$\tan\theta = \frac{V_1\sin\theta_1 + V_2\sin\theta_2}{V_1\cos\theta_1 + V_2\cos\theta_2}$$

To find the magnitude, square Eq-1 and Eq-2 and add them side-by-side.

$$\begin{aligned}
V^2 &= V_1^2 + V_2^2 + 2V_1V_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)\\
&= V_1^2 + V_2^2 + 2V_1V_2\cos(\theta_1-\theta_2)
\end{aligned}$$

Problem 59. Shift the periodic square-wave signal of Example 7 by a constant value and show that the shifted waveform may be represented by an infinite series of the following form:

$$v(t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n t}{T} + \theta_n\right), \quad \text{where } |a_n| = \frac{2A}{\pi n} \quad \text{and } n \text{ odd.}$$

Show that the shift does not affect conclusions regarding power distribution obtained in Example 7.

Solution.

$$v(t-\tau) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n(t-\tau)}{T} + \theta_n\right) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n t}{T} + \phi_n\right),$$

$$\text{where } \phi_n = \theta_n - \frac{2\pi n\tau}{T}, \quad |a_n| = \frac{2A}{\pi n} \quad \text{and } n \text{ odd}$$

In the new description the sinusoids are still orthogonal to each other, and their amplitude has not changed. Its power spectrum remains the same. Time doesn't affect the power spectrum of a periodic signal.

Problem 60. The function $x(t) = t$ can be approximated during $-T/2 < t < T/2$ by the finite series

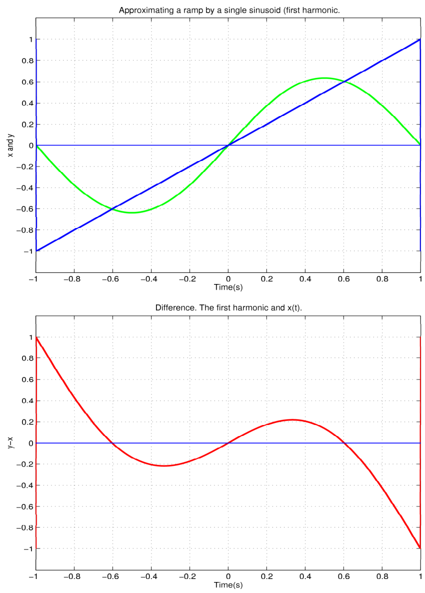
$$y(t) = \frac{T}{\pi}\sum_{n=1}^{N} \frac{T}{\pi n}(-1)^{(n-1)}\sin\left(\frac{2\pi n t}{T}\right)$$

One measure of approximation error is $\mathcal{E} = \frac{1}{T}\int_{-\frac{T}{2}}^{-\frac{T}{2}} |x(t)-y(t)|^2 dt$. Write a program to generate $x(t)$ and $y(t)$, plot them, and compute the error as defined above. Run the program for $N = 1, \cdots, 10$ and plot $\mathcal{E}$ vs. N.

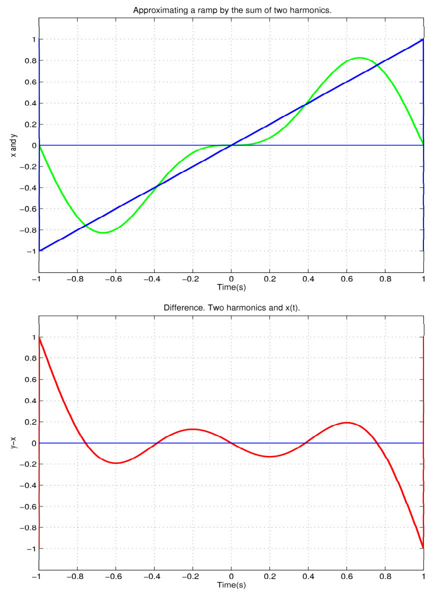
Solution.

```
N=10;
t=linspace(-1,1,1000*N);
x=t;
for n=1:N
   for i=1:1000*N;
      p(n,i)=2*(-1)^(n-1)*sin(n*pi*t(i))/(pi*n);
   end;
end;
y=sum(p,1);
e=x-y;
e2=e.^(2);
AE=sum(e2)/(1000*N)
```

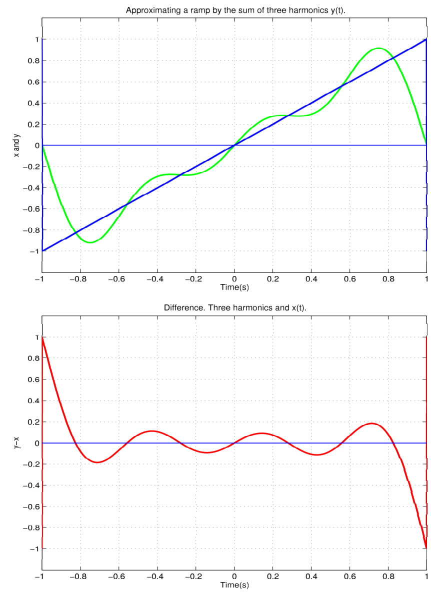
N	1	2	3	4	5	6	7	8	9	10
$\mathcal{E}$	0.1307	0.0800	0.0575	0.0448	0.0367	0.0311	0.0270	0.0238	0.0213	0.0193



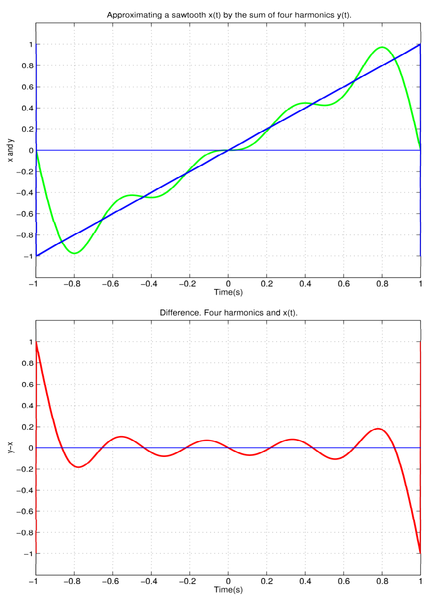
$N = 1$



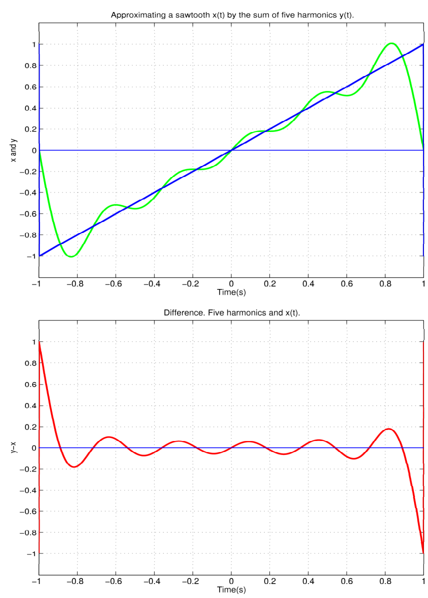
$N = 2$



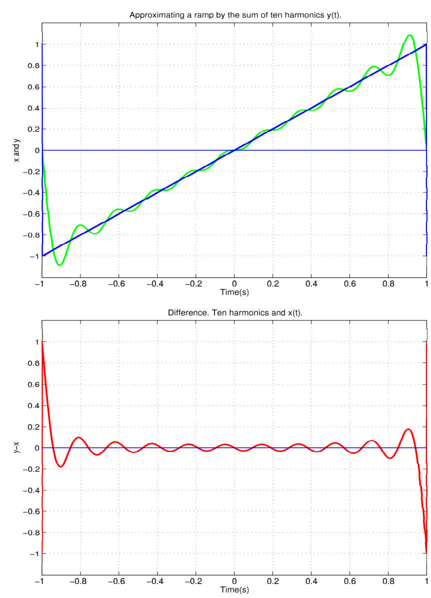
$N = 3$



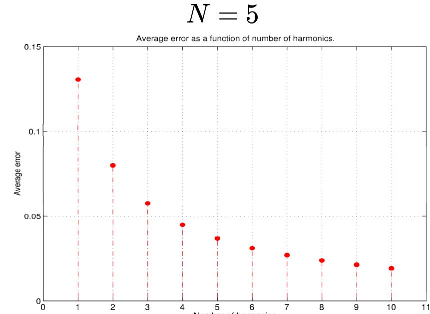
$N = 4$



$N = 5$



$N = 10$



$\mathcal{E}$ vs. $N = 1, 2 \cdots 10$

Problem 61. Motion of a free electron in a sinusoidal electric field. An electron has a negative electric charge of $e = 1.602 \times 10^{-19}$ C and a mass of $m = 9.109 \times 10^{-31}$ kg. When placed in an electric field of field strength $\mathcal{E}$, it experiences a force of $e \times \mathcal{E}$. Determine the span of the motion of an electron in vacuum when subjected to an electric field $\mathcal{E} = 10^{-6}\cos(2\pi f t)$ V/m at frequencies of a) 60 Hz, b) 1 kHz, c) 1 MHz, and d) 1 GHz.

Solution.

$$F = e\mathcal{E} = ma,\ a(t) = \frac{e\mathcal{E}(t)}{m} = a_0\cos(2\pi f t), \quad \text{where } a_0 = 10^{-6}\frac{e}{m}$$

$$x(t) = -X_0\cos(2\pi f t), \quad \text{where } X_0 = \frac{a_0}{(2\pi f)^2} = \frac{4454.8}{f^2}$$

Frequency	60 Hz	1 kHz	1 MHz	1 GHz
Span of oscillations	1.237455 m	4.455 mm	4.455 nm	4.455×10^{-15} m

Problem 62 The following Matlab program is written to sweep from left to right in the xy plane. In each sweep it plots a sinusoid, then moves down an incremental value, similar to the motion of the electron beam in a cathode ray tube or the operation of the recording element in a seismograph.

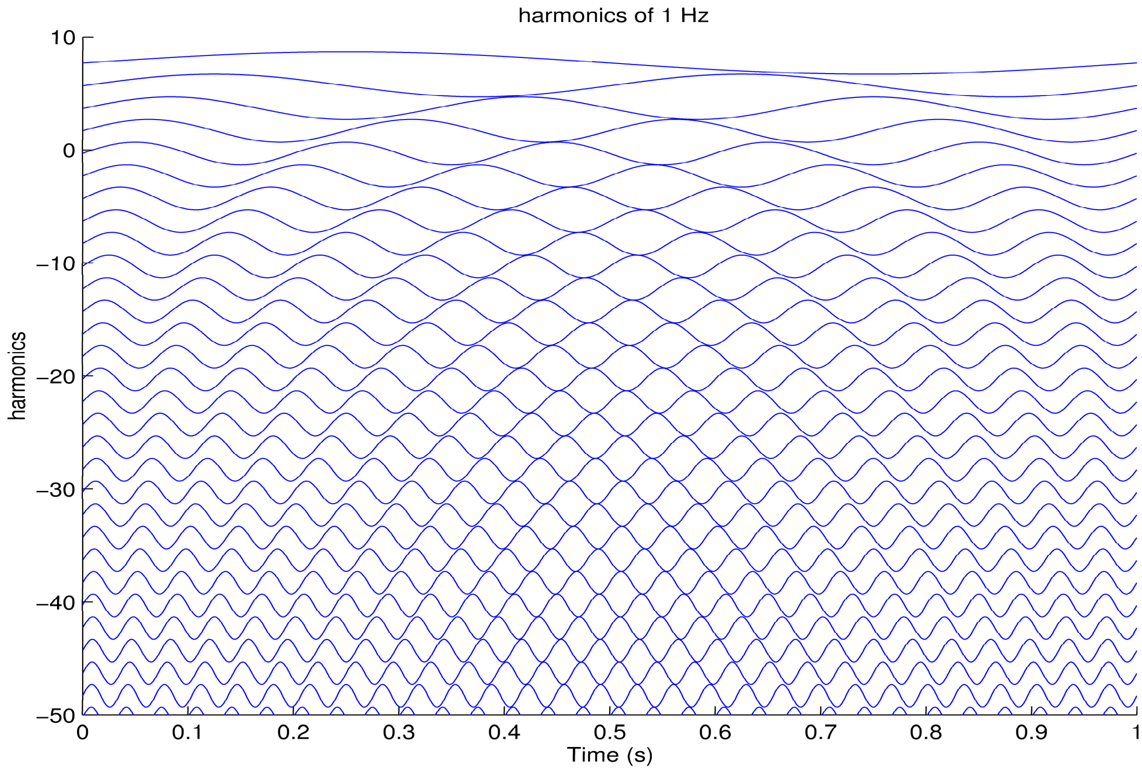
```
t=[0:0.001:1];
hold on
for k=1:30,
    x=sin(2*pi*k*t)+10-2*k-0.3; axis([0 1 -50 10]);
    plot(t,x);
    xlabel('Time (s)'); ylabel('harmonics'); title(' harmonics of 1 Hz');
    grid;
end
hold off
```

a) Execute the program and examine the plot to verify that it agrees with expectation.
b) In successive steps replace the fourth line in the program with a new command line from the following list and note if the resulting plot agrees with your expectation.

```
x=sin(2*pi*k*t)+2*k+0.3;      axis([0 1 0 50]);
x=sin(2*pi*k*t)+50-2*k-0.3;  axis([0 1 0 50]);
x=sin(2*pi*k*t)+60-2*k-0.5;  axis([0 1 0 60]);
x=sin(2*pi*k*t)+90-2*k-1;    axis([0 1 30 90]);
x=sin(2*pi*k*t)+90-4*k-1;    y=cos(2*pi*k*t)+90-4*k-3;  axis([0 1 30 90]);  plot(t,x,t,y);
```

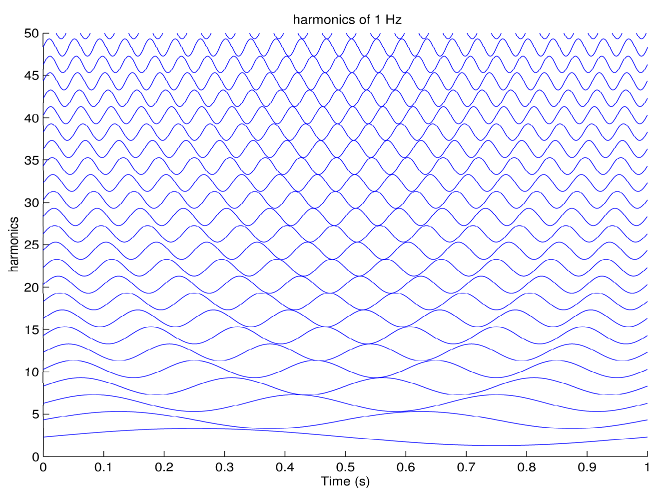
Explore an alternative set of variables which would enhance the wavy appearance of the two-dimensional plot.

Solution. a) The two-dimensional pattern generated by the program is shown below.

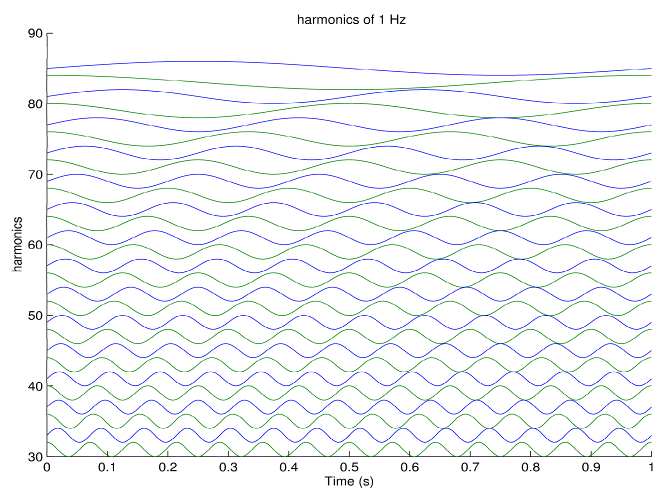


$$x = \sin(2\pi k t) + 10 - 2k - 0.3, \quad k = 1, 2, \cdots, 30$$

b) Examples of plot variations are shown below.



$$x = \sin(2\pi k t) + 90 - 4k + 0.3, \quad k = 1, 2, \cdots, 30$$



$$x_1 = \sin(2\pi k t) + 90 - 4k - 1, \quad k = 1, 2, \cdots, 30$$
$$x_2 = \cos(2\pi k t) + 90 - 4k - 3, \quad k = 1, 2, \cdots, 30$$

The following Matlab file generates and plots the above patterns.

```
t=[0:0.001:1];
figure(1)
hold on
for k=1:30,
      x=sin(2*pi*k*t)+10-2*k-0.3;  axis([0 1 -50 10]); %Constructs the set of sinusoids
plot(t,x);
xlabel(’Time (s)’);
ylabel(’harmonics’);
title(’ harmonics of 1 Hz’);
```

```
grid;
   end
hold off
print -dpsc pattern_a.eps    %Saves the plot in encapsulated post-script format
print -dtiff pattern_a.tiff  %Saves the plot in tiff format
%
figure(2)
hold on
for k=1:30,
      x=sin(2*pi*k*t)+2*k+0.3;  axis([0 1 0 50]);
plot(t,x);
xlabel('Time (s)');
ylabel('harmonics');
title(' harmonics of 1 Hz');
grid;
   end
   hold off
hold off
print -dpsc pattern_b.eps
%.......................
figure(6)
hold on
for k=1:30,
   x=sin(2*pi*k*t)+90-4*k-1;
   y=cos(2*pi*k*t)+90-4*k-3;
   axis([0 1 30 90]);
   plot(t,x,t,y);
xlabel('Time (s)');
ylabel('harmonics');
title(' harmonics of 1 Hz');
grid;
   end
   hold off
hold off
print -dpsc pattern_f.eps
```

Project: Trajectories, Wave Polarization, and Lissajous Patterns.

Purpose. To investigate the trajectories of the sinusoidal motion of a point in the xy plane and obtain parameters of the motion from the patterns of the trajectories.

Introduction and Summary. The motion of a point M in the xy plane can be stated as a time-varying vector drawn from the origin to its tip at point M. The motion is described by two equations which specify the Cartesian coordinates of M as functions of time. The path traversed by the tip of the vector, called its trajectory, may be found by eliminating the variable t from those equations. Some parameters of the motion (such as the amplitude, phase, and frequency) may be deduced from the trajectory In this project you will generate and examine several classes of trajectories where the x and y coordinate values vary sinusoidally with time. The project contains six sections.

i) $x(t)$ and $y(t)$ have the same frequency.
ii) $x(t)$ and $y(t)$ have slightly different frequencies.
iii) The frequencies of $x(t)$ and $y(t)$ are harmonics of a principle frequency.
iv) The frequencies of $x(t)$ and $y(t)$ are not harmonics of a principle frequency.
v) The effect of the sampling rate.
vi) Implementation through an electric circuit.

The present project may be carried out by mathematical simulation or in real time by physical oscillators. Examples from both are included.

Section I. $x(t)$ and $y(t)$ have the same frequency. Consider the time-varying vector $\mathbf{E}$ in the xy plane drawn from the origin to point M whose projections on the x and y axes are given by

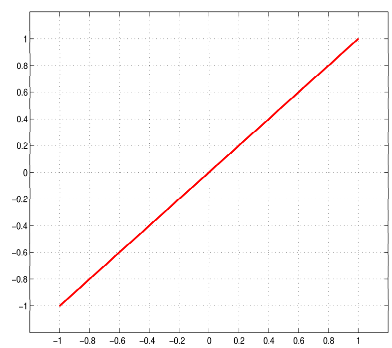
$$\begin{cases} x = E_x \cos\ \omega t \\ y = E_y \cos(\omega t + \theta) \end{cases}$$

Depending on the phase difference and amplitude ratio, three types of trajectories are observed: linear (zero phase or π), circular (equal amplitudes with $\pm\pi/2$ phase) , and elliptical (all other values).

a) Linear Trajectory. For $\theta = 0$ or π, the trajectory is a straight line. See Fig. 14-a. Show that $y = \pm\frac{E_y}{E_x}x$. Rotate the xy coordinate system by an angle γ in the counter-clockwise direction to have a new coordinate system $\alpha\beta$. Show the relationship between the two coordinate systems is

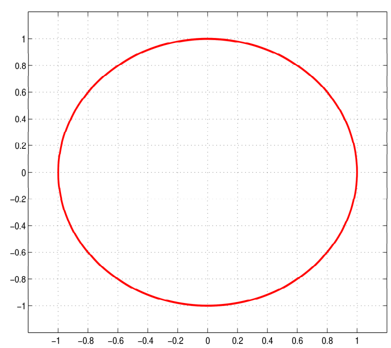
$$\alpha = x\cos\gamma + y\sin\gamma$$
$$\beta = -x\sin\gamma + y\cos\gamma$$

Find the equation of the trajectory in the new coordinate system. Determine the appropriate γ value to represent $\mathbf{E}$ by a one-dimensional vector which oscillates in time along the α-axis only.



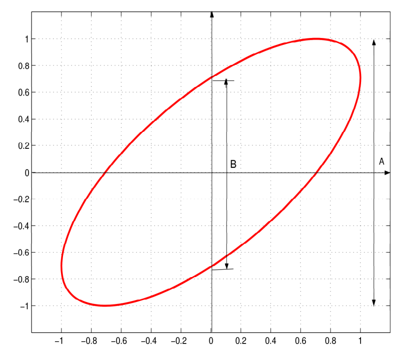
(a) Linear trajectory

$$\begin{cases} x = E_x \cos\ \omega t \\ y = \pm E_y \cos\omega t \end{cases}$$



(b) Circular trajectory

$$\begin{cases} x = E_x \cos\ \omega t \\ y = \pm E_y \sin\omega t \end{cases}$$



(c) Elliptical trajectory

$$\begin{cases} x = E_x \cos\ \omega t \\ y = \pm E_y \cos(\omega t + \pi/4) \end{cases}$$

Fig. 14. Trajectories of the motion of the tip of a rotating vector in the xy plane. The trajectories have the form of simple Lissajous patterns.

b) Circular Trajectory. For $E_y = E_x = E$ and $\theta = \pi/2$ or $-\pi/2$, the motion has a circular trajectory. See Fig. 14-b. Show that $x^2 + y^2 = E^2$ and determine the direction of motion for the trajectory.

c) Elliptical Trajectory. For the general case, the tip of the vector moves along an elliptical trajectory. Show that

$$\left(\frac{x}{E_x}\right)^2 + \left(\frac{y}{E_y}\right)^2 - \frac{2xy}{E_x E_y}\cos\theta = \sin^2\theta$$

and determine the direction of motion for the trajectory. Show that the phase angle θ can be found from

$$\sin\theta = \pm(B/A)$$

where A and B are as shown on Fig. 14-c. Rotate the xy coordinate system by an appropriate angle to align the x-axis along the major axis of the ellipse. Determine the rotation angle. Write the equation for the trajectory in the new coordinate system.

Simulation by Computer. Run the Matlab code given below to generate a linear trajectory.

```
f=1; w=2*pi*f; T=1/f; N=1; a=1; b=1; theta=0;                             % Motion parameters.
t=linspace(0,N*T,100*N*T); x=a*cos(w*t); y=b*cos(w*t+theta); plot(x,y) % Trajectory.
```

Change the parameters of the motion and the variables in the above code to explore how each one may or may not change the shape of the trajectory. From the plots of the trajectories obtain the amplitudes of the horizontal and vertical motions and the phase difference between them. Compare with the values used in the simulation.

Parallels with Electromagnetic Wave Polarization. The electric field vector **E** in an electromagnetic plane wave is a time-varying vector **E** which lies in the plane that is perpendicular to the direction of propagation. It has two components; namely, in the x and y directions. In the case of sinusoidal time variation, the electric field is an **E** vector with x and y components (each of which vary sinusoidally with time) as follows:

$$\begin{cases} x = E_x \cos\ \omega t \\ y = E_y \cos(\omega t + \theta) \end{cases}$$

This is the same vector we discussed at the beginning of this section with three possible tip trajectories. Each trajectory is associated with one type of wave polarization. The electromagnetic wave, therefore, is said to be polarized as any of the above types. In addition, the motion of the tip of the electric field vector can be in the clockwise or counter-clockwise directions, labeled left or right, circular or elliptical.

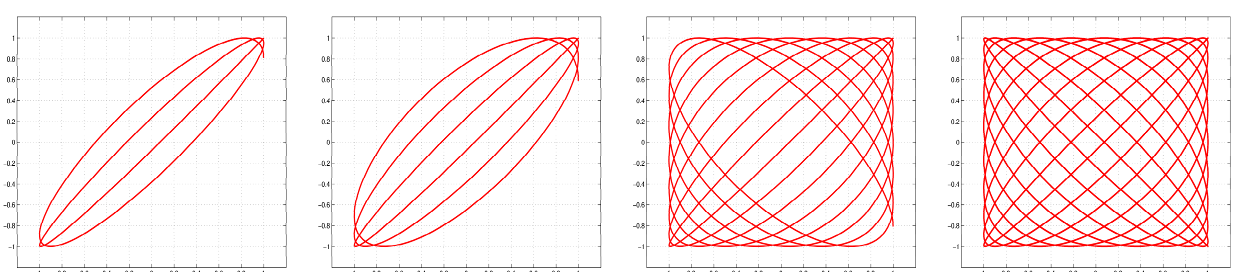
Section II. $x(t)$ and $y(t)$ have slightly different frequencies.

$$\begin{cases} x(t) = a \cos \omega_1 t \\ y(t) = b \cos \omega_2 t \end{cases}$$

With ω_1 and ω_2 approximately the same (but not exactly equal), the difference in frequencies will appear as a time-varying phase difference, resulting in a trajectory which slowly moves between the above three patterns. Construct an example where

$$\begin{cases} x(t) = \cos 2\pi t \\ y(t) = b \cos 2.1\pi t \end{cases}$$

(corresponding to $f_1 = 1$ Hz and $f_2 = 1.05$ Hz, respectively). Modify the Matlab program given in Section I to plot the Lissajous patterns for this example, allowing the plots to be generated for a) 2 seconds, b) 3 seconds, c) 8 seconds, and d) 150 seconds. Determine the time needed for a full cycle in each plot.

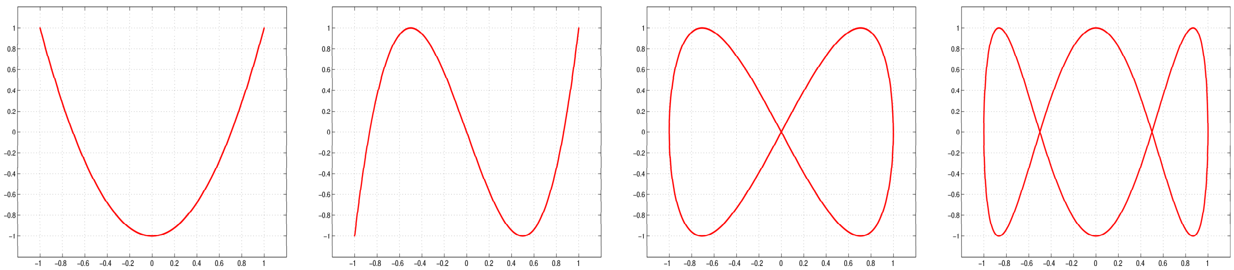


(a) 2 seconds (b) 3 seconds (c) 8 seconds (d) 150 seconds

Four Lissajous patterns, all with $x = \sin \omega t$. How long does a full cycle take?

Explore the effect for $f_2 = (1 + k)f_1$, with $k = 0.2$,0.1, and 0.01.

Section III. The frequencies of $x(t)$ and $y(t)$ are harmonics of a principle frequency. In this case, more complex patterns are generated whose shapes and parameters are associated with the ratio of the two frequencies. Four examples are shown in Fig. 15 for $x = \cos \omega t$ and four different variations of $y(t)$.



(a) $y = \cos 2\omega t$ (b) $y = \cos 3\omega t$ (c) $y = \sin 2\omega t$ (d) $y = \sin 3\omega t$

Fig. 15. Four Lissajous patterns with $x = \cos \omega t$ and various $y(t)$.

Using Matlab, plot trajectories of y versus x with $x = cos(at)$ and $y = cos(bt)$, for $a/b = 3$ and $3/5$. Repeat for $x = cos(at)$ and $y = sin(bt)$. In each case eliminate t between y and x to obtain the equation relating them together and verify its representation by a plot obtained through Matlab. Determine the number of crossings of a horizontal line and a vertical line and relate them to the ratio a/b. Suggest a method to measure the frequency of a sinusoidal signal from Lissajous patterns.

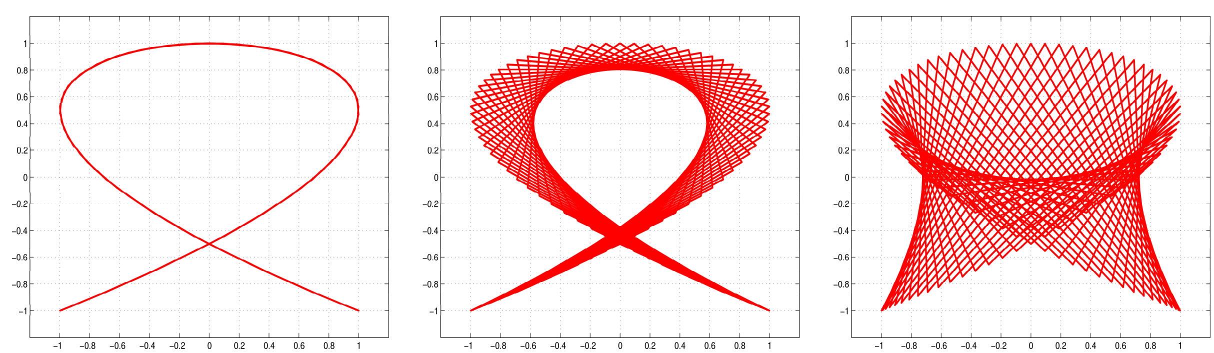
Section IV. The frequencies of $x(t)$ and $y(t)$ are not harmonics of a principle frequency. In theory, the motion is not periodic and it takes $T = \infty$ for it to repeat. In practice (e.g., in simulation by Matlab or in a real-time experiment using two physical oscillators), the Lissajous pattern will eventually repeat itself. Run the following Matlab code, observe the trend, and compare with the observations in Section II. Repeat the procedure after replacing $\cos \pi t$ by $\cos 3.1t$.

```
N=4; %Repeat for N=2, 8, 20, 50
t=linspace(0,N*pi,800);
x=cos(t);  y=cos(pi*t); plot(x,y);
```

Section V. Effect of low sampling rate. In plotting a trajectory, the x and y coordinates need to be sampled sufficiently fast. Otherwise the plot will exhibit the artifacts caused by a low number of samples. This effect is shown by the following two examples.

a) Generate three examples of Lissajous patterns derived from the same motion but with three different sampling rates. You may use the following code for this purpose.

```
N=2; % Repeat for N=20 and 50.
t=linspace(0,N*pi,100);
x= cos(2*t); y=sin(3*t); plot(y,x)
```



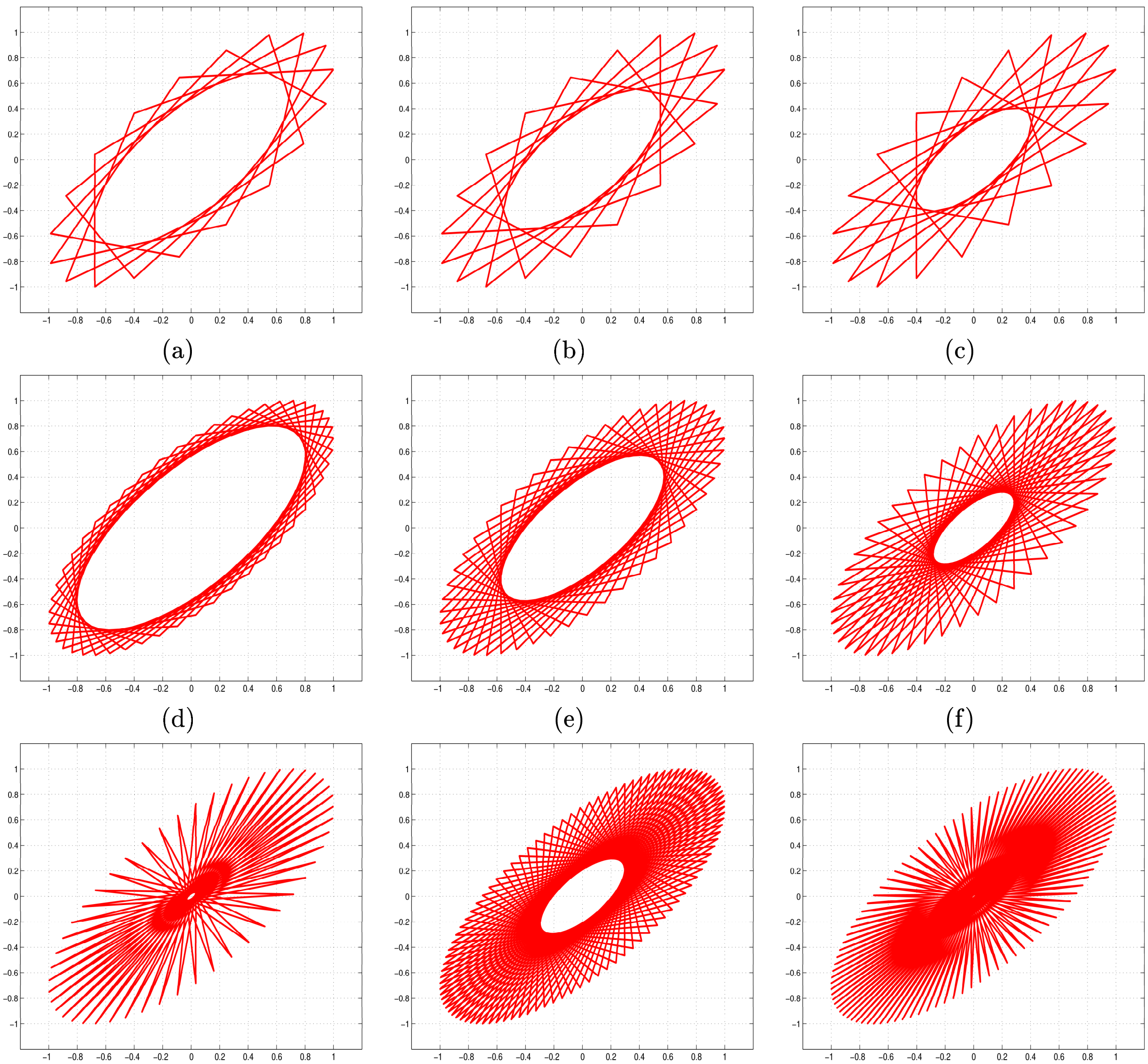
(a) $N = 2$ (b) $N = 20$ (c) $N = 50$

b) For the following set of samples (N) and time (T)

$$\begin{array}{lll} i) & N = 20, & T = 100,\ 50,\ 25 \\ ii) & N = 50, & T = 10,\ 20,\ 25 \\ iii) & N = 100, & T = 40,\ 50 \\ iv) & N = 200, & T = 2 \end{array}$$

run the Matlab code given below and explain trends in the plots.

```
t=linspace(0,T,N);
x=cos(2*pi*t); y=cos(2*pi*t+pi/4); plot(x,y)
```



(a) (b) (c) (d) (e) (f) (g) (h) (i)

Section VI. Implementation through an Electric Circuit. An oscilloscope whose horizontal and vertical deflections are controlled by the signals $x(t)$ and $y(t)$, respectively, eliminates t between them and displays the xy motion trajectory. For this purpose, an ordinary oscilloscope can be used if set in the X deflection mode.

a) Start with the situation described in Section I of this project. Configure the circuit of Fig. 16-a with $R = 1$ kΩ and $C = 1$ μF. Set up the function generator to provide a 2-Volt peak-to-peak sinusoidal voltage signal v_1 at 1590 Hz and connect it to the RC circuit as shown, and also to the horizontal axis of the scope and set it in X deflection mode. Connect the capacitor voltage v_2 to the vertical axis of the scope. The relationship between v_1 and v_2 can be readily found using terminal characteristics of R and C, and Kirchhoff's current and voltage laws. The result is

$$v_1(t) = V_1 \cos \omega t, \quad v_2(t) = V_2 \cos(\omega t + \theta), \quad V_2 = \frac{C^2\omega^2}{\sqrt{1 + R^2C^2\omega^2}} V_1, \quad \text{and } \theta = -\tan^{-1}(RC\omega)$$

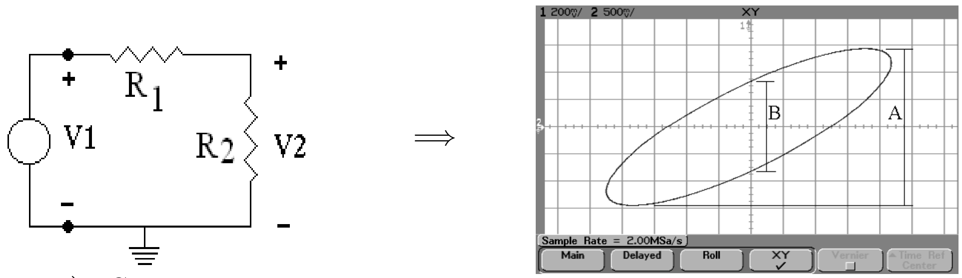
Show that at the above settings,

$$v_1(t) = \cos 10000t \quad \text{and} \quad v_2(t) = 0.707 \cos\left(10000t + \frac{\pi}{4}\right)$$

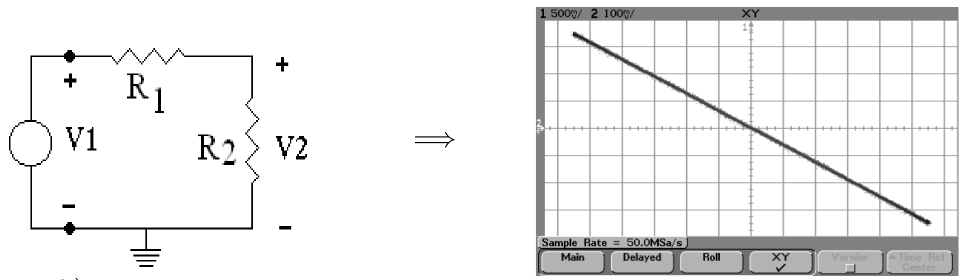
The elliptic pattern shown in Fig. 16-a should appear on the scope. Find the equation of the trajectory on the screen as predicted from theory. Compute the phase angle between v_1 and v_2 from

$$\sin\theta = \pm(B/A)$$

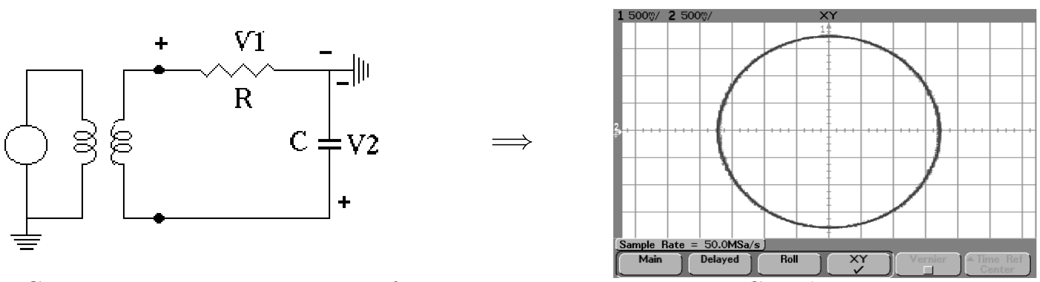
where A and B are shown on the ellipse. The phase angle is expected to be 45^o. The amplitudes of the horizontal and vertical signals may be changed through the horizontal and vertical gains of the scope. The phase angle between the horizontal and vertical signals can be changed by changing the frequency of the signal generator. Explore the effect of the above parameters on the shape of the trajectory.



a) RC circuit. Elliptic trajectory



b) RR circuit Linear trajectory



c) RC circuit and an isolation transformer Circular trajectory

Fig. 16 Three circuits for generating x and y coordinate signals and the resulting trajectories captured off an oscilloscope.

To obtain a linear trajectory, replace the capacitor in the circuit by a resistor. The circuit and resulting trajectory are shown in Fig. 16-b. Relate the slope of the trajectory to the resistor values and the horizontal and vertical gains of the oscilloscope.

To obtain a circular trajectory use the circuit configuration of Fig. 16-c which contains an isolation transformer.

b) Continue with real-time implementations of the situations described in sections II, III, IV, and V. For that purpose you will employ two sinusoidal signal generators directly connected to the horizontal and vertical channels.

Conclusions. Describe your overall conclusions from the project. What applications may it have? Where can one draw the line which distinguishes the results obtained by the digital approach (using the computer) from the analog approach (using the funtion generator and oscilloscope)?