

Ch. 2 Solutions

2.1 Let

$$S_x \doteq \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and write the S_x eigenvalue equations in matrix notation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = +\frac{\hbar}{2} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -\frac{\hbar}{2} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

which yields

$$\begin{aligned} a + b &= +\frac{\hbar}{2} & c + d &= +\frac{\hbar}{2} \\ a - b &= -\frac{\hbar}{2} & c - d &= +\frac{\hbar}{2} \end{aligned}$$

Solve by adding and subtracting the equations to get

$$a = 0 \quad b = \frac{\hbar}{2} \quad c = \frac{\hbar}{2} \quad d = 0$$

Hence the matrix representing S_x in the S_z basis is

$$S_x \doteq \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Let

$$S_y \doteq \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and write the S_y eigenvalue equations in matrix notation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = +\frac{\hbar}{2} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} = -\frac{\hbar}{2} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

which yields

$$\begin{aligned} a + ib &= +\frac{\hbar}{2} & c + id &= +i\frac{\hbar}{2} \\ a - ib &= -\frac{\hbar}{2} & c - id &= +i\frac{\hbar}{2} \end{aligned}$$

Solve by adding and subtracting the equations to get

$$a = 0 \quad b = -i\frac{\hbar}{2} \quad c = i\frac{\hbar}{2} \quad d = 0$$

Hence the matrix representing S_y in the S_z basis is

$$S_y \doteq \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

2.2 Solve the secular equation

$$\det[S_x - \lambda I] = 0$$

$$\begin{vmatrix} -\lambda & \frac{\hbar}{2} \\ \frac{\hbar}{2} & -\lambda \end{vmatrix} = 0$$

Solve to find the eigenvalues

$$\lambda^2 - \left(\frac{\hbar}{2}\right)^2 = 0$$

$$\lambda^2 = \left(\frac{\hbar}{2}\right)^2$$

$$\lambda = \pm \frac{\hbar}{2}$$

which was to be expected, because we know that the only possible results of a measurement of any spin component are $\pm\hbar/2$. Find the eigenvectors. For the positive eigenvalue:

$$\frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = +\frac{\hbar}{2} \begin{pmatrix} a \\ b \end{pmatrix}$$

yields

$$b = a$$

The normalization condition yields

$$|a|^2 + |a|^2 = 1$$

$$|a|^2 = \frac{1}{2}$$

Choose a to be real and positive, resulting in

$$a = \frac{1}{\sqrt{2}}$$

$$b = \frac{1}{\sqrt{2}}$$

so the eigenvector corresponding to the positive eigenvalue is

$$|+\rangle_x \doteq \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Likewise, the eigenvector for the negative eigenvalue is

$$\frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -\frac{\hbar}{2} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$b = -a$$

The normalization condition yields

$$|a|^2 + |a|^2 = 1$$

$$|a|^2 = \frac{1}{2}$$

Choose a to be real and positive, resulting in

$$a = \frac{1}{\sqrt{2}}$$

$$b = -\frac{1}{\sqrt{2}}$$

so the eigenvector corresponding to the negative eigenvalue is

$$|-\rangle_x \doteq \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

2.3 From Eq. (1.37), we know the S_z eigenstates in the S_x basis:

$$|+\rangle = \frac{1}{\sqrt{2}}(|+\rangle_x + |-\rangle_x)$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|+\rangle_x - |-\rangle_x)$$

Let the representation of S_z in the S_x basis be

$$S_z \doteq \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and write the S_z eigenvalue equations in matrix notation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = +\frac{\hbar}{2} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -\frac{\hbar}{2} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

These yield

$$\begin{aligned} a+b &= +\frac{\hbar}{2} & c+d &= +\frac{\hbar}{2} \\ a-b &= -\frac{\hbar}{2} & c-d &= +\frac{\hbar}{2} \end{aligned}$$

Solve by adding and subtracting the equations to get

$$a=0 \quad b=\frac{\hbar}{2} \quad c=\frac{\hbar}{2} \quad d=0$$

Hence the matrix representing S_z in the S_x basis is

$$S_z \doteq \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Now diagonalize:

$$\begin{vmatrix} -\lambda & \frac{\hbar}{2} \\ \frac{\hbar}{2} & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - \left(\frac{\hbar}{2}\right)^2 = 0 \Rightarrow \lambda = \pm \frac{\hbar}{2}$$

as expected. Find the eigenvectors:

$$\frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = +\frac{\hbar}{2} \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow b=a$$

yielding

$$|+\rangle \doteq \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Likewise

$$\frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -\frac{\hbar}{2} \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow b=-a$$

$$|-\rangle \doteq \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Hence the eigenvalue equations are

$$\frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = +\frac{\hbar}{2} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow S_z|+\rangle = +\frac{\hbar}{2}|+\rangle \quad OK$$

$$\frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -\frac{\hbar}{2} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow S_z|-\rangle = -\frac{\hbar}{2}|-\rangle \quad OK$$

2.4 The general matrix is

$$A \doteq \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The matrix elements are

$$\langle +|A|+\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix} = a$$

$$\langle +|A|-\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} b \\ d \end{pmatrix} = b$$

$$\langle -|A|+\rangle = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix} = c$$

$$\langle -|A|-\rangle = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} b \\ d \end{pmatrix} = d$$

Hence we get

$$A \doteq \begin{pmatrix} \langle +|A|+\rangle & \langle +|A|-\rangle \\ \langle -|A|+\rangle & \langle -|A|-\rangle \end{pmatrix}$$

2.5 The commutators are

$$\begin{aligned} [S_x, S_y] &\doteq \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &\doteq \left(\frac{\hbar}{2}\right)^2 \left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \right] \\ &\doteq \left(\frac{\hbar}{2}\right)^2 \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} \doteq i\hbar \left(\frac{\hbar}{2}\right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= i\hbar S_z \end{aligned}$$

$$\begin{aligned} [S_y, S_z] &\doteq \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ &\doteq \left(\frac{\hbar}{2}\right)^2 \left[\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \right] \\ &\doteq \left(\frac{\hbar}{2}\right)^2 \begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} \doteq i\hbar \left(\frac{\hbar}{2}\right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= i\hbar S_x \end{aligned}$$

$$\begin{aligned}
[S_z, S_x] &\doteq \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&\doteq \left(\frac{\hbar}{2}\right)^2 \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right] \\
&\doteq \left(\frac{\hbar}{2}\right)^2 \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \doteq i\hbar \left(\frac{\hbar}{2}\right) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
&= i\hbar S_y
\end{aligned}$$

2.6 The spin component operator S_n is

$$\begin{aligned}
S_n &= \mathbf{S} \cdot \hat{\mathbf{n}} \\
&= S_x \sin \theta \cos \phi + S_y \sin \theta \sin \phi + S_z \cos \theta
\end{aligned}$$

Using the matrix representations for S_x , S_y , and S_z gives

$$\begin{aligned}
S_n &\doteq \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sin \theta \cos \phi + \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \sin \theta \sin \phi + \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cos \theta \\
&\doteq \frac{\hbar}{2} \begin{pmatrix} \cos \theta & \sin \theta \cos \phi - i \sin \theta \sin \phi \\ \sin \theta \cos \phi + i \sin \theta \sin \phi & -\cos \theta \end{pmatrix} \\
&\doteq \frac{\hbar}{2} \begin{pmatrix} \cos \theta & \sin \theta e^{-i\phi} \\ \sin \theta e^{i\phi} & -\cos \theta \end{pmatrix}
\end{aligned}$$

2.7 Diagonalize S_n :

$$S_n \doteq \frac{\hbar}{2} \begin{pmatrix} \cos \theta & \sin \theta e^{-i\phi} \\ \sin \theta e^{i\phi} & -\cos \theta \end{pmatrix}$$

Now diagonalize:

$$\begin{vmatrix} \frac{\hbar}{2} \cos \theta - \lambda & \frac{\hbar}{2} \sin \theta e^{-i\phi} \\ \frac{\hbar}{2} \sin \theta e^{i\phi} & -\frac{\hbar}{2} \cos \theta - \lambda \end{vmatrix} = 0$$

$$\lambda^2 - \left(\frac{\hbar}{2}\right)^2 \cos^2 \theta - \left(\frac{\hbar}{2}\right)^2 \sin^2 \theta = 0 \quad \Rightarrow \quad \lambda^2 - \left(\frac{\hbar}{2}\right)^2 = 0 \quad \Rightarrow \quad \lambda = \pm \frac{\hbar}{2}$$

as expected. Find the eigenvectors:

$$\frac{\hbar}{2} \begin{pmatrix} \cos \theta & \sin \theta e^{-i\phi} \\ \sin \theta e^{i\phi} & -\cos \theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = +\frac{\hbar}{2} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$a \cos \theta + b \sin \theta e^{-i\phi} = a \Rightarrow b = a e^{i\phi} \frac{1 - \cos \theta}{\sin \theta}$$

The normalization condition yields

$$|a|^2 + |a|^2 \left(\frac{1 - \cos \theta}{\sin \theta} \right)^2 = 1$$

$$|a|^2 = \frac{\sin^2 \theta}{2 - 2 \cos \theta} = \frac{4 \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2}}{4 \sin^2 \frac{\theta}{2}} = \cos^2 \frac{\theta}{2}$$

yielding

$$|+\rangle_n = \cos \frac{\theta}{2} |+\rangle + e^{i\phi} \sin \frac{\theta}{2} |-\rangle$$

Likewise

$$\frac{\hbar}{2} \begin{pmatrix} \cos \theta & \sin \theta e^{-i\phi} \\ \sin \theta e^{i\phi} & -\cos \theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -\frac{\hbar}{2} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$a \cos \theta + b \sin \theta e^{-i\phi} = -a \Rightarrow b = -a e^{i\phi} \frac{1 + \cos \theta}{\sin \theta}$$

The normalization condition yields

$$|a|^2 + |a|^2 \left(\frac{1 + \cos \theta}{\sin \theta} \right)^2 = 1$$

$$|a|^2 = \frac{\sin^2 \theta}{2 + 2 \cos \theta} = \frac{4 \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2}}{4 \cos^2 \frac{\theta}{2}} = \sin^2 \frac{\theta}{2}$$

yielding

$$|-\rangle_n = \sin \frac{\theta}{2} |+\rangle - e^{i\phi} \cos \frac{\theta}{2} |-\rangle$$

2.8 The $|+\rangle_n$ eigenstate is

$$|+\rangle_n = \cos \frac{\theta}{2} |+\rangle + e^{i\phi} \sin \frac{\theta}{2} |-\rangle = \cos \frac{\pi}{8} |+\rangle + e^{i5\pi/3} \sin \frac{\pi}{8} |-\rangle$$

The probabilities are

$$\begin{aligned}
\mathcal{P}_{+y} &= \left| {}_y\langle +|+\rangle_n \right|^2 = \left| \left(\frac{1}{\sqrt{2}}\langle +| - \frac{i}{\sqrt{2}}\langle -| \right) \left(\cos \frac{\pi}{8}|+\rangle + e^{i5\pi/3} \sin \frac{\pi}{8}|-\rangle \right) \right|^2 \\
&= \frac{1}{2} \left| \cos \frac{\pi}{8} - ie^{i5\pi/3} \sin \frac{\pi}{8} \right|^2 = \frac{1}{2} \left| \cos \frac{\pi}{8} + \sin \frac{\pi}{8} \sin \frac{5\pi}{3} - i \sin \frac{\pi}{8} \cos \frac{5\pi}{3} \right|^2 \\
&= \frac{1}{2} \left(\cos^2 \frac{\pi}{8} + \sin^2 \frac{\pi}{8} \sin^2 \frac{5\pi}{3} + \sin^2 \frac{\pi}{8} \cos^2 \frac{5\pi}{3} + 2 \cos \frac{\pi}{8} \sin \frac{\pi}{8} \sin \frac{5\pi}{3} \right) \\
&= \frac{1}{2} \left(1 + 2 \cos \frac{\pi}{8} \sin \frac{\pi}{8} \sin \frac{5\pi}{3} \right) \cong 0.194 \\
\mathcal{P}_{-y} &= \left| {}_y\langle -|+\rangle_n \right|^2 = \left| \left(\frac{1}{\sqrt{2}}\langle +| + \frac{i}{\sqrt{2}}\langle -| \right) \left(\cos \frac{\pi}{8}|+\rangle + e^{i5\pi/3} \sin \frac{\pi}{8}|-\rangle \right) \right|^2 \\
&= \frac{1}{2} \left| \cos \frac{\pi}{8} + ie^{i5\pi/3} \sin \frac{\pi}{8} \right|^2 = \frac{1}{2} \left| \cos \frac{\pi}{8} - \sin \frac{\pi}{8} \sin \frac{5\pi}{3} + i \sin \frac{\pi}{8} \cos \frac{5\pi}{3} \right|^2 \\
&= \frac{1}{2} \left(\cos^2 \frac{\pi}{8} + \sin^2 \frac{\pi}{8} \sin^2 \frac{5\pi}{3} + \sin^2 \frac{\pi}{8} \cos^2 \frac{5\pi}{3} - 2 \cos \frac{\pi}{8} \sin \frac{\pi}{8} \sin \frac{5\pi}{3} \right) \\
&= \frac{1}{2} \left(1 - 2 \cos \frac{\pi}{8} \sin \frac{\pi}{8} \sin \frac{5\pi}{3} \right) \cong 0.806
\end{aligned}$$

2.9 The expectation value of S_z is easy to do in Dirac notation:

$$\langle S_z \rangle = \langle + | S_z | + \rangle = \langle + | \frac{\hbar}{2} | + \rangle = \frac{\hbar}{2} \langle + | + \rangle = \frac{\hbar}{2}$$

The expectation values of S_x and S_y are easier in matrix notation:

$$\langle S_x \rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0\hbar$$

$$\langle S_y \rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ i \end{pmatrix} = 0\hbar$$

To find the uncertainties, we need the expectation values of the squares:

$$\langle S_z^2 \rangle = \langle + | S_z^2 | + \rangle = \langle + | \frac{\hbar}{2} \frac{\hbar}{2} | + \rangle = \left(\frac{\hbar}{2} \right)^2 \langle + | + \rangle = \left(\frac{\hbar}{2} \right)^2$$

$$\langle S_x^2 \rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left(\frac{\hbar}{2} \right)^2 \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left(\frac{\hbar}{2} \right)^2$$

$$\langle S_y^2 \rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left(\frac{\hbar}{2} \right)^2 \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left(\frac{\hbar}{2} \right)^2$$

The uncertainties are

$$\Delta S_z = \sqrt{\langle S_z^2 \rangle - \langle S_z \rangle^2} = \sqrt{\left(\frac{\hbar}{2}\right)^2 - \left(\frac{\hbar}{2}\right)^2} = 0\hbar$$

$$\Delta S_x = \sqrt{\langle S_x^2 \rangle - \langle S_x \rangle^2} = \sqrt{\left(\frac{\hbar}{2}\right)^2 - 0} = \left(\frac{\hbar}{2}\right)$$

$$\Delta S_y = \sqrt{\langle S_y^2 \rangle - \langle S_y \rangle^2} = \sqrt{\left(\frac{\hbar}{2}\right)^2 - 0} = \left(\frac{\hbar}{2}\right)$$

2.10 These expectation values are easier in matrix notation:

$$\langle S_x \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{\hbar}{4} \begin{pmatrix} 1 & -i \end{pmatrix} \begin{pmatrix} i \\ 1 \end{pmatrix} = 0\hbar$$

$$\langle S_y \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{\hbar}{4} \begin{pmatrix} 1 & -i \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{\hbar}{2}$$

$$\langle S_z \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{\hbar}{4} \begin{pmatrix} 1 & -i \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = 0\hbar$$

To find the uncertainties, we need the expectation values of the squares:

$$\langle S_x^2 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2} \left(\frac{\hbar}{2}\right)^2 \begin{pmatrix} 1 & -i \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \left(\frac{\hbar}{2}\right)^2$$

$$\langle S_y^2 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2} \left(\frac{\hbar}{2}\right)^2 \begin{pmatrix} 1 & -i \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \left(\frac{\hbar}{2}\right)^2$$

$$\langle S_z^2 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2} \left(\frac{\hbar}{2}\right)^2 \begin{pmatrix} 1 & -i \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \left(\frac{\hbar}{2}\right)^2$$

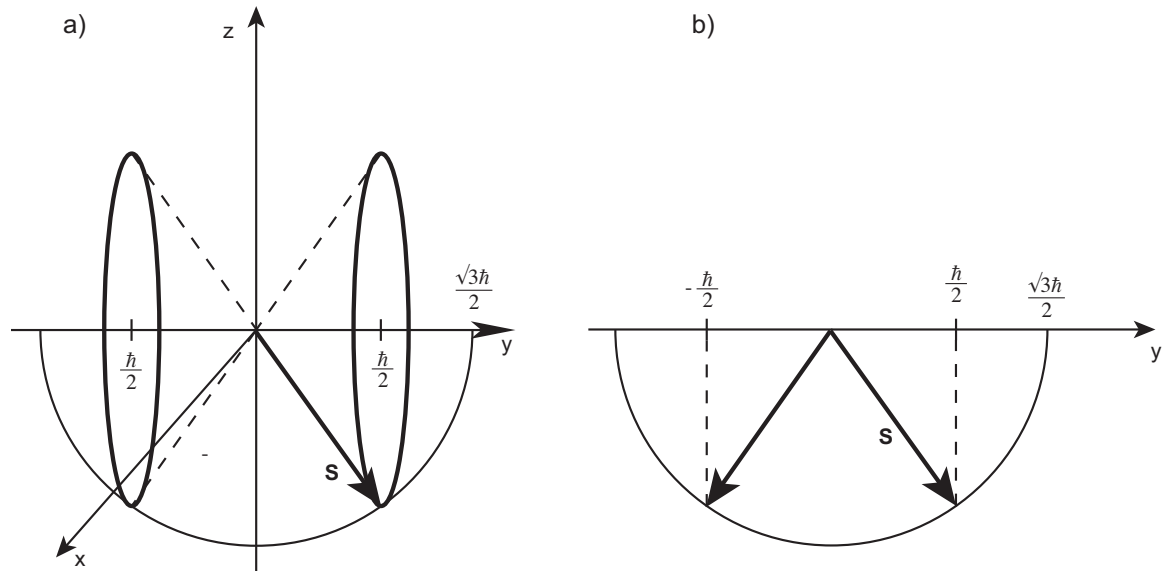
The uncertainties are

$$\Delta S_x = \sqrt{\langle S_x^2 \rangle - \langle S_x \rangle^2} = \sqrt{\left(\frac{\hbar}{2}\right)^2 - 0} = \left(\frac{\hbar}{2}\right)$$

$$\Delta S_y = \sqrt{\langle S_y^2 \rangle - \langle S_y \rangle^2} = \sqrt{\left(\frac{\hbar}{2}\right)^2 - \left(\frac{\hbar}{2}\right)^2} = 0\hbar$$

$$\Delta S_z = \sqrt{\langle S_z^2 \rangle - \langle S_z \rangle^2} = \sqrt{\left(\frac{\hbar}{2}\right)^2 - 0} = \left(\frac{\hbar}{2}\right)$$

In the vector model, shown below, the spin is precessing around the y-axis at a constant angle such the y-component of the spin is constant and x- and z-components oscillate about zero.



2.11 The commutators in matrix notation are

$$\begin{aligned}
 [\mathbf{S}^2, S_x] &\doteq \frac{3\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{3\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
 &\doteq \frac{3\hbar^3}{8} \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] \\
 &= 0 \\
 [\mathbf{S}^2, S_y] &\doteq \frac{3\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{3\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
 &\doteq \frac{3\hbar^3}{8} \left[\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right] \\
 &= 0 \\
 [\mathbf{S}^2, S_z] &\doteq \frac{3\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{3\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
 &\doteq \frac{3\hbar^3}{8} \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] \\
 &= 0
 \end{aligned}$$

In abstract notation, the commutators are

$$\begin{aligned}
[\mathbf{S}^2, S_x] &= [S_x^2 + S_y^2 + S_z^2, S_x] = [S_x^2, S_x] + [S_y^2, S_x] + [S_z^2, S_x] \\
&= S_x^2 S_x - S_x S_x^2 + S_y^2 S_x - S_x S_y^2 + S_z^2 S_x - S_x S_z^2 \\
&= S_x^3 - S_x^3 + S_y S_y S_x - S_x S_y S_y + S_z S_z S_x - S_x S_z S_z \\
&= 0 + S_y (S_x S_y - i\hbar S_z) - (S_y S_x + i\hbar S_z) S_y + S_z (S_x S_z + i\hbar S_y) - (S_z S_x - i\hbar S_y) S_z \\
&= S_y S_x S_y - i\hbar S_y S_z - S_y S_x S_y - i\hbar S_z S_y + S_z S_x S_z + i\hbar S_z S_y - S_z S_x S_z + i\hbar S_y S_z \\
&= 0 \\
[\mathbf{S}^2, S_y] &= [S_x^2 + S_y^2 + S_z^2, S_y] = [S_x^2, S_y] + [S_y^2, S_y] + [S_z^2, S_y] \\
&= S_x^2 S_y - S_y S_x^2 + S_y^2 S_y - S_y S_y^2 + S_z^2 S_y - S_y S_z^2 \\
&= S_x S_x S_y - S_y S_x S_x + S_y^3 - S_y^3 + S_z S_z S_y - S_y S_z S_z \\
&= S_x (S_y S_x + i\hbar S_z) - (S_x S_y - i\hbar S_z) S_x + 0 + S_z (S_y S_z - i\hbar S_x) - (S_z S_y + i\hbar S_x) S_z \\
&= S_x S_y S_x + i\hbar S_x S_z - S_x S_y S_x + i\hbar S_z S_x + S_z S_y S_z - i\hbar S_z S_x - S_z S_y S_z - i\hbar S_x S_z \\
&= 0 \\
[\mathbf{S}^2, S_z] &= [S_x^2 + S_y^2 + S_z^2, S_z] = [S_x^2, S_z] + [S_y^2, S_z] + [S_z^2, S_z] \\
&= S_x^2 S_z - S_z S_x^2 + S_y^2 S_z - S_z S_y^2 + S_z^2 S_z - S_z S_z^2 \\
&= S_x S_x S_z - S_z S_x S_x + S_y S_y S_z - S_z S_y S_y + S_z^3 - S_z^3 \\
&= S_x (S_z S_x - i\hbar S_y) - (S_x S_z + i\hbar S_y) S_x + S_y (S_z S_y + i\hbar S_x) - (S_y S_z - i\hbar S_x) S_y + 0 \\
&= S_x S_z S_x - i\hbar S_x S_y - S_x S_z S_x - i\hbar S_y S_x + S_y S_z S_y + i\hbar S_y S_x - S_y S_z S_y + i\hbar S_x S_y \\
&= 0
\end{aligned}$$

2.12 For S_x the diagonalization yields the eigenvalues

$$\begin{aligned}
S_x &\doteq \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
\begin{pmatrix} -\lambda & \frac{\hbar}{\sqrt{2}} & 0 \\ \frac{\hbar}{\sqrt{2}} & -\lambda & \frac{\hbar}{\sqrt{2}} \\ 0 & \frac{\hbar}{\sqrt{2}} & -\lambda \end{pmatrix} &= 0 \Rightarrow -\lambda \left(\lambda^2 - \frac{\hbar^2}{2} \right) - \frac{\hbar}{\sqrt{2}} \left(-\lambda \frac{\hbar}{\sqrt{2}} \right) = 0 \\
\lambda \left(\lambda^2 - \hbar^2 \right) &= 0 \Rightarrow \lambda = \hbar, 0, -\hbar
\end{aligned}$$

and the eigenvectors

$$\frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \hbar \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow \begin{aligned} b &= a\sqrt{2} \\ a+c &= b\sqrt{2} \\ b &= c\sqrt{2} \end{aligned}$$

$$|a|^2 + |b|^2 + |c|^2 = 1 \Rightarrow |b|^2 \left(\frac{1}{2} + 1 + \frac{1}{2}\right) = 1 \Rightarrow b = \frac{1}{\sqrt{2}}, a = \frac{1}{2}, c = \frac{1}{2}$$

$$|1\rangle_x = \frac{1}{2}|1\rangle + \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|-1\rangle$$

$$\frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \hbar \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow \begin{aligned} b &= 0 \\ a+c &= 0 \\ b &= 0 \end{aligned}$$

$$|a|^2 + |b|^2 + |c|^2 = 1 \Rightarrow |a|^2 (1+1) = 1 \Rightarrow a = \frac{1}{\sqrt{2}}, b = 0, c = -\frac{1}{\sqrt{2}}$$

$$|0\rangle_x = \frac{1}{\sqrt{2}}|1\rangle - \frac{1}{\sqrt{2}}|-1\rangle$$

$$\frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = -\hbar \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow \begin{aligned} b &= -a\sqrt{2} \\ a+c &= -b\sqrt{2} \\ b &= -c\sqrt{2} \end{aligned}$$

$$|a|^2 + |b|^2 + |c|^2 = 1 \Rightarrow |b|^2 \left(\frac{1}{2} + 1 + \frac{1}{2}\right) = 1 \Rightarrow b = -\frac{1}{\sqrt{2}}, a = \frac{1}{2}, c = \frac{1}{2}$$

$$|-1\rangle_x = \frac{1}{2}|1\rangle - \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|-1\rangle$$

For S_y the diagonalization yields

$$S_y \doteq \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\begin{pmatrix} -\lambda & \frac{-i\hbar}{\sqrt{2}} & 0 \\ \frac{i\hbar}{\sqrt{2}} & -\lambda & \frac{-i\hbar}{\sqrt{2}} \\ 0 & \frac{i\hbar}{\sqrt{2}} & -\lambda \end{pmatrix} = 0 \Rightarrow -\lambda \left(\lambda^2 - \frac{\hbar^2}{2}\right) - \frac{-i\hbar}{\sqrt{2}} \left(-\lambda \frac{i\hbar}{\sqrt{2}}\right) = 0$$

$$\lambda(\lambda^2 - \hbar^2) = 0 \Rightarrow \lambda = \hbar, 0, -\hbar$$

and the eigenvectors

$$\frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \hbar \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow \begin{aligned} -ib &= a\sqrt{2} \\ ia - ic &= b\sqrt{2} \\ ib &= c\sqrt{2} \end{aligned}$$

$$|a|^2 + |b|^2 + |c|^2 = 1 \Rightarrow |b|^2 \left(\frac{1}{2} + 1 + \frac{1}{2}\right) = 1 \Rightarrow b = \frac{i}{\sqrt{2}}, a = \frac{1}{2}, c = -\frac{1}{2}$$

$$|1\rangle_y = \frac{1}{2}|1\rangle + \frac{i}{\sqrt{2}}|0\rangle - \frac{1}{2}|-1\rangle$$

$$\frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \hbar \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow \begin{aligned} -ib &= 0 \\ ia - ic &= 0 \\ ib &= 0 \end{aligned}$$

$$|a|^2 + |b|^2 + |c|^2 = 1 \Rightarrow |a|^2(1+1) = 1 \Rightarrow a = \frac{1}{\sqrt{2}}, b = 0, c = \frac{1}{\sqrt{2}}$$

$$|0\rangle_y = \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|-1\rangle$$

$$\frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = -1 \hbar \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow \begin{aligned} -ib &= -a\sqrt{2} \\ ia - ic &= -b\sqrt{2} \\ ib &= -c\sqrt{2} \end{aligned}$$

$$|a|^2 + |b|^2 + |c|^2 = 1 \Rightarrow |b|^2\left(\frac{1}{2} + 1 + \frac{1}{2}\right) = 1 \Rightarrow b = -\frac{i}{\sqrt{2}}, a = \frac{1}{2}, c = -\frac{1}{2}$$

$$|-1\rangle_y = \frac{1}{2}|1\rangle - \frac{i}{\sqrt{2}}|0\rangle - \frac{1}{2}|-1\rangle$$

2.13 The commutators are

$$\begin{aligned} [S_x, S_y] &\doteq \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} - \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ &\doteq \frac{\hbar^2}{2} \left[\begin{pmatrix} i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & i \end{pmatrix} \right] \\ &\doteq \frac{\hbar^2}{2} \begin{pmatrix} 2i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2i \end{pmatrix} \doteq i\hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\ &= i\hbar S_z \end{aligned}$$

$$\begin{aligned} [S_y, S_z] &\doteq \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} - \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \\ &\doteq \frac{\hbar^2}{\sqrt{2}} \left[\begin{pmatrix} 0 & 0 & 0 \\ i & 0 & i \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i & 0 \\ 0 & 0 & 0 \\ 0 & -i & 0 \end{pmatrix} \right] \\ &\doteq \frac{\hbar^2}{\sqrt{2}} \begin{pmatrix} 0 & i & 0 \\ i & 0 & i \\ 0 & i & 0 \end{pmatrix} \doteq i\hbar \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ &= i\hbar S_x \end{aligned}$$

$$\begin{aligned}
[S_z, S_x] &\doteq \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} - \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\
&\doteq \frac{\hbar^2}{\sqrt{2}} \left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \right] \\
&\doteq \frac{\hbar^2}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \doteq i\hbar \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \\
&= i\hbar S_y
\end{aligned}$$

2.14 Using the component matrices we find

$$\begin{aligned}
\mathbf{S}^2 &= S_x^2 + S_y^2 + S_z^2 \\
&\doteq \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \\
&\quad + \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\
&\doteq \frac{\hbar^2}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{\hbar^2}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&\doteq 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

The eigenvalue equation is

$$\mathbf{S}^2 |sm\rangle = s(s+1)\hbar^2 |sm\rangle$$

For spin-1 this is

$$\mathbf{S}^2 |1m\rangle = 2\hbar^2 |1m\rangle$$

Hence the $\mathbf{S}^2$ operator must be $2\hbar^2$ times the identity matrix:

$$\mathbf{S}^2 \doteq 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2.15 a) The possible results of a measurement of the spin component S_z are always $+1\hbar, 0\hbar, -1\hbar$ for a spin-1 particle. The probabilities are

$$\begin{aligned}\mathcal{P}_1 &= \left| \langle 1 | \psi_{in} \rangle \right|^2 = \left| \langle 1 | \left[\frac{2}{\sqrt{29}} |1\rangle + \frac{3i}{\sqrt{29}} |0\rangle - \frac{4}{\sqrt{29}} |-1\rangle \right] \right|^2 \\ &= \left| \frac{2}{\sqrt{29}} \langle 1|1\rangle + \frac{3i}{\sqrt{29}} \langle 1|0\rangle - \frac{4}{\sqrt{29}} \langle 1|-1\rangle \right|^2 = \left| \frac{2}{\sqrt{29}} \right|^2 = \frac{4}{29} \\ \mathcal{P}_0 &= \left| \langle 0 | \psi_{in} \rangle \right|^2 = \left| \langle 0 | \left[\frac{2}{\sqrt{29}} |1\rangle + \frac{3i}{\sqrt{29}} |0\rangle - \frac{4}{\sqrt{29}} |-1\rangle \right] \right|^2 \\ &= \left| \frac{2}{\sqrt{29}} \langle 0|1\rangle + \frac{3i}{\sqrt{29}} \langle 0|0\rangle - \frac{4}{\sqrt{29}} \langle 0|-1\rangle \right|^2 = \left| \frac{3i}{\sqrt{29}} \right|^2 = \frac{9}{29} \\ \mathcal{P}_{-1} &= \left| \langle -1 | \psi_{in} \rangle \right|^2 = \left| \langle -1 | \left[\frac{2}{\sqrt{29}} |1\rangle + \frac{3i}{\sqrt{29}} |0\rangle - \frac{4}{\sqrt{29}} |-1\rangle \right] \right|^2 \\ &= \left| \frac{2}{\sqrt{29}} \langle -1|1\rangle + \frac{3i}{\sqrt{29}} \langle -1|0\rangle - \frac{4}{\sqrt{29}} \langle -1|-1\rangle \right|^2 = \left| -\frac{4}{\sqrt{29}} \right|^2 = \frac{16}{29}\end{aligned}$$

The three probabilities add to unity, as they must.

b) The possible results of a measurement of the spin component S_x are always $+1\hbar, 0\hbar, -1\hbar$ for a spin-1 particle. The probabilities are

$$\begin{aligned}\mathcal{P}_{1x} &= \left| {}_x\langle 1 | \psi_{in} \rangle \right|^2 = \left| \left(\frac{1}{2} \langle 1 | + \frac{1}{\sqrt{2}} \langle 0 | + \frac{1}{2} \langle -1 | \right) \left(\frac{2}{\sqrt{29}} |1\rangle + \frac{3i}{\sqrt{29}} |0\rangle - \frac{4}{\sqrt{29}} |-1\rangle \right) \right|^2 \\ &= \left| \frac{1}{\sqrt{29}} + \frac{3i}{\sqrt{2}\sqrt{29}} - \frac{2}{\sqrt{29}} \right|^2 = \frac{1}{58} \left| -\sqrt{2} + 3i \right|^2 = \frac{11}{58} \\ \mathcal{P}_{0x} &= \left| {}_x\langle 0 | \psi_{in} \rangle \right|^2 = \left| \left(\frac{1}{\sqrt{2}} \langle 1 | - \frac{1}{\sqrt{2}} \langle -1 | \right) \left(\frac{2}{\sqrt{29}} |1\rangle + \frac{3i}{\sqrt{29}} |0\rangle - \frac{4}{\sqrt{29}} |-1\rangle \right) \right|^2 \\ &= \left| \frac{2}{\sqrt{2}\sqrt{29}} + \frac{4}{\sqrt{2}\sqrt{29}} \right|^2 = \frac{36}{58} \\ \mathcal{P}_{-1x} &= \left| {}_x\langle -1 | \psi_{in} \rangle \right|^2 = \left| \left(\frac{1}{2} \langle 1 | - \frac{1}{\sqrt{2}} \langle 0 | + \frac{1}{2} \langle -1 | \right) \left(\frac{2}{\sqrt{29}} |1\rangle + \frac{3i}{\sqrt{29}} |0\rangle - \frac{4}{\sqrt{29}} |-1\rangle \right) \right|^2 \\ &= \left| \frac{1}{\sqrt{29}} - \frac{3i}{\sqrt{2}\sqrt{29}} - \frac{2}{\sqrt{29}} \right|^2 = \frac{1}{58} \left| -\sqrt{2} - 3i \right|^2 = \frac{11}{58}\end{aligned}$$

The three probabilities add to unity, as they must.

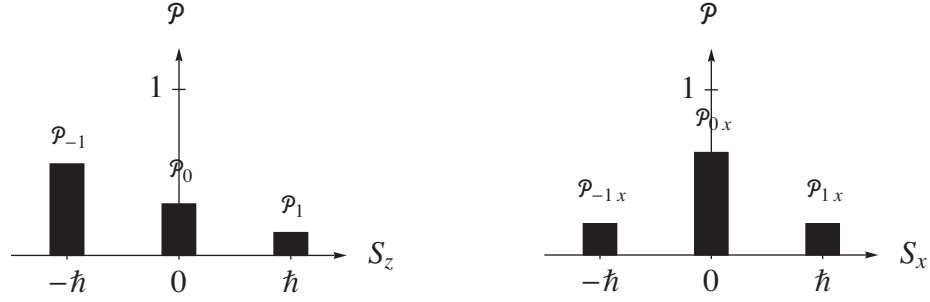
c) For the first measurement, the expectation value is

$$\langle S_z \rangle = \sum_m m\hbar \mathcal{P}_m = 1\hbar \frac{4}{29} + 0\hbar \frac{9}{29} + (-1)\hbar \frac{16}{29} = -\hbar \frac{12}{29}$$

For the second measurement, the expectation value is

$$\langle S_x \rangle = \sum_m m\hbar \mathcal{P}_{mx} = 1\hbar \frac{11}{58} + 0\hbar \frac{36}{58} + (-1)\hbar \frac{11}{58} = 0\hbar$$

The histograms are shown below.



2.16 a) The possible results of a measurement of the spin component S_z are always $+1\hbar, 0\hbar, -1\hbar$ for a spin-1 particle. The probabilities are

$$\begin{aligned}
 \mathcal{P}_1 &= |\langle 1 | \psi_{in} \rangle|^2 = \left| \langle 1 | \left[\frac{2}{\sqrt{29}} |1\rangle_y + \frac{3i}{\sqrt{29}} |0\rangle_y - \frac{4}{\sqrt{29}} |-1\rangle_y \right] \right|^2 \\
 &= \left| \frac{2}{\sqrt{29}} \langle 1 | 1 \rangle_y + \frac{3i}{\sqrt{29}} \langle 1 | 0 \rangle_y - \frac{4}{\sqrt{29}} \langle 1 | -1 \rangle_y \right|^2 \\
 &= \left| \frac{2}{\sqrt{29}} \frac{1}{2} + \frac{3i}{\sqrt{29}} \frac{1}{\sqrt{2}} - \frac{4}{\sqrt{29}} \frac{1}{2} \right|^2 = \frac{1}{58} |-\sqrt{2} + 3i|^2 = \frac{11}{58} \\
 \mathcal{P}_0 &= |\langle 0 | \psi_{in} \rangle|^2 = \left| \langle 0 | \left[\frac{2}{\sqrt{29}} |1\rangle_y + \frac{3i}{\sqrt{29}} |0\rangle_y - \frac{4}{\sqrt{29}} |-1\rangle_y \right] \right|^2 \\
 &= \left| \frac{2}{\sqrt{29}} \langle 0 | 1 \rangle_y + \frac{3i}{\sqrt{29}} \langle 0 | 0 \rangle_y - \frac{4}{\sqrt{29}} \langle 0 | -1 \rangle_y \right|^2 \\
 &= \left| \frac{2}{\sqrt{29}} \frac{i}{\sqrt{2}} + \frac{3i}{\sqrt{29}} 0 - \frac{4}{\sqrt{29}} \frac{-i}{\sqrt{2}} \right|^2 = \frac{1}{58} |i(2+4)|^2 = \frac{36}{58} \\
 \mathcal{P}_{-1} &= |\langle -1 | \psi_{in} \rangle|^2 = \left| \langle -1 | \left[\frac{2}{\sqrt{29}} |1\rangle_y + \frac{3i}{\sqrt{29}} |0\rangle_y - \frac{4}{\sqrt{29}} |-1\rangle_y \right] \right|^2 \\
 &= \left| \frac{2}{\sqrt{29}} \langle -1 | 1 \rangle_y + \frac{3i}{\sqrt{29}} \langle -1 | 0 \rangle_y - \frac{4}{\sqrt{29}} \langle -1 | -1 \rangle_y \right|^2 \\
 &= \left| \frac{2}{\sqrt{29}} \frac{-1}{2} + \frac{3i}{\sqrt{29}} \frac{1}{\sqrt{2}} - \frac{4}{\sqrt{29}} \frac{-1}{2} \right|^2 = \frac{1}{58} |\sqrt{2} + 3i|^2 = \frac{11}{58}
 \end{aligned}$$

The three probabilities add to unity, as they must.

b) The possible results of a measurement of the spin component S_y are always $+1\hbar, 0\hbar, -1\hbar$ for a spin-1 particle. The probabilities are

$$\begin{aligned}
 \mathcal{P}_{1y} &= \left| {}_y\langle 1 | \psi_{in} \rangle \right|^2 = \left| {}_y\langle 1 | \left[\frac{2}{\sqrt{29}} |1\rangle_y + \frac{3i}{\sqrt{29}} |0\rangle_y - \frac{4}{\sqrt{29}} |-1\rangle_y \right] \right|^2 \\
 &= \left| \frac{2}{\sqrt{29}} {}_y\langle 1 | 1 \rangle_y + \frac{3i}{\sqrt{29}} {}_y\langle 1 | 0 \rangle_y - \frac{4}{\sqrt{29}} {}_y\langle 1 | -1 \rangle_y \right|^2 = \left| \frac{2}{\sqrt{29}} \right|^2 = \frac{4}{29} \\
 \mathcal{P}_{0y} &= \left| {}_y\langle 0 | \psi_{in} \rangle \right|^2 = \left| {}_y\langle 0 | \left[\frac{2}{\sqrt{29}} |1\rangle_y + \frac{3i}{\sqrt{29}} |0\rangle_y - \frac{4}{\sqrt{29}} |-1\rangle_y \right] \right|^2 \\
 &= \left| \frac{2}{\sqrt{29}} {}_y\langle 0 | 1 \rangle_y + \frac{3i}{\sqrt{29}} {}_y\langle 0 | 0 \rangle_y - \frac{4}{\sqrt{29}} {}_y\langle 0 | -1 \rangle_y \right|^2 = \left| \frac{3i}{\sqrt{29}} \right|^2 = \frac{9}{29}
 \end{aligned}$$

$$\begin{aligned}\mathcal{P}_{-1y} &= \left| {}_y\langle -1 | \psi_{in} \rangle \right|^2 = \left| {}_y\langle -1 | \left[\frac{2}{\sqrt{29}} |1\rangle_y + \frac{3i}{\sqrt{29}} |0\rangle_y - \frac{4}{\sqrt{29}} |-1\rangle_y \right] \right|^2 \\ &= \left| \frac{2}{\sqrt{29}} {}_y\langle -1 | 1 \rangle + \frac{3i}{\sqrt{29}} {}_y\langle -1 | 0 \rangle - \frac{4}{\sqrt{29}} {}_y\langle -1 | -1 \rangle \right|^2 = \left| -\frac{4}{\sqrt{29}} \right|^2 = \frac{16}{29}\end{aligned}$$

The three probabilities add to unity, as they must.

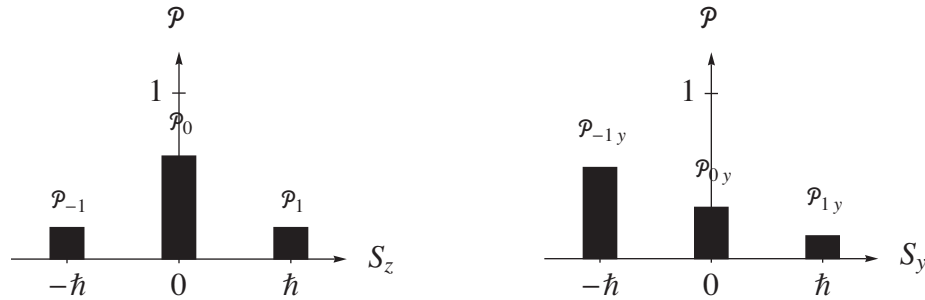
c) For the first measurement, the expectation value is

$$\langle S_z \rangle = \sum_m m \hbar \mathcal{P}_m = 1\hbar \frac{11}{58} + 0\hbar \frac{36}{58} + (-1)\hbar \frac{11}{58} = 0\hbar$$

For the second measurement, the expectation value is

$$\langle S_y \rangle = \sum_m m \hbar \mathcal{P}_{my} = 1\hbar \frac{4}{29} + 0\hbar \frac{9}{29} + (-1)\hbar \frac{16}{29} = -\hbar \frac{12}{29}$$

The histograms are shown below.



2.17 a) The possible results of a measurement of the spin component S_z are always $+1\hbar$, $0\hbar$, $-1\hbar$ for a spin-1 particle. The probabilities are

$$\mathcal{P}_h = |\langle 1 | \psi \rangle|^2 = \left| \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{30}} \begin{pmatrix} 1 \\ 2 \\ 5i \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{30}} 1 \right|^2 = \frac{1}{30}$$

$$\mathcal{P}_0 = |\langle 0 | \psi \rangle|^2 = \left| \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{30}} \begin{pmatrix} 1 \\ 2 \\ 5i \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{30}} 2 \right|^2 = \frac{4}{30}$$

$$\mathcal{P}_{-h} = |\langle -1 | \psi \rangle|^2 = \left| \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \frac{1}{\sqrt{30}} \begin{pmatrix} 1 \\ 2 \\ 5i \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{30}} 5i \right|^2 = \frac{25}{30}$$

The expectation value of S_z is

$$\langle S_z \rangle = \mathcal{P}_h \hbar + \mathcal{P}_0 0 + \mathcal{P}_{-h} (-\hbar) = \frac{1}{30} \hbar + \frac{4}{30} 0 + \frac{25}{30} (-\hbar) = -\frac{24}{30} \hbar = -\frac{4}{5} \hbar$$

b) The expectation value of S_x is

$$\begin{aligned}
\langle S_x \rangle &= \langle \psi | S_x | \psi \rangle = \frac{1}{\sqrt{30}} \begin{pmatrix} 1 & 2 & -5i \end{pmatrix} \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{30}} \begin{pmatrix} 1 \\ 2 \\ 5i \end{pmatrix} \\
&= \frac{1}{30} \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 1 & 2 & -5i \end{pmatrix} \begin{pmatrix} 2 \\ 1+5i \\ 2 \end{pmatrix} = \frac{1}{30} \frac{\hbar}{\sqrt{2}} (2 + 2(1+5i) - 5i \times 2) = \frac{\sqrt{2}}{15} \hbar
\end{aligned}$$

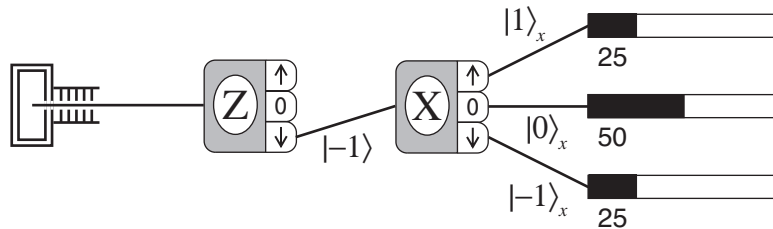
2.18 a) The possible results of a measurement of the spin component S_z are always $+1\hbar$, $0\hbar$, $-1\hbar$ for a spin-1 particle. The probabilities are

$$\begin{aligned}
\mathcal{P}_1 &= |\langle 1 | \psi_{in} \rangle|^2 = \left| \langle 1 | \left[\frac{1}{\sqrt{14}} |1\rangle - \frac{3}{\sqrt{14}} |0\rangle + \frac{2i}{\sqrt{14}} |-1\rangle \right] \right|^2 \\
&= \left| \frac{1}{\sqrt{14}} \langle 1|1\rangle - \frac{3}{\sqrt{14}} \langle 1|0\rangle + \frac{2i}{\sqrt{14}} \langle 1|-1\rangle \right|^2 = \left| \frac{1}{\sqrt{14}} \right|^2 = \frac{1}{14} \\
\mathcal{P}_0 &= |\langle 0 | \psi_{in} \rangle|^2 = \left| \langle 0 | \left[\frac{1}{\sqrt{14}} |1\rangle - \frac{3}{\sqrt{14}} |0\rangle + \frac{2i}{\sqrt{14}} |-1\rangle \right] \right|^2 \\
&= \left| \frac{1}{\sqrt{14}} \langle 0|1\rangle - \frac{3}{\sqrt{14}} \langle 0|0\rangle + \frac{2i}{\sqrt{14}} \langle 0|-1\rangle \right|^2 = \left| -\frac{3}{\sqrt{14}} \right|^2 = \frac{9}{14} \\
\mathcal{P}_{-1} &= |\langle -1 | \psi_{in} \rangle|^2 = \left| \langle -1 | \left[\frac{1}{\sqrt{14}} |1\rangle - \frac{3}{\sqrt{14}} |0\rangle + \frac{2i}{\sqrt{14}} |-1\rangle \right] \right|^2 \\
&= \left| \frac{1}{\sqrt{14}} \langle -1|1\rangle - \frac{3}{\sqrt{14}} \langle -1|0\rangle + \frac{2i}{\sqrt{14}} \langle -1|-1\rangle \right|^2 = \left| \frac{2i}{\sqrt{14}} \right|^2 = \frac{4}{14}
\end{aligned}$$

b) After the S_z measurement, the system is in the state $|-1\rangle$. The possible results of a measurement of the spin component S_x are always $+1\hbar$, $0\hbar$, $-1\hbar$ for a spin-1 particle. The probabilities are

$$\begin{aligned}
\mathcal{P}_{1x} &= |\langle 1 | \psi_{in} \rangle|^2 = \left| \left(\frac{1}{2} \langle 1 | + \frac{1}{\sqrt{2}} \langle 0 | + \frac{1}{2} \langle -1 | \right) |-1\rangle \right|^2 = \left| \frac{1}{2} \right|^2 = \frac{1}{4} \\
\mathcal{P}_{0x} &= |\langle 0 | \psi_{in} \rangle|^2 = \left| \left(\frac{1}{\sqrt{2}} \langle 1 | - \frac{1}{\sqrt{2}} \langle -1 | \right) |-1\rangle \right|^2 = \left| \frac{-1}{\sqrt{2}} \right|^2 = \frac{1}{2} \\
\mathcal{P}_{-1x} &= |\langle -1 | \psi_{in} \rangle|^2 = \left| \left(\frac{1}{2} \langle 1 | - \frac{1}{\sqrt{2}} \langle 0 | + \frac{1}{2} \langle -1 | \right) |-1\rangle \right|^2 = \left| \frac{1}{2} \right|^2 = \frac{1}{4}
\end{aligned}$$

c) Schematic of experiment.



2.19 The probability is

$$\begin{aligned}
 \mathcal{P}_{\psi_f} &= \left| \langle \psi_f | \psi_i \rangle \right|^2 = \left| \left(\frac{1-i}{\sqrt{7}} \langle 1| + \frac{2}{\sqrt{7}} \langle 0| + \frac{i}{\sqrt{7}} \langle -1| \right) \left(\frac{1}{\sqrt{6}} |1\rangle - \frac{\sqrt{2}}{\sqrt{6}} |0\rangle + \frac{i\sqrt{3}}{\sqrt{6}} |-1\rangle \right) \right|^2 \\
 &= \left| \frac{1-i}{\sqrt{7}} \frac{1}{\sqrt{6}} \langle 1|1\rangle - \frac{1-i}{\sqrt{7}} \frac{\sqrt{2}}{\sqrt{6}} \langle 1|0\rangle + \frac{1-i}{\sqrt{7}} \frac{i\sqrt{3}}{\sqrt{6}} \langle 1|-1\rangle + \frac{2}{\sqrt{7}} \frac{1}{\sqrt{6}} \langle 0|1\rangle - \frac{2}{\sqrt{7}} \frac{\sqrt{2}}{\sqrt{6}} \langle 0|0\rangle + \frac{2}{\sqrt{7}} \frac{i\sqrt{3}}{\sqrt{6}} \langle 0|-1\rangle + \right. \\
 &\quad \left. + \frac{i}{\sqrt{7}} \frac{1}{\sqrt{6}} \langle -1|1\rangle - \frac{i}{\sqrt{7}} \frac{\sqrt{2}}{\sqrt{6}} \langle -1|0\rangle + \frac{i}{\sqrt{7}} \frac{i\sqrt{3}}{\sqrt{6}} \langle -1|-1\rangle \right|^2 \\
 &= \left| \frac{1-i}{\sqrt{7}} \frac{1}{\sqrt{6}} \frac{1}{2} + \frac{1-i}{\sqrt{7}} \frac{\sqrt{2}}{\sqrt{6}} \frac{i}{\sqrt{2}} - \frac{1-i}{\sqrt{7}} \frac{i\sqrt{3}}{\sqrt{6}} \frac{1}{2} + \frac{2}{\sqrt{7}} \frac{1}{\sqrt{6}} \frac{1}{\sqrt{2}} - \frac{2}{\sqrt{7}} \frac{\sqrt{2}}{\sqrt{6}} 0 + \frac{2}{\sqrt{7}} \frac{i\sqrt{3}}{\sqrt{6}} \frac{1}{\sqrt{2}} + \right. \\
 &\quad \left. + \frac{i}{\sqrt{7}} \frac{1}{\sqrt{6}} \frac{1}{2} - \frac{i}{\sqrt{7}} \frac{\sqrt{2}}{\sqrt{6}} \frac{i}{\sqrt{2}} - \frac{i}{\sqrt{7}} \frac{i\sqrt{3}}{\sqrt{6}} \frac{1}{2} \right|^2 \\
 &= \frac{1}{168} \left| 1 - i + 2 + 2i - \sqrt{3} - i\sqrt{3} + 2\sqrt{2} - 0 + i2\sqrt{6} + i + 2 + \sqrt{3} \right|^2 \\
 &= \frac{1}{168} \left| 5 + 2\sqrt{2} + 2i - i\sqrt{3} + i2\sqrt{6} \right|^2 = \frac{1}{168} \left\{ \left(5 + 2\sqrt{2} \right)^2 + \left(2 - \sqrt{3} + 2\sqrt{6} \right)^2 \right\} \\
 &= \frac{1}{168} \left(64 + 8\sqrt{2} - 4\sqrt{3} + 8\sqrt{6} \right) \cong 0.524
 \end{aligned}$$

or in matrix notation

$$\begin{aligned}
 \mathcal{P}_{\psi_f} &= \left| \langle \psi_f | \psi_i \rangle \right|^2 = \left| \left\{ \frac{1-i}{\sqrt{7}} \begin{pmatrix} \frac{1}{2} & \frac{-i}{\sqrt{2}} & \frac{-1}{2} \end{pmatrix} + \frac{2}{\sqrt{7}} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} + \frac{i}{\sqrt{7}} \begin{pmatrix} \frac{1}{2} & \frac{i}{\sqrt{2}} & \frac{-1}{2} \end{pmatrix} \right\} \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -\sqrt{2} \\ i\sqrt{3} \end{pmatrix} \right|^2 \\
 &= \frac{1}{168} \left| \begin{pmatrix} 1 + 2\sqrt{2} & -2\sqrt{2} - i\sqrt{2} & -1 + 2\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 \\ -\sqrt{2} \\ i\sqrt{3} \end{pmatrix} \right|^2 \\
 &= \frac{1}{168} \left| 1 + 2\sqrt{2} + 4 + 2i - i\sqrt{3} + i2\sqrt{6} \right|^2 = \frac{1}{168} \left(64 + 8\sqrt{2} - 4\sqrt{3} + 8\sqrt{6} \right) \cong 0.524
 \end{aligned}$$

2.20 Spin 1 unknowns. Follow the solution method given in the lab handout. (i) For unknown number 1, the measured probabilities are

$$\begin{aligned}
 \mathcal{P}_1 &= \frac{1}{4} & \mathcal{P}_{1x} &= \frac{1}{4} & \mathcal{P}_{1y} &= 1 \\
 \mathcal{P}_0 &= \frac{1}{2} & \mathcal{P}_{0x} &= \frac{1}{2} & \mathcal{P}_{0y} &= 0 \\
 \mathcal{P}_{-1} &= \frac{1}{4} & \mathcal{P}_{-1x} &= \frac{1}{4} & \mathcal{P}_{-1y} &= 0
 \end{aligned}$$

Write the unknown state as

$$|\psi_1\rangle = a|1\rangle + b|0\rangle + c|-1\rangle$$

Equating the predicted S_z probabilities and the experimental results gives

$$\begin{aligned}\mathcal{P}_1 &= |\langle 1|\psi_1\rangle|^2 = |\langle 1|\{a|1\rangle + b|0\rangle + c|-1\rangle\}|^2 = |a|^2 = \frac{1}{4} \Rightarrow a = \frac{1}{2} \\ \mathcal{P}_0 &= |\langle 0|\psi_1\rangle|^2 = |\langle 0|\{a|1\rangle + b|0\rangle + c|-1\rangle\}|^2 = |b|^2 = \frac{1}{2} \Rightarrow b = \frac{1}{\sqrt{2}}e^{i\alpha} \\ \mathcal{P}_{-1} &= |\langle -1|\psi_1\rangle|^2 = |\langle -1|\{a|1\rangle + b|0\rangle + c|-1\rangle\}|^2 = |c|^2 = \frac{1}{4} \Rightarrow c = \frac{1}{2}e^{i\beta}\end{aligned}$$

allowing for possible relative phases. So now the unknown state is

$$|\psi_1\rangle = \frac{1}{2}|1\rangle + \frac{1}{\sqrt{2}}e^{i\alpha}|0\rangle + \frac{1}{2}e^{i\beta}|-1\rangle$$

Equating the predicted S_x probabilities and the experimental results gives

$$\begin{aligned}\mathcal{P}_{0x} &= |\langle 0|\psi_1\rangle|^2 = \left| \frac{1}{\sqrt{2}}\left\{ \langle 1| - \langle -1| \right\} \left\{ \frac{1}{2}|1\rangle + \frac{1}{\sqrt{2}}e^{i\alpha}|0\rangle + \frac{1}{2}e^{i\beta}|-1\rangle \right\} \right|^2 = \left| \frac{1}{2\sqrt{2}}\{1 - e^{i\beta}\} \right|^2 \\ &= \frac{1}{8}\{1 - e^{i\beta}\}\{1 - e^{-i\beta}\} = \frac{1}{8}\{1 + 1 - e^{i\beta} - e^{-i\beta}\} = \frac{1}{4}\{1 - \cos\beta\} = \frac{1}{2} \Rightarrow \cos\beta = -1 \Rightarrow \beta = \pi\end{aligned}$$

Giving the state

$$|\psi_1\rangle = \frac{1}{2}|1\rangle + \frac{1}{\sqrt{2}}e^{i\alpha}|0\rangle - \frac{1}{2}|-1\rangle$$

Equating the predicted S_y probabilities and the experimental results gives

$$\begin{aligned}\mathcal{P}_{1y} &= |\langle 1|\psi_1\rangle|^2 = \left| \left\{ \frac{1}{2}\langle 1| - \frac{i}{\sqrt{2}}\langle 0| - \frac{1}{2}\langle -1| \right\} \left\{ \frac{1}{2}|1\rangle + \frac{1}{\sqrt{2}}e^{i\alpha}|0\rangle - \frac{1}{2}|-1\rangle \right\} \right|^2 = \left| \frac{1}{4} - \frac{i}{2}e^{i\alpha} + \frac{1}{4} \right|^2 \\ &= \frac{1}{4}\{1 - ie^{i\alpha}\}\{1 + ie^{-i\alpha}\} = \frac{1}{4}\{1 + 1 - ie^{i\alpha} + ie^{-i\alpha}\} = \frac{1}{2}\{1 + \sin\alpha\} = 1 \Rightarrow \sin\alpha = 1 \Rightarrow \alpha = \frac{\pi}{2}\end{aligned}$$

Hence the unknown state is

$$|\psi_1\rangle = \frac{1}{2}|1\rangle + \frac{1}{\sqrt{2}}e^{i\frac{\pi}{2}}|0\rangle - \frac{1}{2}|-1\rangle = \frac{1}{2}|1\rangle + \frac{i}{\sqrt{2}}|0\rangle - \frac{1}{2}|-1\rangle = |1\rangle_y$$

(ii) For unknown number 2, the measured probabilities are

$$\begin{aligned}\mathcal{P}_1 &= \frac{1}{4} & \mathcal{P}_{1x} &= \frac{9}{16} & \mathcal{P}_{1y} &= 0.870 \\ \mathcal{P}_0 &= \frac{1}{2} & \mathcal{P}_{0x} &= \frac{3}{8} & \mathcal{P}_{0y} &= 0.125 \\ \mathcal{P}_{-1} &= \frac{1}{4} & \mathcal{P}_{-1x} &= \frac{1}{16} & \mathcal{P}_{-1y} &= 0.005\end{aligned}$$

Write the unknown state as

$$|\psi_2\rangle = a|1\rangle + b|0\rangle + c|-1\rangle$$

Equating the predicted S_z probabilities and the experimental results gives

$$\begin{aligned}\mathcal{P}_1 &= \left| \langle 1 | \psi_2 \rangle \right|^2 = \left| \langle 1 | \{a|1\rangle + b|0\rangle + c|-1\rangle \} \right|^2 = |a|^2 = \frac{1}{4} \Rightarrow a = \frac{1}{2} \\ \mathcal{P}_0 &= \left| \langle 0 | \psi_2 \rangle \right|^2 = \left| \langle 0 | \{a|1\rangle + b|0\rangle + c|-1\rangle \} \right|^2 = |b|^2 = \frac{1}{2} \Rightarrow b = \frac{1}{\sqrt{2}} e^{i\alpha} \\ \mathcal{P}_{-1} &= \left| \langle -1 | \psi_2 \rangle \right|^2 = \left| \langle -1 | \{a|1\rangle + b|0\rangle + c|-1\rangle \} \right|^2 = |c|^2 = \frac{1}{4} \Rightarrow c = \frac{1}{2} e^{i\beta}\end{aligned}$$

allowing for possible relative phases. So now the unknown state is

$$|\psi_2\rangle = \frac{1}{2}|1\rangle + \frac{1}{\sqrt{2}}e^{i\alpha}|0\rangle + \frac{1}{2}e^{i\beta}|-1\rangle$$

Equating the predicted S_x probabilities and the experimental results gives

$$\begin{aligned}\mathcal{P}_{0x} &= \left| {}_x\langle 0 | \psi_2 \rangle \right|^2 = \left| \frac{1}{\sqrt{2}} \{ \langle 1 | - \langle -1 | \} \left[\frac{1}{2}|1\rangle + \frac{1}{\sqrt{2}}e^{i\alpha}|0\rangle + \frac{1}{2}e^{i\beta}|-1\rangle \right] \right|^2 = \left| \frac{1}{2\sqrt{2}} \{1 - e^{i\beta}\} \right|^2 \\ &= \frac{1}{8} \{1 - e^{i\beta}\} \{1 - e^{-i\beta}\} = \frac{1}{8} \{1 + 1 - e^{i\beta} - e^{-i\beta}\} = \frac{1}{4} \{1 - \cos \beta\} = \frac{3}{8} \\ \Rightarrow \cos \beta &= -\frac{1}{2} \Rightarrow \beta = \frac{2\pi}{3}, \frac{4\pi}{3}\end{aligned}$$

$$\begin{aligned}\mathcal{P}_{1x} &= \left| {}_x\langle 1 | \psi_2 \rangle \right|^2 = \left| \left[\frac{1}{2}\langle 1 | + \frac{1}{\sqrt{2}}\langle 0 | + \frac{1}{2}\langle -1 | \right] \left[\frac{1}{2}|1\rangle + \frac{1}{\sqrt{2}}e^{i\alpha}|0\rangle + \frac{1}{2}e^{i\beta}|-1\rangle \right] \right|^2 = \left| \frac{1}{4} + \frac{1}{2}e^{i\alpha} + \frac{1}{4}e^{i\beta} \right|^2 \\ &= \frac{1}{16} \{1 + 2e^{i\alpha} + e^{i\beta}\} \{1 + 2e^{-i\alpha} + e^{-i\beta}\} = \frac{1}{16} \{6 + 4\cos \alpha + 2\cos \beta + 4\cos(\alpha - \beta)\} \\ &= \frac{1}{16} \{6 + 4\cos \alpha + 2\cos \beta + 4\cos \alpha \cos \beta + 4\sin \alpha \sin \beta\} \\ &= \frac{1}{16} \{5 + 2\cos \alpha + 4\sin \alpha \sin \beta\} = \frac{9}{16}\end{aligned}$$

which yields

$$\begin{aligned}\cos \alpha + 2\sin \alpha \sin \beta &= 2 \\ 2\sin \alpha \sin \beta &= 2 - \cos \alpha \\ 4\sin^2 \alpha \sin^2 \beta &= 4 - 4\cos \alpha + \cos^2 \alpha \\ 4(1 - \cos^2 \alpha) \frac{3}{4} &= 4 - 4\cos \alpha + \cos^2 \alpha \\ 4\cos^2 \alpha - 4\cos \alpha + 1 &= 0 \\ 2\cos \alpha - 1 &= 0 \\ \cos \alpha = \frac{1}{2} &\Rightarrow \alpha = \frac{\pi}{3}, \frac{5\pi}{3}\end{aligned}$$

Equating the predicted S_y probabilities and the experimental results gives

$$\begin{aligned}\mathcal{P}_{1y} &= \left| {}_y\langle 1 | \psi_2 \rangle \right|^2 = \left| \left[\frac{1}{2}\langle 1 | - \frac{i}{\sqrt{2}}\langle 0 | - \frac{1}{2}\langle -1 | \right] \left[\frac{1}{2}|1\rangle + \frac{1}{\sqrt{2}}e^{i\alpha}|0\rangle + \frac{1}{2}e^{i\beta}|-1\rangle \right] \right|^2 = \left| \frac{1}{4} - \frac{i}{2}e^{i\alpha} - \frac{1}{4}e^{i\beta} \right|^2 \\ &= \frac{1}{16} \{1 - 2ie^{i\alpha} - e^{i\beta}\} \{1 + 2ie^{-i\alpha} - e^{-i\beta}\} = \frac{1}{16} \{6 + 4\sin \alpha - 2\cos \beta - 4\sin(\alpha - \beta)\} \\ &= \frac{1}{16} \{6 + 4\sin \alpha - 2\cos \beta - 4\sin \alpha \cos \beta + 4\cos \alpha \sin \beta\} \\ &= \frac{1}{16} \{7 + 6\sin \alpha + 4\cos \alpha \sin \beta\} = 0.87\end{aligned}$$

which gives

$$\begin{aligned}
 3\sin\alpha + 2\cos\alpha\sin\beta &= 3.46 \cong 2\sqrt{3} \\
 2\cos\alpha\sin\beta &= 2\sqrt{3} - 3\sin\alpha \\
 4\cos^2\alpha\sin^2\beta &= 12 - 12\sqrt{3}\sin\alpha + 9\sin^2\alpha \\
 4(1 - \sin^2\alpha)\frac{3}{4} &= 12 - 12\sqrt{3}\sin\alpha + 9\sin^2\alpha \\
 \sin^2\alpha - \sqrt{3}\sin\alpha + \frac{3}{4} &= 0 \\
 \sin\alpha - \frac{\sqrt{3}}{2} &= 0 \\
 \sin\alpha = \frac{\sqrt{3}}{2} &\Rightarrow \alpha = \frac{\pi}{3}, \frac{2\pi}{3} \Rightarrow \alpha = \frac{\pi}{3} \Rightarrow \beta = \frac{2\pi}{3}
 \end{aligned}$$

Hence the unknown state is

$$|\psi_2\rangle = \frac{1}{2}|1\rangle + \frac{1}{\sqrt{2}}e^{i\frac{\pi}{3}}|0\rangle + \frac{1}{2}e^{i\frac{2\pi}{3}}|-1\rangle = |1\rangle_{n(\theta=\frac{\pi}{2}, \phi=\frac{\pi}{3})}$$

(iii) For unknown number 3, the measured probabilities are

$$\begin{aligned}
 \mathcal{P}_1 &= \frac{1}{3} & \mathcal{P}_{1x} &= \frac{1}{6} & \mathcal{P}_{1y} &= 0.0286 \\
 \mathcal{P}_0 &= \frac{1}{3} & \mathcal{P}_{0x} &= \frac{2}{3} & \mathcal{P}_{0y} &= 0 \\
 \mathcal{P}_{-1} &= \frac{1}{3} & \mathcal{P}_{-1x} &= \frac{1}{6} & \mathcal{P}_{-1y} &= 0.9714
 \end{aligned}$$

Write the unknown state as

$$|\psi_3\rangle = a|1\rangle + b|0\rangle + c|-1\rangle$$

Equating the predicted S_z probabilities and the experimental results gives

$$\begin{aligned}
 \mathcal{P}_1 &= |\langle 1|\psi_3\rangle|^2 = |\langle 1|\{a|1\rangle + b|0\rangle + c|-1\rangle\}|^2 = |a|^2 = \frac{1}{3} \Rightarrow a = \frac{1}{\sqrt{3}} \\
 \mathcal{P}_0 &= |\langle 0|\psi_3\rangle|^2 = |\langle 0|\{a|1\rangle + b|0\rangle + c|-1\rangle\}|^2 = |b|^2 = \frac{1}{3} \Rightarrow b = \frac{1}{\sqrt{3}}e^{i\alpha} \\
 \mathcal{P}_{-1} &= |\langle -1|\psi_3\rangle|^2 = |\langle -1|\{a|1\rangle + b|0\rangle + c|-1\rangle\}|^2 = |c|^2 = \frac{1}{3} \Rightarrow c = \frac{1}{\sqrt{3}}e^{i\beta}
 \end{aligned}$$

allowing for possible relative phases. So now the unknown state is

$$|\psi_3\rangle = \frac{1}{\sqrt{3}}|1\rangle + \frac{1}{\sqrt{3}}e^{i\alpha}|0\rangle + \frac{1}{\sqrt{3}}e^{i\beta}|-1\rangle$$

Equating the predicted S_x probabilities and the experimental results gives

$$\begin{aligned}
 \mathcal{P}_{0x} &= \left| {}_x\langle 0|\psi_3\rangle \right|^2 = \left| \frac{1}{\sqrt{2}}\{\langle 1| - \langle -1|\}\frac{1}{\sqrt{3}}\{ |1\rangle + e^{i\alpha}|0\rangle + e^{i\beta}|-1\rangle\} \right|^2 = \left| \frac{1}{\sqrt{6}}\{1 - e^{i\beta}\} \right|^2 \\
 &= \frac{1}{6}\{1 - e^{i\beta}\}\{1 - e^{-i\beta}\} = \frac{1}{6}\{1 + 1 - e^{i\beta} - e^{-i\beta}\} = \frac{1}{3}\{1 - \cos\beta\} = \frac{2}{3} \\
 &\Rightarrow \cos\beta = -1 \Rightarrow \beta = \pi
 \end{aligned}$$

Equating the predicted S_y probabilities and the experimental results gives

$$\begin{aligned}
\mathcal{P}_{1y} &= \left| {}_y\langle 1 | \psi_3 \rangle \right|^2 = \left| \left\{ \frac{1}{2} \langle 1 | - \frac{i}{\sqrt{2}} \langle 0 | - \frac{1}{2} \langle -1 | \right\} \frac{1}{\sqrt{3}} \{ |1\rangle + e^{i\alpha} |0\rangle - |-1\rangle \} \right|^2 = \left| \frac{1}{\sqrt{3}} \left(\frac{1}{2} - \frac{i}{\sqrt{2}} e^{i\alpha} + \frac{1}{2} \right) \right|^2 \\
&= \frac{1}{3} \left\{ 1 - \frac{i}{\sqrt{2}} e^{i\alpha} \right\} \left\{ 1 + \frac{i}{\sqrt{2}} e^{-i\alpha} \right\} = \frac{1}{3} \left\{ \frac{3}{2} + \sqrt{2} \sin \alpha \right\} = 0.0286 \\
&\Rightarrow \sin \alpha = -1 \Rightarrow \alpha = \frac{3\pi}{2}
\end{aligned}$$

Hence the unknown state is

$$|\psi_3\rangle = \frac{1}{\sqrt{3}}|1\rangle - \frac{i}{\sqrt{3}}|0\rangle - \frac{1}{\sqrt{3}}|-1\rangle \quad \{ \neq |m\rangle_n \}$$

(iv) For unknown number 4, the measured probabilities are

$$\begin{aligned}
\mathcal{P}_1 &= \frac{1}{2} & \mathcal{P}_{1x} &= \frac{1}{4} & \mathcal{P}_{1y} &= \frac{1}{4} \\
\mathcal{P}_0 &= 0 & \mathcal{P}_{0x} &= \frac{1}{2} & \mathcal{P}_{0y} &= \frac{1}{2} \\
\mathcal{P}_{-1} &= \frac{1}{2} & \mathcal{P}_{-1x} &= \frac{1}{4} & \mathcal{P}_{-1y} &= \frac{1}{4}
\end{aligned}$$

Write the unknown state as

$$|\psi_4\rangle = a|1\rangle + b|0\rangle + c|-1\rangle$$

Equating the predicted S_z probabilities and the experimental results gives

$$\begin{aligned}
\mathcal{P}_1 &= \left| \langle 1 | \psi_4 \rangle \right|^2 = \left| \langle 1 | \{ a|1\rangle + b|0\rangle + c|-1\rangle \} \right|^2 = |a|^2 = \frac{1}{2} \Rightarrow a = \frac{1}{\sqrt{2}} \\
\mathcal{P}_0 &= \left| \langle 0 | \psi_4 \rangle \right|^2 = \left| \langle 0 | \{ a|1\rangle + b|0\rangle + c|-1\rangle \} \right|^2 = |b|^2 = 0 \Rightarrow b = 0 \\
\mathcal{P}_{-1} &= \left| \langle -1 | \psi_4 \rangle \right|^2 = \left| \langle -1 | \{ a|1\rangle + b|0\rangle + c|-1\rangle \} \right|^2 = |c|^2 = \frac{1}{2} \Rightarrow c = \frac{1}{\sqrt{2}} e^{i\beta}
\end{aligned}$$

allowing for possible relative phases. So now the unknown state is

$$|\psi_4\rangle = \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}e^{i\beta}|-1\rangle$$

Equating the predicted S_x probabilities and the experimental results gives

$$\begin{aligned}
\mathcal{P}_{0x} &= \left| {}_x\langle 0 | \psi_4 \rangle \right|^2 = \left| \frac{1}{\sqrt{2}} \{ |1\rangle - \langle -1| \} \frac{1}{\sqrt{2}} \{ |1\rangle + e^{i\beta} |-1\rangle \} \right|^2 = \left| \frac{1}{2} \{ 1 - e^{i\beta} \} \right|^2 \\
&= \frac{1}{4} \{ 1 - e^{i\beta} \} \{ 1 - e^{-i\beta} \} = \frac{1}{4} \{ 1 + 1 - e^{i\beta} - e^{-i\beta} \} = \frac{1}{2} \{ 1 - \cos \beta \} = \frac{1}{2} \\
&\Rightarrow \cos \beta = 0 \Rightarrow \beta = \frac{\pi}{2}, \frac{3\pi}{2}
\end{aligned}$$

Giving

$$|\psi_4\rangle = \frac{1}{\sqrt{2}} \{ |1\rangle \pm i |-1\rangle \}$$

with no more info from S_x , S_y , S_z measurements. In the SPINS program choose $\mathbf{n}$ at angles $\theta = 90^\circ$, $\phi = 45^\circ$, 225° to see that the unknown state is

$$|\psi_4\rangle = \frac{1}{\sqrt{2}} \{ |1\rangle - i |-1\rangle \} = |0\rangle_{n(\theta=\frac{\pi}{2}, \phi=\frac{\pi}{4})} = |0\rangle_{n(\theta=\frac{\pi}{2}, \phi=\frac{5\pi}{4})}$$

2.21. The spin-1 interferometer had an S_z SG device, an S_x device, and an S_z SG device. The S_z eigenstates are $|1\rangle, |0\rangle, |-1\rangle$. The S_x eigenstates are

$$\begin{aligned} |1\rangle_x &= \frac{1}{2}|1\rangle + \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|-1\rangle \\ |0\rangle_x &= \frac{1}{\sqrt{2}}|1\rangle - \frac{1}{\sqrt{2}}|-1\rangle \\ |-1\rangle_x &= \frac{1}{2}|1\rangle - \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|-1\rangle \end{aligned} .$$

Let $|\psi_i\rangle$ be the quantum state after the i^{th} Stern-Gerlach device. The first SG device transmit particles with $S_z = +\hbar$, so the state $|\psi_1\rangle$ is

$$|\psi_1\rangle = |1\rangle.$$

The second SG device transmits particles in 1, 2, or 3 of the S_x eigenstates $|1\rangle_x, |0\rangle_x, |-1\rangle_x$. To find $|\psi_2\rangle$, we use the projection postulate:

$$|\psi_2\rangle = \frac{P_n |\psi_1\rangle}{\sqrt{\langle \psi_1 | P_n | \psi_1 \rangle}}$$

where P_n is the projection operator onto the measured states. For example, if the second SG device transmits particles with $S_x = +\hbar$, we get

$$|\psi_2\rangle = \frac{P_{1x} |\psi_1\rangle}{\sqrt{\langle \psi_1 | P_{1x} | \psi_1 \rangle}} = \frac{|1\rangle_x \langle 1|1\rangle}{\sqrt{\langle 1|1\rangle_x \langle 1|1\rangle}} = |1\rangle_x$$

as expected. In matrix notation, the S_x eigenstates are

$$|1\rangle_x \doteq \begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{pmatrix} \quad |0\rangle_x \doteq \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix} \quad |-1\rangle_x \doteq \begin{pmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{pmatrix}$$

and the projection operators are

$$\begin{aligned}
 P_{1x} &= |1\rangle_{xx} \langle 1| \doteq \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{1}{2\sqrt{2}} & \frac{1}{4} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2} & \frac{1}{2\sqrt{2}} \\ \frac{1}{4} & \frac{1}{2\sqrt{2}} & \frac{1}{4} \end{pmatrix} \\
 P_{0x} &= |0\rangle_{xx} \langle 0| \doteq \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \\
 P_{-1x} &= |-1\rangle_{xx} \langle -1| \doteq \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{2\sqrt{2}} & \frac{1}{4} \\ -\frac{1}{2\sqrt{2}} & \frac{1}{2} & -\frac{1}{2\sqrt{2}} \\ \frac{1}{4} & -\frac{1}{2\sqrt{2}} & \frac{1}{4} \end{pmatrix}
 \end{aligned}$$

The probability of measuring a result after the third SG device is $\mathcal{P} = |\langle \psi_2 | \psi_3 \rangle|^2$. We want to calculate the three probabilities

$$\begin{aligned}
 \mathcal{P}_1 &= |\langle 1 | \psi_2 \rangle|^2 \\
 \mathcal{P}_0 &= |\langle 0 | \psi_2 \rangle|^2 \\
 \mathcal{P}_{-1} &= |\langle -1 | \psi_2 \rangle|^2
 \end{aligned}$$

for all possible (7) cases of 1 beam, 2 beams, or 3 beams from SG2.

(i) When the second SG device transmits particles with $S_x = +1\hbar$ only:

$$\begin{aligned}
 |\psi_2\rangle &= \frac{P_{1x}|\psi_1\rangle}{\sqrt{\langle \psi_1 | P_{1x} | \psi_1 \rangle}} = \frac{P_{1x}|1\rangle}{\sqrt{\langle 1 | P_{1x} | 1 \rangle}} \\
 P_{1x}|1\rangle &\doteq \begin{pmatrix} \frac{1}{4} & \frac{1}{2\sqrt{2}} & \frac{1}{4} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2} & \frac{1}{2\sqrt{2}} \\ \frac{1}{4} & \frac{1}{2\sqrt{2}} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2\sqrt{2}} \\ \frac{1}{4} \end{pmatrix} \\
 \langle 1 | P_{1x} | 1 \rangle &\doteq \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2\sqrt{2}} \\ \frac{1}{4} \end{pmatrix} = 1/4 \\
 |\psi_2\rangle &\doteq \frac{1}{\sqrt{1/4}} \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2\sqrt{2}} \\ \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}
 \end{aligned}$$

which is $|1\rangle_x$ as expected. The three probabilities are

$$\begin{aligned}\mathcal{P}_1 &= |\langle 1 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} \right|^2 = \frac{1}{4} \\ \mathcal{P}_0 &= |\langle 0 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} \right|^2 = \frac{1}{2} \\ \mathcal{P}_{-1} &= |\langle -1 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} \right|^2 = \frac{1}{4}\end{aligned}$$

(ii) When the second SG device transmits particles with $S_x = 0\hbar$, $|\psi_2\rangle = |0\rangle_x$ and the three probabilities are

$$\begin{aligned}\mathcal{P}_1 &= |\langle 1 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \right|^2 = \frac{1}{2} \\ \mathcal{P}_0 &= |\langle 0 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \right|^2 = 0 \\ \mathcal{P}_{-1} &= |\langle -1 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \right|^2 = \frac{1}{2}\end{aligned}$$

(iii) When the second SG device transmits particles with $S_x = -1\hbar$, $|\psi_2\rangle = |-1\rangle_x$ and the three probabilities are

$$\begin{aligned}\mathcal{P}_1 &= |\langle 1 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} \right|^2 = \frac{1}{4} \\ \mathcal{P}_0 &= |\langle 0 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} \right|^2 = \frac{1}{2} \\ \mathcal{P}_{-1} &= |\langle -1 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} \right|^2 = \frac{1}{4}\end{aligned}$$

(iv) When the second SG device transmits particles with $S_x = +\hbar$ **and** $S_x = 0\hbar$ in a coherent beam

$$\begin{aligned}|\psi_2\rangle &= \frac{(P_{1x} + P_{0x})|\psi_1\rangle}{\sqrt{\langle \psi_1 | (P_{1x} + P_{0x}) | \psi_1 \rangle}} = \frac{(P_{1x} + P_{0x})|1\rangle}{\sqrt{\langle 1 | (P_{1x} + P_{0x}) | 1 \rangle}} \\ P_{0x}|1\rangle &\doteq \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix} \\ (P_{1x} + P_{0x})|1\rangle &\doteq \begin{pmatrix} 1/4 \\ 1/2\sqrt{2} \\ 1/4 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix} = \begin{pmatrix} 3/4 \\ 1/2\sqrt{2} \\ -1/4 \end{pmatrix} \\ \langle 1 | (P_{1x} + P_{0x}) | 1 \rangle &\doteq \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3/4 \\ 1/2\sqrt{2} \\ -1/4 \end{pmatrix} = 3/4 \\ |\psi_2\rangle &\doteq \frac{1}{\sqrt{3/4}} \begin{pmatrix} 3/4 \\ 1/2\sqrt{2} \\ -1/4 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 \\ 1/\sqrt{6} \\ -1/2\sqrt{3} \end{pmatrix}\end{aligned}$$

The three probabilities are

$$\begin{aligned}\mathcal{P}_1 &= |\langle 1 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 \\ 1/\sqrt{6} \\ -1/2\sqrt{3} \end{pmatrix} \right|^2 = \frac{3}{4} \\ \mathcal{P}_0 &= |\langle 0 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 \\ 1/\sqrt{6} \\ -1/2\sqrt{3} \end{pmatrix} \right|^2 = \frac{1}{6} \\ \mathcal{P}_{-1} &= |\langle -1 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 \\ 1/\sqrt{6} \\ -1/2\sqrt{3} \end{pmatrix} \right|^2 = \frac{1}{12}\end{aligned}$$

(v) When the second SG device transmits particles with $S_x = +\hbar$ **and** $S_x = -\hbar$ in a coherent beam

$$\begin{aligned}|\psi_2\rangle &= \frac{(P_{1x} + P_{-1x})|\psi_1\rangle}{\sqrt{\langle\psi_1|(P_{1x} + P_{-1x})|\psi_1\rangle}} = \frac{(P_{1x} + P_{-1x})|1\rangle}{\sqrt{\langle 1|(P_{1x} + P_{-1x})|1\rangle}} \\ P_{-1x}|1\rangle &\doteq \begin{pmatrix} 1/4 & -1/2\sqrt{2} & 1/4 \\ -1/2\sqrt{2} & 1/2 & -1/2\sqrt{2} \\ 1/4 & -1/2\sqrt{2} & 1/4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/4 \\ -1/2\sqrt{2} \\ 1/4 \end{pmatrix} \\ (P_{1x} + P_{-1x})|1\rangle &\doteq \begin{pmatrix} 1/4 \\ 1/2\sqrt{2} \\ 1/4 \end{pmatrix} + \begin{pmatrix} 1/4 \\ -1/2\sqrt{2} \\ 1/4 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix} \\ \langle 1|(P_{1x} + P_{-1x})|1\rangle &\doteq \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix} = 1/2 \\ |\psi_2\rangle &\doteq \frac{1}{\sqrt{1/2}} \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}\end{aligned}$$

The three probabilities are

$$\begin{aligned}\mathcal{P}_1 &= |\langle 1 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \right|^2 = \frac{1}{2} \\ \mathcal{P}_0 &= |\langle 0 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \right|^2 = 0 \\ \mathcal{P}_{-1} &= |\langle -1 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \right|^2 = \frac{1}{2}\end{aligned}$$

(vi) When the second SG device transmits particles with $S_x = 0\hbar$ **and** $S_x = -1\hbar$ in a coherent beam

$$\begin{aligned}|\psi_2\rangle &= \frac{(P_{0x} + P_{-1x})|\psi_1\rangle}{\sqrt{\langle\psi_1|(P_{0x} + P_{-1x})|\psi_1\rangle}} = \frac{(P_{0x} + P_{-1x})|1\rangle}{\sqrt{\langle 1|(P_{0x} + P_{-1x})|1\rangle}} \\ (P_{0x} + P_{-1x})|1\rangle &\doteq \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix} + \begin{pmatrix} 1/4 \\ -1/2\sqrt{2} \\ 1/4 \end{pmatrix} = \begin{pmatrix} 3/4 \\ -1/2\sqrt{2} \\ -1/4 \end{pmatrix} \\ \langle 1|(P_{0x} + P_{-1x})|1\rangle &\doteq \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3/4 \\ -1/2\sqrt{2} \\ -1/4 \end{pmatrix} = 3/4 \\ |\psi_2\rangle &\doteq \frac{1}{\sqrt{3/4}} \begin{pmatrix} 3/4 \\ -1/2\sqrt{2} \\ -1/4 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 \\ -1/\sqrt{6} \\ -1/2\sqrt{3} \end{pmatrix}\end{aligned}$$

The three probabilities are

$$\begin{aligned}\mathcal{P}_1 &= |\langle 1 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 \\ -1/\sqrt{6} \\ -1/2\sqrt{3} \end{pmatrix} \right|^2 = \frac{3}{4} \\ \mathcal{P}_0 &= |\langle 0 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 \\ -1/\sqrt{6} \\ -1/2\sqrt{3} \end{pmatrix} \right|^2 = \frac{1}{6} \\ \mathcal{P}_{-1} &= |\langle -1 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 \\ -1/\sqrt{6} \\ -1/2\sqrt{3} \end{pmatrix} \right|^2 = \frac{1}{12}\end{aligned}$$

(vii) When the second SG device transmits particles with $S_x = +\hbar$, $S_x = 0\hbar$, **and** $S_x = -\hbar$ in a coherent beam.

$$\begin{aligned}|\psi_2\rangle &= \frac{(P_{1x} + P_{0x} + P_{-1x})|\psi_1\rangle}{\sqrt{\langle\psi_1|(P_{1x} + P_{0x} + P_{-1x})|\psi_1\rangle}} = \frac{(P_{1x} + P_{0x} + P_{-1x})|1\rangle}{\sqrt{\langle 1|(P_{1x} + P_{0x} + P_{-1x})|1\rangle}} \\ P_{-1x}|1\rangle &\doteq \begin{pmatrix} 1/4 & -1/2\sqrt{2} & 1/4 \\ -1/2\sqrt{2} & 1/2 & -1/2\sqrt{2} \\ 1/4 & -1/2\sqrt{2} & 1/4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/4 \\ -1/2\sqrt{2} \\ 1/4 \end{pmatrix} \\ (P_{1x} + P_{0x} + P_{-1x})|1\rangle &\doteq \begin{pmatrix} 1/4 \\ 1/2\sqrt{2} \\ 1/4 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix} + \begin{pmatrix} 1/4 \\ -1/2\sqrt{2} \\ 1/4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ \langle 1|(P_{1x} + P_{0x} + P_{-1x})|1\rangle &\doteq \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1 \\ |\psi_2\rangle &\doteq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\end{aligned}$$

which is $|1\rangle$ as expected. The three probabilities are

$$\mathcal{P}_1 = |\langle 1 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right|^2 = 1$$

$$\mathcal{P}_0 = |\langle 0 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right|^2 = 0$$

$$\mathcal{P}_{-1} = |\langle -1 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right|^2 = 0$$

The cumulated results are

$ \psi_2\rangle$	$\mathcal{P}_1$	$\mathcal{P}_0$	$\mathcal{P}_{-1}$
$ 1\rangle_x$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
$ 0\rangle_x$	$\frac{1}{2}$	0	$\frac{1}{2}$
$ -1\rangle_x$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
$ 1\rangle_x \& 0\rangle_x$	$\frac{3}{4}$	$\frac{1}{6}$	$\frac{1}{12}$
$ 1\rangle_x \& -1\rangle_x$	$\frac{1}{2}$	0	$\frac{1}{2}$
$ 0\rangle_x \& -1\rangle_x$	$\frac{3}{4}$	$\frac{1}{6}$	$\frac{1}{12}$
$ 1\rangle_x \& 0\rangle_x \& -1\rangle_x$	1	0	0

2.22 a) The probability of measuring spin up at the 2nd Stern-Gerlach analyzer and spin down at the 3rd Stern-Gerlach analyzer is the product of the individual probabilities:

$$\mathcal{P}_{+\rightarrow+n\rightarrow-z} = \mathcal{P}_{+\rightarrow+n} \mathcal{P}_{+n\rightarrow-z} = |{}_n\langle + | + \rangle|^2 |{}_n\langle - | + \rangle|^2$$

The $|+\rangle_n$ eigenstate is

$$|+\rangle_n = \cos \frac{\theta}{2} |+\rangle + e^{i\phi} \sin \frac{\theta}{2} |-\rangle$$

so the probability is

$$\begin{aligned} \mathcal{P}_{+\rightarrow+n\rightarrow-z} &= \left| \left(\cos \frac{\theta}{2} \langle + | + \rangle + e^{-i\phi} \sin \frac{\theta}{2} \langle - | + \rangle \right) | + \rangle \right|^2 \left| \langle - | \left(\cos \frac{\theta}{2} |+\rangle + e^{i\phi} \sin \frac{\theta}{2} |-\rangle \right) \right|^2 \\ &= \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2} = \frac{1}{4} \sin^2 \theta \end{aligned}$$

b) To maximize the probability requires that $\theta = \pi/2$, and the probability is

$$\mathcal{P}_{+\rightarrow+n\rightarrow-z} = \frac{1}{4} \sin^2 \frac{\pi}{2} = \frac{1}{4}$$

c) If the 2nd Stern-Gerlach analyzer is removed, then the probability is

$$\mathcal{P}_{+ \rightarrow -z} = |\langle -|+\rangle|^2 = 0$$

because the two states are orthogonal.

2.23 (a) The commutator is

$$\begin{aligned} [A, B] &= AB - BA \doteq \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \begin{pmatrix} b_1 & 0 & 0 \\ 0 & 0 & b_2 \\ 0 & b_2 & 0 \end{pmatrix} - \begin{pmatrix} b_1 & 0 & 0 \\ 0 & 0 & b_2 \\ 0 & b_2 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \\ &\doteq \begin{pmatrix} a_1 b_1 & 0 & 0 \\ 0 & 0 & a_2 b_2 \\ 0 & a_3 b_2 & 0 \end{pmatrix} - \begin{pmatrix} a_1 b_1 & 0 & 0 \\ 0 & 0 & a_3 b_2 \\ 0 & a_2 b_2 & 0 \end{pmatrix} \\ &\doteq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & b_2(a_2 - a_3) \\ 0 & b_2(a_3 - a_2) & 0 \end{pmatrix} \neq 0 \end{aligned}$$

so they do not commute.

(b) A is already diagonal, so the eigenvalues and eigenvectors are obtained by inspection.

The eigenvalues are

$$a_1, a_2, a_3$$

and the eigenvectors are

$$|a_1\rangle = |1\rangle \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |a_2\rangle = |2\rangle \doteq \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |a_3\rangle = |3\rangle \doteq \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

For B , diagonalization yields the eigenvalues

$$\begin{aligned} \begin{pmatrix} b_1 - \lambda & 0 & 0 \\ 0 & -\lambda & b_2 \\ 0 & b_2 & -\lambda \end{pmatrix} = 0 &\Rightarrow (b_1 - \lambda)(\lambda^2 - b_2^2) = 0 \\ &\Rightarrow \lambda = b_1, b_2, -b_2 \end{aligned}$$

and the eigenvectors

$$\begin{pmatrix} b_1 & 0 & 0 \\ 0 & 0 & b_2 \\ 0 & b_2 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = b_1 \begin{pmatrix} u \\ v \\ w \end{pmatrix} \Rightarrow \begin{aligned} b_1 u &= b_1 u \\ b_2 w &= b_1 v \\ b_2 v &= b_1 w \end{aligned} \Rightarrow w = v = 0$$

$$|u|^2 + |v|^2 + |w|^2 = 1 \Rightarrow |u|^2 = 1 \Rightarrow u = 1, v = 0, w = 0 \Rightarrow |b_1\rangle = |1\rangle \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} b_1 & 0 & 0 \\ 0 & 0 & b_2 \\ 0 & b_2 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = b_2 \begin{pmatrix} u \\ v \\ w \end{pmatrix} \Rightarrow \begin{aligned} b_1 u &= b_2 u \\ b_2 w &= b_2 v \\ b_2 v &= b_2 w \end{aligned} \Rightarrow u = 0, w = v$$

$$\langle b_2 | b_2 \rangle = 1 \Rightarrow |v|^2 + |w|^2 = 1 \Rightarrow u = 0, v = \frac{1}{\sqrt{2}}, w = \frac{1}{\sqrt{2}} \Rightarrow |b_2\rangle = \frac{1}{\sqrt{2}}(|2\rangle + |3\rangle) \doteq \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\begin{pmatrix} b_1 & 0 & 0 \\ 0 & 0 & b_2 \\ 0 & b_2 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = -b_2 \begin{pmatrix} u \\ v \\ w \end{pmatrix} \Rightarrow \begin{aligned} b_1 u &= -b_2 u \\ b_2 w &= -b_2 v \\ b_2 v &= -b_2 w \end{aligned} \Rightarrow u = 0, w = -v$$

$$\langle -b_2 | -b_2 \rangle = 1 \Rightarrow |v|^2 + |w|^2 = 1 \Rightarrow u = 0, v = \frac{1}{\sqrt{2}}, w = -\frac{1}{\sqrt{2}} \Rightarrow |-b_2\rangle = \frac{1}{\sqrt{2}}(|2\rangle - |3\rangle) \doteq \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

c) If B is measured, the possible results are the allowed eigenvalues $b_1, b_2, -b_2$. If the initial state is $|\psi_i\rangle = |2\rangle$, then the probabilities are

$$\mathcal{P}_{b_1} = |\langle b_1 | \psi_i \rangle|^2 = |\langle 1 | 2 \rangle|^2 = 0$$

$$\mathcal{P}_{b_2} = |\langle b_2 | \psi_i \rangle|^2 = \left| \frac{1}{\sqrt{2}} (\langle 2 | + \langle 3 |) | 2 \rangle \right|^2 = \frac{1}{2}$$

$$\mathcal{P}_{-b_2} = |\langle -b_2 | \psi_i \rangle|^2 = \left| \frac{1}{\sqrt{2}} (\langle 2 | - \langle 3 |) | 2 \rangle \right|^2 = \frac{1}{2}$$

If A is then measured, the possible results are the allowed eigenvalues a_1, a_2, a_3 . If b_2 was the first result, then the new state is $|b_2\rangle$ and when A is measured the subsequent probabilities are

$$\mathcal{P}_{a_1} = |\langle a_1 | b_2 \rangle|^2 = \left| \langle 1 | \frac{1}{\sqrt{2}}(|2\rangle + |3\rangle) \rangle \right|^2 = 0$$

$$\mathcal{P}_{a_2} = |\langle a_2 | b_2 \rangle|^2 = \left| \langle 2 | \frac{1}{\sqrt{2}}(|2\rangle + |3\rangle) \rangle \right|^2 = \frac{1}{2}$$

$$\mathcal{P}_{a_3} = |\langle a_3 | b_2 \rangle|^2 = \left| \langle 3 | \frac{1}{\sqrt{2}}(|2\rangle + |3\rangle) \rangle \right|^2 = \frac{1}{2}$$

If $-b_2$ was the first result, then the new state is $|-b_2\rangle$ and when A is measured the subsequent probabilities are

$$\mathcal{P}_{a_1} = |\langle a_1 | -b_2 \rangle|^2 = \left| \langle 1 | \frac{1}{\sqrt{2}}(|2\rangle - |3\rangle) \rangle \right|^2 = 0$$

$$\mathcal{P}_{a_2} = |\langle a_2 | -b_2 \rangle|^2 = \left| \langle 2 | \frac{1}{\sqrt{2}}(|2\rangle - |3\rangle) \rangle \right|^2 = \frac{1}{2}$$

$$\mathcal{P}_{a_3} = |\langle a_3 | -b_2 \rangle|^2 = \left| \langle 3 | \frac{1}{\sqrt{2}}(|2\rangle - |3\rangle) \rangle \right|^2 = \frac{1}{2}$$

d) If two operators do not commute, then the corresponding observables cannot be measured simultaneously. Part (a) tells us that the operators A and B not commute. Part (c) tells us that measurement B "disturbs" the measurement of A so the two measurements are not compatible (cannot be made simultaneously).

2.24 (a) The eigenvalue equations for the S_z operator and the four eigenstates are

$$S_z |+\frac{3}{2}\rangle = +\frac{3}{2}\hbar |+\frac{3}{2}\rangle$$

$$S_z |+\frac{1}{2}\rangle = +\frac{1}{2}\hbar |+\frac{1}{2}\rangle$$

$$S_z |-\frac{1}{2}\rangle = -\frac{1}{2}\hbar |-\frac{1}{2}\rangle$$

$$S_z |-\frac{3}{2}\rangle = -\frac{3}{2}\hbar |-\frac{3}{2}\rangle$$

(b) The matrix representations of the S_z eigenstates are the unit vectors

$$|+\frac{3}{2}\rangle \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |+\frac{1}{2}\rangle \doteq \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |-\frac{1}{2}\rangle \doteq \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |-\frac{3}{2}\rangle \doteq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(c) The matrix representation of the S_z operator has the eigenvalues along the diagonal:

$$S_z \doteq \begin{pmatrix} +\frac{3}{2}\hbar & 0 & 0 & 0 \\ 0 & +\frac{1}{2}\hbar & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\hbar & 0 \\ 0 & 0 & 0 & -\frac{3}{2}\hbar \end{pmatrix}$$

(d) The eigenvalue equations for the $\mathbf{S}^2$ operator follow from the general equation $\mathbf{S}^2 |sm_s\rangle = s(s+1)\hbar^2 |sm_s\rangle$

Ch. 2 Solutions

$$\mathbf{S}^2 \left| +\frac{3}{2} \right\rangle = \frac{15}{4} \hbar^2 \left| +\frac{3}{2} \right\rangle$$

$$\mathbf{S}^2 \left| +\frac{1}{2} \right\rangle = \frac{15}{4} \hbar^2 \left| +\frac{1}{2} \right\rangle$$

$$\mathbf{S}^2 \left| -\frac{1}{2} \right\rangle = \frac{15}{4} \hbar^2 \left| -\frac{1}{2} \right\rangle$$

$$\mathbf{S}^2 \left| -\frac{3}{2} \right\rangle = \frac{15}{4} \hbar^2 \left| -\frac{3}{2} \right\rangle$$

where we have suppressed the s label.

(e) The matrix representation of the $\mathbf{S}^2$ operator has the eigenvalues along the diagonal:

$$\mathbf{S}^2 \doteq \begin{pmatrix} \frac{15}{4} \hbar^2 & 0 & 0 & 0 \\ 0 & \frac{15}{4} \hbar^2 & 0 & 0 \\ 0 & 0 & \frac{15}{4} \hbar^2 & 0 \\ 0 & 0 & 0 & \frac{15}{4} \hbar^2 \end{pmatrix}$$

2.25 The projection operators P_+ and P_- are represented by the matrices

$$P_+ \doteq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_- \doteq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The Hermitian adjoints of these matrices are obtained by transposing and complex conjugating them, yielding

$$P_+^\dagger \doteq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_-^\dagger \doteq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Since the Hermitian adjoints are equal to the original matrices, these operators are Hermitian.
