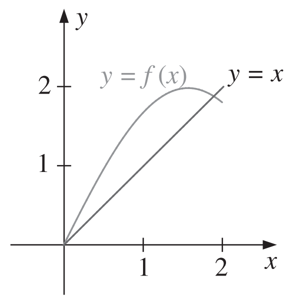
Solutions of Equations of One Variable

Exercise Set 2.1, page 54

1. $p_3 = 0.625$
2. (a) $p_3 = -0.6875$
   (b) $p_3 = 1.09375$
3. The Bisection method gives:
   (a) $p_7 = 0.5859$
   (b) $p_8 = 3.002$
   (c) $p_7 = 3.419$
4. The Bisection method gives:
   (a) $p_7 = -1.414$
   (b) $p_8 = 1.414$
   (c) $p_7 = 2.727$
   (d) $p_7 = -0.7265$
5. The Bisection method gives:
   (a) $p_{17} = 0.641182$
   (b) $p_{17} = 0.257530$
   (c) For the interval $[-3, -2]$, we have $p_{17} = -2.191307$, and for the interval $[-1, 0]$, we have $p_{17} = -0.798164$.
   (d) For the interval $[0.2, 0.3]$, we have $p_{14} = 0.297528$, and for the interval $[1.2, 1.3]$, we have $p_{14} = 1.256622$.
6. (a) $p_{17} = 1.51213837$
   (b) $p_{18} = 1.239707947$
   (c) For the interval $[1, 2]$, we have $p_{17} = 1.41239166$, and for the interval $[2, 4]$, we have $p_{18} = 3.05710602$.

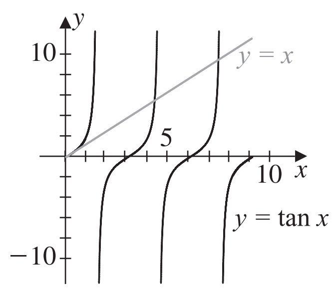
(d) For the interval $[0, 0.5]$, we have $p_{16} = 0.20603180$, and for the interval $[0.5, 1]$, we have $p_{16} = 0.68196869$.

7. (a)



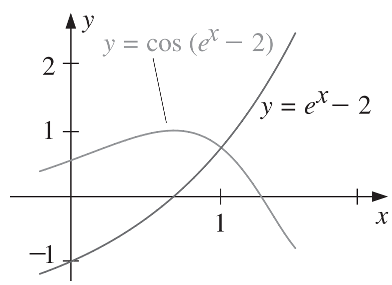
(b) Using $[1.5, 2]$ from part (a) gives $p_{16} = 1.89550018$.

8. (a)



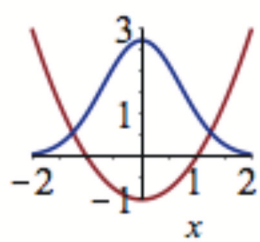
(b) Using $[4.2, 4.6]$ from part (a) gives $p_{16} = 4.4934143$.

9. (a)



(b) $p_{17} = 1.00762177$

10. (a)



(b) $p_{11} = -1.250976563$

11. (a) 2
(b) -2
(c) -1
(d) 1

12. (a) 0
(b) 0
(c) 2
(d) -2

13. The cube root of 25 is approximately $p_{14} = 2.92401$, using $[2, 3]$.

14. We have $\sqrt{3} \approx p_{14} = 1.7320$, using $[1, 2]$.

15. The depth of the water is 0.838 ft.

16. The angle θ changes at the approximate rate $w = -0.317059$.

17. A bound is $n \geq 14$, and $p_{14} = 1.32477$.

18. A bound for the number of iterations is $n \geq 12$ and $p_{12} = 1.3787$.

19. Since $\lim_{n\to\infty}(p_n - p_{n-1}) = \lim_{n\to\infty} 1/n = 0$, the difference in the terms goes to zero. However, p_n is the nth term of the divergent harmonic series, so $\lim_{n\to\infty} p_n = \infty$.

20. For $n > 1$,

$$|f(p_n)| = \left(\frac{1}{n}\right)^{10} \leq \left(\frac{1}{2}\right)^{10} = \frac{1}{1024} < 10^{-3},$$

so

$$|p - p_n| = \frac{1}{n} < 10^{-3} \Leftrightarrow 1000 < n.$$

21. Since $-1 < a < 0$ and $2 < b < 3$, we have $1 < a + b < 3$ or $1/2 < 1/2(a + b) < 3/2$ in all cases. Further,

$$\begin{aligned} f(x) < 0, &\quad \text{for } -1 < x < 0 \quad \text{and} \quad 1 < x < 2; \\ f(x) > 0, &\quad \text{for } \ 0 < x < 1 \quad \text{and} \quad 2 < x < 3. \end{aligned}$$

Thus, $a_1 = a$, $f(a_1) < 0$, $b_1 = b$, and $f(b_1) > 0$.

(a) Since $a + b < 2$, we have $p_1 = \frac{a+b}{2}$ and $1/2 < p_1 < 1$. Thus, $f(p_1) > 0$. Hence, $a_2 = a_1 = a$ and $b_2 = p_1$. The only zero of f in $[a_2, b_2]$ is $p = 0$, so the convergence will be to 0.

(b) Since $a + b > 2$, we have $p_1 = \frac{a+b}{2}$ and $1 < p_1 < 3/2$. Thus, $f(p_1) < 0$. Hence, $a_2 = p_1$ and $b_2 = b_1 = b$. The only zero of f in $[a_2, b_2]$ is $p = 2$, so the convergence will be to 2.

(c) Since $a + b = 2$, we have $p_1 = \frac{a+b}{2} = 1$ and $f(p_1) = 0$. Thus, a zero of f has been found on the first iteration. The convergence is to $p = 1$.

Exercise Set 2.2, page 64

1. For the value of x under consideration we have

   (a) $x=(3+x-2x^2)^{1/4} \Leftrightarrow x^4=3+x-2x^2 \Leftrightarrow f(x)=0$

   (b) $x=\left(\dfrac{x+3-x^4}{2}\right)^{1/2} \Leftrightarrow 2x^2=x+3-x^4 \Leftrightarrow f(x)=0$

   (c) $x=\left(\dfrac{x+3}{x^2+2}\right)^{1/2} \Leftrightarrow x^2(x^2+2)=x+3 \Leftrightarrow f(x)=0$

   (d) $x=\dfrac{3x^4+2x^2+3}{4x^3+4x-1} \Leftrightarrow 4x^4+4x^2-x=3x^4+2x^2+3 \Leftrightarrow f(x)=0$

2. (a) $p_4=1.10782$; (b) $p_4=0.987506$; (c) $p_4=1.12364$; (d) $p_4=1.12412$;

   (b) Part (d) gives the best answer since $|p_4-p_3|$ is the smallest for (d).

3. (a) Solve for $2x$ then divide by 2. $p_1=0.5625, p_2=0.58898926, p_3=0.60216264, p_4=0.60917204$

   (b) Solve for x^3, divide by x^2. $p_1=0, p_2$ undefined

   (c) Solve for x^3, divide by x, then take positive square root. $p_1=0, p_2$ undefined

   (d) Solve for x^3, then take negative of the cubed root. $p_1=0, p_2=-1, p_3=-1.4422496, p_4=-1.57197274$. Parts (a) and (d) seem promising.

4. (a) $x^4+3x^2-2=0 \Leftrightarrow 3x^2=2-x^4 \Leftrightarrow x=\sqrt{\frac{2-x^4}{3}}$; $p_0=1$, $p_1=0.577350269$,$p_2=0.79349204$,$p_3=0.73111023$, $p_4=0.75592901$.

   (b) $x^4+3x^2-2=0 \Leftrightarrow x^4=2-3x^2 \Leftrightarrow x=\sqrt[4]{2-3x^2}$; $p_0=1$, p_1 undefined.

   (c) $x^4+3x^2-2=0 \Leftrightarrow 3x^2=2-x^4 \Leftrightarrow x=\frac{2-x^4}{3x}$; $p_0=1$, $p_1=\frac{1}{3}$, $p_2=1.9876543$,$p_3=-2.2821844$, $p_4=3.6700326$.

   (d) $x^4+3x^2-2=0 \Leftrightarrow x^4=2-3x^2 \Leftrightarrow x^3=\frac{2-3x^2}{x} \Leftrightarrow x=\sqrt[3]{\frac{2-3x^2}{x}}$; $p_0=1$, $p_1=-1$, $p_2=1$, $p_3=-1$, $p_4=1$.

   Only the method of part (a) seems promising.

5. The order in descending speed of convergence is (b), (d), and (a). The sequence in (c) does not converge.

6. The sequence in (c) converges faster than in (d). The sequences in (a) and (b) diverge.

7. With $g(x)=(3x^2+3)^{1/4}$ and $p_0=1$, $p_6=1.94332$ is accurate to within 0.01.

8. With $g(x)=\sqrt{1+\frac{1}{x}}$ and $p_0=1$, we have $p_4=1.324$.

9. Since $g'(x)=\frac{1}{4}\cos\frac{x}{2}$, g is continuous and g' exists on $[0,2\pi]$. Further, $g'(x)=0$ only when $x=\pi$, so that $g(0)=g(2\pi)=\pi\le g(x)=\le g(\pi)=\pi+\frac{1}{2}$ and $|g'(x)|\le\frac{1}{4}$, for $0\le x\le 2\pi$. Theorem 2.3 implies that a unique fixed point p exists in $[0,2\pi]$. With $k=\frac{1}{4}$ and $p_0=\pi$, we have $p_1=\pi+\frac{1}{2}$. Corollary 2.5 implies that

$$|p_n-p|\le\frac{k^n}{1-k}|p_1-p_0|=\frac{2}{3}\left(\frac{1}{4}\right)^n.$$

For the bound to be less than 0.1, we need $n \geq 4$. However, $p_3 = 3.626996$ is accurate to within 0.01.

10. Using $p_0 = 1$ gives $p_{12} = 0.6412053$. Since $|g'(x)| = 2^{-x} \ln 2 \leq 0.551$ on $\left[\frac{1}{3}, 1\right]$ with $k = 0.551$, Corollary 2.5 gives a bound of 16 iterations.

11. For $p_0 = 1.0$ and $g(x) = 0.5(x + \frac{3}{x})$, we have $\sqrt{3} \approx p_4 = 1.73205$.

12. For $g(x) = 5/\sqrt{x}$ and $p_0 = 2.5$, we have $p_{14} = 2.92399$.

13. (a) With $[0, 1]$ and $p_0 = 0$, we have $p_9 = 0.257531$.
    (b) With $[2.5, 3.0]$ and $p_0 = 2.5$, we have $p_{17} = 2.690650$.
    (c) With $[0.25, 1]$ and $p_0 = 0.25$, we have $p_{14} = 0.909999$.
    (d) With $[0.3, 0.7]$ and $p_0 = 0.3$, we have $p_{39} = 0.469625$.
    (e) With $[0.3, 0.6]$ and $p_0 = 0.3$, we have $p_{48} = 0.448059$.
    (f) With $[0, 1]$ and $p_0 = 0$, we have $p_6 = 0.704812$.

14. The inequalities in Corollary 2.4 give $|p_n - p| < k^n \max(p_0 - a, b - p_0)$. We want

$$k^n \max(p_0 - a, b - p_0) < 10^{-5} \quad \text{so we need} \quad n > \frac{\ln(10^{-5}) - \ln(\max(p_0 - a, b - p_0))}{\ln k}.$$

    (a) Using $g(x) = 2 + \sin x$ we have $k = 0.9899924966$ so that with $p_0 = 2$ we have $n > \ln(0.00001)/\ln k = 1144.663221$. However, our tolerance is met with $p_{63} = 2.5541998$.
    (b) Using $g(x) = \sqrt[3]{2x + 5}$ we have $k = 0.1540802832$ so that with $p_0 = 2$ we have $n > \ln(0.00001)/\ln k = 6.155718005$. However, our tolerance is met with $p_6 = 2.0945503$.
    (c) Using $g(x) = \sqrt{e^x/3}$ and the interval $[0, 1]$ we have $k = 0.4759448347$ so that with $p_0 = 1$ we have $n > \ln(0.00001)/\ln k = 15.50659829$. However, our tolerance is met with $p_{12} = 0.91001496$.
    (d) Using $g(x) = \cos x$ and the interval $[0, 1]$ we have $k = 0.8414709848$ so that with $p_0 = 0$ we have $n > \ln(0.00001)/\ln k > 66.70148074$. However, our tolerance is met with $p_{30} = 0.73908230$.

15. For $g(x) = (2x^2 - 10 \cos x)/(3x)$, we have the following:

$$p_0 = 3 \Rightarrow p_8 = 3.16193; \quad p_0 = -3 \Rightarrow p_8 = -3.16193.$$

For $g(x) = \arccos(-0.1x^2)$, we have the following:

$$p_0 = 1 \Rightarrow p_{11} = 1.96882; \quad p_0 = -1 \Rightarrow p_{11} = -1.96882.$$

16. For $g(x) = \dfrac{1}{\tan x} - \dfrac{1}{x} + x$ and $p_0 = 4$, we have $p_4 = 4.493409$.

17. With $g(x) = \dfrac{1}{\pi} \arcsin\left(-\dfrac{x}{2}\right) + 2$, we have $p_5 = 1.683855$.

18. With $g(t) = 501.0625 - 201.0625 e^{-0.4t}$ and $p_0 = 5.0$, $p_3 = 6.0028$ is within 0.01 s of the actual time.

19. Since g' is continuous at p and $|g'(p)| > 1$, by letting $\epsilon = |g'(p)| - 1$ there exists a number $\delta > 0$ such that $|g'(x) - g'(p)| < |g'(p)| - 1$ whenever $0 < |x - p| < \delta$. Hence, for any x satisfying $0 < |x - p| < \delta$, we have

$$|g'(x)| \geq |g'(p)| - |g'(x) - g'(p)| > |g'(p)| - (|g'(p)| - 1) = 1.$$

If p_0 is chosen so that $0 < |p - p_0| < \delta$, we have by the Mean Value Theorem that

$$|p_1 - p| = |g(p_0) - g(p)| = |g'(\xi)||p_0 - p|,$$

for some ξ between p_0 and p. Thus, $0 < |p - \xi| < \delta$ so $|p_1 - p| = |g'(\xi)||p_0 - p| > |p_0 - p|$.

20. (a) If fixed-point iteration converges to the limit p, then

$$p = \lim_{n\to\infty} p_n = \lim_{n\to\infty} 2p_{n-1} - Ap_{n-1}^2 = 2p - Ap^2.$$

Solving for p gives $p = \dfrac{1}{A}$.

(b) Any subinterval $[c, d]$ of $\left(\dfrac{1}{2A}, \dfrac{3}{2A}\right)$ containing $\dfrac{1}{A}$ suffices.

Since

$$g(x) = 2x - Ax^2, \quad g'(x) = 2 - 2Ax,$$

so $g(x)$ is continuous, and $g'(x)$ exists. Further, $g'(x) = 0$ only if $x = \dfrac{1}{A}$.

Since

$$g\left(\frac{1}{A}\right) = \frac{1}{A}, \quad g\left(\frac{1}{2A}\right) = g\left(\frac{3}{2A}\right) = \frac{3}{4A}, \quad \text{and we have} \quad \frac{3}{4A} \leq g(x) \leq \frac{1}{A}.$$

For x in $\left(\frac{1}{2A}, \frac{3}{2A}\right)$, we have

$$\left|x - \frac{1}{A}\right| < \frac{1}{2A} \quad \text{so} \quad |g'(x)| = 2A\left|x - \frac{1}{A}\right| < 2A\left(\frac{1}{2A}\right) = 1.$$

21. One of many examples is $g(x) = \sqrt{2x - 1}$ on $\left[\frac{1}{2}, 1\right]$.

22. (a) The proof of existence is unchanged. For uniqueness, suppose p and q are fixed points in $[a, b]$ with $p \neq q$. By the Mean Value Theorem, a number ξ in (a, b) exists with

$$p - q = g(p) - g(q) = g'(\xi)(p - q) \leq k(p - q) < p - q,$$

giving the same contradiction as in Theorem 2.3.

(b) Consider $g(x) = 1 - x^2$ on $[0, 1]$. The function g has the unique fixed point

$$p = \frac{1}{2}\left(-1 + \sqrt{5}\right).$$

With $p_0 = 0.7$, the sequence eventually alternates between 0 and 1.

23. (a) Suppose that $x_0 > \sqrt{2}$. Then

$$x_1 - \sqrt{2} = g(x_0) - g\left(\sqrt{2}\right) = g'(\xi)\left(x_0 - \sqrt{2}\right),$$

where $\sqrt{2} < \xi < x$. Thus, $x_1 - \sqrt{2} > 0$ and $x_1 > \sqrt{2}$. Further,

$$x_1 = \frac{x_0}{2} + \frac{1}{x_0} < \frac{x_0}{2} + \frac{1}{\sqrt{2}} = \frac{x_0 + \sqrt{2}}{2}$$

and $\sqrt{2} < x_1 < x_0$. By an inductive argument,

$$\sqrt{2} < x_{m+1} < x_m < \ldots < x_0.$$

Thus, $\{x_m\}$ is a decreasing sequence which has a lower bound and must converge. Suppose $p = \lim_{m\to\infty} x_m$. Then

$$p = \lim_{m\to\infty}\left(\frac{x_{m-1}}{2} + \frac{1}{x_{m-1}}\right) = \frac{p}{2} + \frac{1}{p}. \quad \text{Thus} \quad p = \frac{p}{2} + \frac{1}{p},$$

which implies that $p = \pm\sqrt{2}$. Since $x_m > \sqrt{2}$ for all m, we have $\lim_{m\to\infty} x_m = \sqrt{2}$.

(b) We have

$$0 < \left(x_0 - \sqrt{2}\right)^2 = x_0^2 - 2x_0\sqrt{2} + 2,$$

so $2x_0\sqrt{2} < x_0^2 + 2$ and $\sqrt{2} < \frac{x_0}{2} + \frac{1}{x_0} = x_1$.

(c) Case 1: $0 < x_0 < \sqrt{2}$, which implies that $\sqrt{2} < x_1$ by part (b). Thus,

$$0 < x_0 < \sqrt{2} < x_{m+1} < x_m < \ldots < x_1 \quad \text{and} \quad \lim_{m\to\infty} x_m = \sqrt{2}.$$

Case 2: $x_0 = \sqrt{2}$, which implies that $x_m = \sqrt{2}$ for all m and $\lim_{m\to\infty} x_m = \sqrt{2}$.
Case 3: $x_0 > \sqrt{2}$, which by part (a) implies that $\lim_{m\to\infty} x_m = \sqrt{2}$.

24. (a) Let

$$g(x) = \frac{x}{2} + \frac{A}{2x}.$$

Note that $g\left(\sqrt{A}\right) = \sqrt{A}$. Also,

$$g'(x) = 1/2 - A/\left(2x^2\right) \text{ if } x \neq 0 \quad \text{and} \quad g'(x) > 0 \text{ if } x > \sqrt{A}.$$

If $x_0 = \sqrt{A}$, then $x_m = \sqrt{A}$ for all m and $\lim_{m\to\infty} x_m = \sqrt{A}$.
If $x_0 > A$, then

$$x_1 - \sqrt{A} = g(x_0) - g\left(\sqrt{A}\right) = g'(\xi)\left(x_0 - \sqrt{A}\right) > 0.$$

Further,

$$x_1 = \frac{x_0}{2} + \frac{A}{2x_0} < \frac{x_0}{2} + \frac{A}{2\sqrt{A}} = \frac{1}{2}\left(x_0 + \sqrt{A}\right).$$

Thus, $\sqrt{A} < x_1 < x_0$. Inductively,

$$\sqrt{A} < x_{m+1} < x_m < \ldots < x_0$$

and $\lim_{m\to\infty} x_m = \sqrt{A}$ by an argument similar to that in Exercise 23(a).
If $0 < x_0 < \sqrt{A}$, then

$$0 < \left(x_0 - \sqrt{A}\right)^2 = x_0^2 - 2x_0\sqrt{A} + A \quad \text{and} \quad 2x_0\sqrt{A} < x_0^2 + A,$$

which leads to

$$\sqrt{A} < \frac{x_0}{2} + \frac{A}{2x_0} = x_1.$$

Thus

$$0 < x_0 < \sqrt{A} < x_{m+1} < x_m < \ldots < x_1,$$

and by the preceding argument, $\lim_{m\to\infty} x_m = \sqrt{A}$.

(b) If $x_0 < 0$, then $\lim_{m\to\infty} x_m = -\sqrt{A}$.

25. Replace the second sentence in the proof with: "Since g satisfies a Lipschitz condition on $[a, b]$ with a Lipschitz constant $L < 1$, we have, for each n,

$$|p_n - p| = |g(p_{n-1}) - g(p)| \le L|p_{n-1} - p|."$$

The rest of the proof is the same, with k replaced by L.

26. Let $\varepsilon = (1 - |g'(p)|)/2$. Since g' is continuous at p, there exists a number $\delta > 0$ such that for $x \in [p-\delta, p+\delta]$, we have $|g'(x) - g'(p)| < \varepsilon$. Thus, $|g'(x)| < |g'(p)| + \varepsilon < 1$ for $x \in [p-\delta, p+\delta]$. By the Mean Value Theorem

$$|g(x) - g(p)| = |g'(c)||x - p| < |x - p|,$$

for $x \in [p - \delta, p + \delta]$. Applying the Fixed-Point Theorem completes the problem.

Exercise Set 2.3, page 75

1. $p_2 = 2.60714$

2. $p_2 = -0.865684$; If $p_0 = 0$, $f'(p_0) = 0$ and p_1 cannot be computed.

3. (a) 2.45454
   (b) 2.44444
   (c) Part (a) is better.

4. (a) -1.25208
   (b) -0.841355

5. (a) For $p_0 = 2$, we have $p_5 = 2.69065$.
   (b) For $p_0 = -3$, we have $p_3 = -2.87939$.
   (c) For $p_0 = 0$, we have $p_4 = 0.73909$.

(d) For $p_0 = 0$, we have $p_3 = 0.96434$.

6. (a) For $p_0 = 1$, we have $p_8 = 1.829384$.
   (b) For $p_0 = 1.5$, we have $p_4 = 1.397748$.
   (c) For $p_0 = 2$, we have $p_4 = 2.370687$; and for $p_0 = 4$, we have $p_4 = 3.722113$.
   (d) For $p_0 = 1$, we have $p_4 = 1.412391$; and for $p_0 = 4$, we have $p_5 = 3.057104$.
   (e) For $p_0 = 1$, we have $p_4 = 0.910008$; and for $p_0 = 3$, we have $p_9 = 3.733079$.
   (f) For $p_0 = 0$, we have $p_4 = 0.588533$; for $p_0 = 3$, we have $p_3 = 3.096364$; and for $p_0 = 6$, we have $p_3 = 6.285049$.

7. Using the endpoints of the intervals as p_0 and p_1, we have:
   (a) $p_{11} = 2.69065$
   (b) $p_7 = -2.87939$
   (c) $p_6 = 0.73909$
   (d) $p_5 = 0.96433$

8. Using the endpoints of the intervals as p_0 and p_1, we have:
   (a) $p_7 = 1.829384$
   (b) $p_9 = 1.397749$
   (c) $p_6 = 2.370687; p_7 = 3.722113$
   (d) $p_8 = 1.412391; p_7 = 3.057104$
   (e) $p_6 = 0.910008; p_{10} = 3.733079$
   (f) $p_6 = 0.588533; p_5 = 3.096364; p_5 = 6.285049$

9. Using the endpoints of the intervals as p_0 and p_1, we have:
   (a) $p_{16} = 2.69060$
   (b) $p_6 = -2.87938$
   (c) $p_7 = 0.73908$
   (d) $p_6 = 0.96433$

10. Using the endpoints of the intervals as p_0 and p_1, we have:
    (a) $p_8 = 1.829383$
    (b) $p_9 = 1.397749$
    (c) $p_6 = 2.370687; p_8 = 3.722112$
    (d) $p_{10} = 1.412392; p_{12} = 3.057099$
    (e) $p_7 = 0.910008; p_{29} = 3.733065$
    (f) $p_9 = 0.588533; p_5 = 3.096364; p_5 = 6.285049$

11. (a) Newton's method with $p_0 = 1.5$ gives $p_3 = 1.51213455$.
    The Secant method with $p_0 = 1$ and $p_1 = 2$ gives $p_{10} = 1.51213455$.
    The Method of False Position with $p_0 = 1$ and $p_1 = 2$ gives $p_{17} = 1.51212954$.

(b) Newton's method with $p_0 = 0.5$ gives $p_5 = 0.976773017$.
The Secant method with $p_0 = 0$ and $p_1 = 1$ gives $p_5 = 10.976773017$.
The Method of False Position with $p_0 = 0$ and $p_1 = 1$ gives $p_5 = 0.976772976$.

12. (a) We have

	Initial Approximation	Result	Initial Approximation	Result
Newton's	$p_0 = 1.5$	$p_4 = 1.41239117$	$p_0 = 3.0$	$p_4 = 3.05710355$
Secant	$p_0 = 1, p_1 = 2$	$p_8 = 1.41239117$	$p_0 = 2, p_1 = 4$	$p_{10} = 3.05710355$
False Position	$p_0 = 1, p_1 = 2$	$p_{13} = 1.41239119$	$p_0 = 2, p_1 = 4$	$p_{19} = 3.05710353$

(b) We have

	Initial Approximation	Result	Initial Approximation	Result
Newton's	$p_0 = 0.25$	$p_4 = 0.206035120$	$p_0 = 0.75$	$p_4 = 0.681974809$
Secant	$p_0 = 0, p_1 = 0.5$	$p_9 = 0.206035120$	$p_0 = 0.5, p_1 = 1$	$p_8 = 0.681974809$
False Position	$p_0 = 0, p_1 = 0.5$	$p_{12} = 0.206035125$	$p_0 = 0.5, p_1 = 1$	$p_{15} = 0.681974791$

13. (a) For $p_0 = -1$ and $p_1 = 0$, we have $p_{17} = -0.04065850$, and for $p_0 = 0$ and $p_1 = 1$, we have $p_9 = 0.9623984$.

(b) For $p_0 = -1$ and $p_1 = 0$, we have $p_5 = -0.04065929$, and for $p_0 = 0$ and $p_1 = 1$, we have $p_{12} = -0.04065929$.

(c) For $p_0 = -0.5$, we have $p_5 = -0.04065929$, and for $p_0 = 0.5$, we have $p_{21} = 0.9623989$.

14. (a) The Bisection method yields $p_{10} = 0.4476563$.

(b) The method of False Position yields $p_{10} = 0.442067$.

(c) The Secant method yields $p_{10} = -195.8950$.

15. Newton's method for the various values of p_0 gives the following results.

(a) $p_0 = -10, p_{11} = -4.30624527$

(b) $p_0 = -5, p_5 = -4.30624527$

(c) $p_0 = -3, p_5 = 0.824498585$

(d) $p_0 = -1, p_4 = -0.824498585$

(e) $p_0 = 0$, p_1 cannot be computed because $f'(0) = 0$

(f) $p_0 = 1, p_4 = 0.824498585$

(g) $p_0 = 3, p_5 = -0.824498585$

(h) $p_0 = 5, p_5 = 4.30624527$

(i) $p_0 = 10, p_{11} = 4.30624527$

16. Newton's method for the various values of p_0 gives the following results.

    (a) $p_8 = -1.379365$
    (b) $p_7 = -1.379365$
    (c) $p_7 = 1.379365$
    (d) $p_7 = -1.379365$
    (e) $p_7 = 1.379365$
    (f) $p_8 = 1.379365$

17. For $f(x) = \ln(x^2 + 1) - e^{0.4x} \cos \pi x$, we have the following roots.

    (a) For $p_0 = -0.5$, we have $p_3 = -0.4341431$.
    (b) For $p_0 = 0.5$, we have $p_3 = 0.4506567$.
        For $p_0 = 1.5$, we have $p_3 = 1.7447381$.
        For $p_0 = 2.5$, we have $p_5 = 2.2383198$.
        For $p_0 = 3.5$, we have $p_4 = 3.7090412$.
    (c) The initial approximation $n - 0.5$ is quite reasonable.
    (d) For $p_0 = 24.5$, we have $p_2 = 24.4998870$.

18. Newton's method gives $p_{15} = 1.895488$, for $p_0 = \frac{\pi}{2}$; and $p_{19} = 1.895489$, for $p_0 = 5\pi$. The sequence does not converge in 200 iterations for $p_0 = 10\pi$. The results do not indicate the fast convergence usually associated with Newton's method.

19. For $p_0 = 1$, we have $p_5 = 0.589755$. The point has the coordinates $(0.589755, 0.347811)$.

20. For $p_0 = 2$, we have $p_2 = 1.866760$. The point is $(1.866760, 0.535687)$.

21. The two numbers are approximately 6.512849 and 13.487151.

22. We have $\lambda \approx 0.100998$ and $N(2) \approx 2{,}187{,}950$.

23. The borrower can afford to pay at most 8.10%.

24. The minimal annual interest rate is 6.67%.

25. We have $P_L = 363432$, $c = -1.0266939$, and $k = 0.026504522$. The 1990 population is $P(30) = 248{,}319$, and the 2020 population is $P(60) = 300{,}528$.

26. We have $P_L = 446505$, $c = 0.91226292$, and $k = 0.014800625$. The 1990 population is $P(30) = 248{,}707$, and the 2020 population is $P(60) = 306{,}528$.

27. Using $p_0 = 0.5$ and $p_1 = 0.9$, the Secant method gives $p_5 = 0.842$.

28. (a) $\frac{1}{3}e, t = 3$ hours
    (b) 11 hours and 5 minutes
    (c) 21 hours and 14 minutes

29. (a) We have, approximately,

$$A = 17.74, \quad B = 87.21, \quad C = 9.66, \quad \text{and} \quad E = 47.47$$

With these values we have

$$A \sin\alpha\cos\alpha + B\sin^2\alpha - C\cos\alpha - E\sin\alpha = 0.02.$$

(b) Newton's method gives $\alpha \approx 33.2^\circ$.

30. This formula involves the subtraction of nearly equal numbers in both the numerator and denominator if p_{n-1} and p_{n-2} are nearly equal.

31. The equation of the tangent line is

$$y - f(p_{n-1}) = f'(p_{n-1})(x - p_{n-1}).$$

To complete this problem, set $y = 0$ and solve for $x = p_n$.

32. For some ξ_n between p_n and p,

$$f(p) = f(p_n) + (p - p_n)f'(p_n) + \frac{(p-p_n)^2}{2}f''(\xi_n)$$

$$0 = f(p_n) + (p - p_n)f'(p_n) + \frac{(p-p_n)^2}{2}f''(\xi_n)$$

Since $f'(p_n) \neq 0$,

$$0 = \frac{f(p_n)}{f'(p_n)} + p - p_n + \frac{(p-p_n)^2}{2f'(p_n)}f''(\xi_n)$$

we have

$$p - [p_n - \frac{f(p_n)}{f'(p_n)}] = -\frac{(p-p_n)^2}{2f'(p_n)}f''(\xi_n)$$

and

$$p - p_{n+1} = -\frac{(p-p_n)^2}{2f'(p_n)}f''(p_n).$$

So

$$|p - p_{n+1}| \leq \frac{M^2}{2|f'(p_n)|}(p - p_n)^2.$$

Exercise Set 2.4, page 85

1. (a) For $p_0 = 0.5$, we have $p_{13} = 0.567135$.
   (b) For $p_0 = -1.5$, we have $p_{23} = -1.414325$.
   (c) For $p_0 = 0.5$, we have $p_{22} = 0.641166$.
   (d) For $p_0 = -0.5$, we have $p_{23} = -0.183274$.

2. (a) For $p_0 = 0.5$, we have $p_{15} = 0.739076589$.
   (b) For $p_0 = -2.5$, we have $p_9 = -1.33434594$.
   (c) For $p_0 = 3.5$, we have $p_5 = 3.14156793$.
   (d) For $p_0 = 4.0$, we have $p_{44} = 3.37354190$.

3. Modified Newton's method in Eq. (2.11) gives the following:

   (a) For $p_0 = 0.5$, we have $p_3 = 0.567143$.
   (b) For $p_0 = -1.5$, we have $p_2 = -1.414158$.
   (c) For $p_0 = 0.5$, we have $p_3 = 0.641274$.
   (d) For $p_0 = -0.5$, we have $p_5 = -0.183319$.

4. (a) For $p_0 = 0.5$, we have $p_4 = 0.739087439$.
   (b) For $p_0 = -2.5$, we have $p_{53} = -1.33434594$.
   (c) For $p_0 = 3.5$, we have $p_5 = 3.14156793$.
   (d) For $p_0 = 4.0$, we have $p_3 = -3.72957639$.

5. Newton's method with $p_0 = -0.5$ gives $p_{13} = -0.169607$. Modified Newton's method in Eq. (2.11) with $p_0 = -0.5$ gives $p_{11} = -0.169607$.

6. (a) Since

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1,$$

   we have linear convergence. To have $|p_n - p| < 5 \times 10^{-2}$, we need $n \geq 20$.

   (b) Since

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = 1,$$

   we have linear convergence. To have $|p_n - p| < 5 \times 10^{-2}$, we need $n \geq 5$.

7. (a) For $k > 0$,

$$\lim_{n\to\infty} \frac{|p_{n+1} - 0|}{|p_n - 0|} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^k}}{\frac{1}{n^k}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^k = 1,$$

   so the convergence is linear.

   (b) We need to have $N > 10^{m/k}$.

8. (a) Since

$$\lim_{n\to\infty} \frac{|p_{n+1} - 0|}{|p_n - 0|^2} = \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{(10^{-2^n})^2} = \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{10^{-2^{n+1}}} = 1,$$

   the sequence is quadratically convergent.

(b) We have

$$\lim_{n\to\infty} \frac{|p_{n+1}-0|}{|p_n-0|^2} = \lim_{n\to\infty} \frac{10^{-(n+1)^k}}{\left(10^{-n^k}\right)^2} = \lim_{n\to\infty} \frac{10^{-(n+1)^k}}{10^{-2n^k}}$$
$$= \lim_{n\to\infty} 10^{2n^k-(n+1)^k} = \lim_{n\to\infty} 10^{n^k\left(2-\left(\frac{n+1}{n}\right)^k\right)} = \infty,$$

so the sequence $p_n = 10^{-n^k}$ does not converge quadratically.

9. Typical examples are

(a) $p_n = 10^{-3^n}$

(b) $p_n = 10^{-\alpha^n}$

10. Suppose $f(x) = (x-p)^m q(x)$. Since

$$g(x) = x - \frac{m(x-p)q(x)}{mq(x)+(x-p)q'(x)},$$

we have $g'(p) = 0$.

11. This follows from the fact that

$$\lim_{n\to\infty} \frac{\left|\dfrac{b-a}{2^{n+1}}\right|}{\left|\dfrac{b-a}{2^n}\right|} = \frac{1}{2}.$$

12. If f has a zero of multiplicity m at p, then f can be written as

$$f(x) = (x-p)^m q(x),$$

for $x \neq p$, where

$$\lim_{x\to p} q(x) \neq 0.$$

Thus,

$$f'(x) = m(x-p)^{m-1}q(x) + (x-p)^m q'(x)$$

and $f'(p) = 0$. Also,

$$f''(x) = m(m-1)(x-p)^{m-2}q(x) + 2m(x-p)^{m-1}q'(x) + (x-p)^m q''(x)$$

and $f''(p) = 0$. In general, for $k \leq m$,

$$f^{(k)}(x) = \sum_{j=0}^{k} \binom{k}{j} \frac{d^j(x-p)^m}{dx^j} q^{(k-j)}(x) = \sum_{j=0}^{k} \binom{k}{j} m(m-1)\cdots(m-j+1)(x-p)^{m-j}q^{(k-j)}(x).$$

Thus, for $0 \leq k \leq m-1$, we have $f^{(k)}(p) = 0$, but $f^{(m)}(p) = m! \lim_{x\to p} q(x) \neq 0$.

Conversely, suppose that

$$f(p) = f'(p) = \ldots = f^{(m-1)}(p) = 0 \quad \text{and} \quad f^{(m)}(p) \neq 0.$$

Consider the $(m-1)$th Taylor polynomial of f expanded about p:

$$\begin{aligned} f(x) =& f(p) + f'(p)(x-p) + \ldots + \frac{f^{(m-1)}(p)(x-p)^{m-1}}{(m-1)!} + \frac{f^{(m)}(\xi(x))(x-p)^m}{m!} \\ =& (x-p)^m \frac{f^{(m)}(\xi(x))}{m!}, \end{aligned}$$

where $\xi(x)$ is between x and p.

Since $f^{(m)}$ is continuous, let

$$q(x) = \frac{f^{(m)}(\xi(x))}{m!}.$$

Then $f(x) = (x-p)^m q(x)$ and

$$\lim_{x \to p} q(x) = \frac{f^{(m)}(p)}{m!} \neq 0.$$

Hence f has a zero of multiplicity m at p.

13. If

$$\frac{|p_{n+1}-p|}{|p_n-p|^3} = 0.75 \quad \text{and} \quad |p_0 - p| = 0.5, \quad \text{then} \quad |p_n - p| = (0.75)^{(3^n-1)/2} |p_0 - p|^{3^n}.$$

To have $|p_n - p| \leq 10^{-8}$ requires that $n \geq 3$.

14. Let $e_n = p_n - p$. If

$$\lim_{n \to \infty} \frac{|e_{n+1}|}{|e_n|^\alpha} = \lambda > 0,$$

then for sufficiently large values of n, $|e_{n+1}| \approx \lambda |e_n|^\alpha$. Thus,

$$|e_n| \approx \lambda |e_{n-1}|^\alpha \quad \text{and} \quad |e_{n-1}| \approx \lambda^{-1/\alpha} |e_n|^{1/\alpha}.$$

Using the hypothesis gives

$$\lambda |e_n|^\alpha \approx |e_{n+1}| \approx C |e_n| \lambda^{-1/\alpha} |e_n|^{1/\alpha}, \quad \text{so} \quad |e_n|^\alpha \approx C \lambda^{-1/\alpha - 1} |e_n|^{1+1/\alpha}.$$

Since the powers of $|e_n|$ must agree,

$$\alpha = 1 + 1/\alpha \;\; \text{and} \;\; \alpha = \frac{1+\sqrt{5}}{2} \approx 1.62.$$

The number α is the *golden ratio* that appeared in Exercise 11 of section 1.3.

Exercise Set 2.5, page 90

1. The results are listed in the following table.

	(a)	(b)	(c)	(d)
$\hat{p}_0$	0.258684	0.907859	0.548101	0.731385
$\hat{p}_1$	0.257613	0.909568	0.547915	0.736087
$\hat{p}_2$	0.257536	0.909917	0.547847	0.737653
$\hat{p}_3$	0.257531	0.909989	0.547823	0.738469
$\hat{p}_4$	0.257530	0.910004	0.547814	0.738798
$\hat{p}_5$	0.257530	0.910007	0.547810	0.738958

2. Newton's Method gives $p_{16} = -0.1828876$ and $\hat{p}_7 = -0.183387$.

3. Steffensen's method gives $p_0^{(1)} = 0.826427$.

4. Steffensen's method gives $p_0^{(1)} = 2.152905$ and $p_0^{(2)} = 1.873464$.

5. Steffensen's method gives $p_1^{(0)} = 1.5$.

6. Steffensen's method gives $p_2^{(0)} = 1.73205$.

7. For $g(x) = \sqrt{1 + \frac{1}{x}}$ and $p_0^{(0)} = 1$, we have $p_0^{(3)} = 1.32472$.

8. For $g(x) = 2^{-x}$ and $p_0^{(0)} = 1$, we have $p_0^{(3)} = 0.64119$.

9. For $g(x) = 0.5(x + \frac{3}{x})$ and $p_0^{(0)} = 0.5$, we have $p_0^{(4)} = 1.73205$.

10. For $g(x) = \frac{5}{\sqrt{x}}$ and $p_0^{(0)} = 2.5$, we have $p_0^{(3)} = 2.92401774$.

11. (a) For $g(x) = \left(2 - e^x + x^2\right)/3$ and $p_0^{(0)} = 0$, we have $p_0^{(3)} = 0.257530$.
    (b) For $g(x) = 0.5(\sin x + \cos x)$ and $p_0^{(0)} = 0$, we have $p_0^{(4)} = 0.704812$.
    (c) With $p_0^{(0)} = 0.25$, $p_0^{(4)} = 0.910007572$.
    (d) With $p_0^{(0)} = 0.3$, $p_0^{(4)} = 0.469621923$.

12. (a) For $g(x) = 2 + \sin x$ and $p_0^{(0)} = 2$, we have $p_0^{(4)} = 2.55419595$.
    (b) For $g(x) = \sqrt[3]{2x + 5}$ and $p_0^{(0)} = 2$, we have $p_0^{(2)} = 2.09455148$.
    (c) With $g(x) = \sqrt{\frac{e^x}{3}}$ and $p_0^{(0)} = 1$, we have $p_0^{(3)} = 0.910007574$.
    (d) With $g(x) = \cos x$, and $p_0^{(0)} = 0$, we have $p_0^{(4)} = 0.739085133$.

13. Aitken's Δ^2 method gives:
    (a) $\hat{p}_{10} = 0.0\overline{45}$
    (b) $\hat{p}_2 = 0.0363$

14. (a) A positive constant λ exists with

$$\lambda = \lim_{n \to \infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha}.$$

Hence

$$\lim_{n\to\infty}\left|\frac{p_{n+1}-p}{p_n-p}\right| = \lim_{n\to\infty}\frac{|p_{n+1}-p|}{|p_n-p|^\alpha}\cdot|p_n-p|^{\alpha-1} = \lambda\cdot 0 = 0 \quad\text{and}\quad \lim_{n\to\infty}\frac{p_{n+1}-p}{p_n-p} = 0.$$

(b) One example is $p_n = \frac{1}{n^n}$.

15. We have

$$\frac{|p_{n+1}-p_n|}{|p_n-p|} = \frac{|p_{n+1}-p+p-p_n|}{|p_n-p|} = \left|\frac{p_{n+1}-p}{p_n-p}-1\right|,$$

so

$$\lim_{n\to\infty}\frac{|p_{n+1}-p_n|}{|p_n-p|} = \lim_{n\to\infty}\left|\frac{p_{n+1}-p}{p_n-p}-1\right| = 1.$$

16.

$$\frac{\hat{p}_n-p}{p_n-p} = \frac{\lambda\left(\delta_n+\delta_{n+1}\right)-2\delta_n+\delta_n\delta_{n+1}-2\delta_n(\lambda-1)-\delta_n^2}{(\lambda-1)^2+\lambda\left(\delta_n+\delta_{n+1}\right)-2\delta_n+\delta_n\delta_{n+1}}$$

17. (a) Since $p_n = P_n(x) = \sum_{k=0}^{n}\frac{1}{k!}x^k$, we have

$$p_n - p = P_n(x) - e^x = \frac{-e^\xi}{(n+1)!}x^{n+1},$$

where ξ is between 0 and x. Thus, $p_n - p \neq 0$, for all $n \geq 0$. Further,

$$\frac{p_{n+1}-p}{p_n-p} = \frac{\frac{-e^{\xi_1}}{(n+2)!}x^{n+2}}{\frac{-e^{\xi}}{(n+1)!}x^{n+1}} = \frac{e^{(\xi_1-\xi)}x}{n+2},$$

where ξ_1 is between 0 and 1. Thus, $\lambda = \lim_{n\to\infty}\frac{e^{(\xi_1-\xi)}x}{n+2} = 0 < 1$.

(b)

n	p_n	$\hat{p}_n$
0	1	3
1	2	2.75
2	2.5	$2.7\overline{2}$
3	$2.\overline{6}$	2.71875
4	$2.708\overline{3}$	$2.718\overline{3}$
5	$2.71\overline{6}$	2.7182870
6	$2.7180\overline{5}$	2.7182823
7	2.7182539	2.7182818
8	2.7182787	2.7182818
9	2.7182815	
10	2.7182818	

(c) Aitken's Δ^2 method gives quite an improvement for this problem. For example, $\hat{p}_6$ is accurate to within 5×10^{-7}. We need p_{10} to have this accuracy.

Exercise Set 2.6, page 100

1. (a) For $p_0 = 1$, we have $p_{22} = 2.69065$.

   (b) For $p_0 = 1$, we have $p_5 = 0.53209$; for $p_0 = -1$, we have $p_3 = -0.65270$; and for $p_0 = -3$, we have $p_3 = -2.87939$.

   (c) For $p_0 = 1$, we have $p_5 = 1.32472$.

   (d) For $p_0 = 1$, we have $p_4 = 1.12412$; and for $p_0 = 0$, we have $p_8 = -0.87605$.

   (e) For $p_0 = 0$, we have $p_6 = -0.47006$; for $p_0 = -1$, we have $p_4 = -0.88533$; and for $p_0 = -3$, we have $p_4 = -2.64561$.

   (f) For $p_0 = 0$, we have $p_{10} = 1.49819$.

2. (a) For $p_0 = 0$, we have $p_9 = -4.123106$; and for $p_0 = 3$, we have $p_6 = 4.123106$. The complex roots are $-2.5 \pm 1.322879i$.

   (b) For $p_0 = 1$, we have $p_7 = -3.548233$; and for $p_0 = 4$, we have $p_5 = 4.38111$. The complex roots are $0.5835597 \pm 1.494188i$.

   (c) The only roots are complex, and they are $\pm\sqrt{2}i$ and $-0.5 \pm 0.5\sqrt{3}i$.

   (d) For $p_0 = 1$, we have $p_5 = -0.250237$; for $p_0 = 2$, we have $p_5 = 2.260086$; and for $p_0 = -11$, we have $p_6 = -12.612430$. The complex roots are $-0.1987094 \pm 0.8133125i$.

   (e) For $p_0 = 0$, we have $p_8 = 0.846743$; and for $p_0 = -1$, we have $p_9 = -3.358044$. The complex roots are $-1.494350 \pm 1.744219i$.

   (f) For $p_0 = 0$, we have $p_8 = 2.069323$; and for $p_0 = 1$, we have $p_3 = 0.861174$. The complex roots are $-1.465248 \pm 0.8116722i$.

   (g) For $p_0 = 0$, we have $p_6 = -0.732051$; for $p_0 = 1$, we have $p_4 = 1.414214$; for $p_0 = 3$, we have $p_5 = 2.732051$; and for $p_0 = -2$, we have $p_6 = -1.414214$.

   (h) For $p_0 = 0$, we have $p_5 = 0.585786$; for $p_0 = 2$, we have $p_2 = 3$; and for $p_0 = 4$, we have $p_6 = 3.414214$.

3. The following table lists the initial approximation and the roots.

	p_0	p_1	p_2	Approximate roots	Complex Conjugate roots
(a)	-1	0	1	$p_7 = -0.34532 - 1.31873i$	$-0.34532 + 1.31873i$
	0	1	2	$p_6 = 2.69065$	
(b)	0	1	2	$p_6 = 0.53209$	
	1	2	3	$p_9 = -0.65270$	
	-2	-3	-2.5	$p_4 = -2.87939$	
(c)	0	1	2	$p_5 = 1.32472$	
	-2	-1	0	$p_7 = -0.66236 - 0.56228i$	$-0.66236 + 0.56228i$
(d)	0	1	2	$p_5 = 1.12412$	
	2	3	4	$p_{12} = -0.12403 + 1.74096i$	$-0.12403 - 1.74096i$
	-2	0	-1	$p_5 = -0.87605$	
(e)	0	1	2	$p_{10} = -0.88533$	
	1	0	-0.5	$p_5 = -0.47006$	
	-1	-2	-3	$p_5 = -2.64561$	
(f)	0	1	2	$p_6 = 1.49819$	
	-1	-2	-3	$p_{10} = -0.51363 - 1.09156i$	$-0.51363 + 1.09156i$
	1	0	-1	$p_8 = 0.26454 - 1.32837i$	$0.26454 + 1.32837i$

4. The following table lists the initial approximation and the roots.

	p_0	p_1	p_2	Approximate roots	Complex Conjugate roots
(a)	0	1	2	$p_{11} = -2.5 - 1.322876i$	$-2.5 + 1.322876i$
	1	2	3	$p_6 = 4.123106$	
	-3	-4	-5	$p_5 = -4.123106$	
(b)	0	1	2	$p_7 = 0.583560 - 1.494188i$	$0.583560 + 1.494188i$
	2	3	4	$p_6 = 4.381113$	
	-2	-3	-4	$p_5 = -3.548233$	
(c)	0	1	2	$p_{11} = 1.414214i$	$-1.414214i$
	-1	-2	-3	$p_{10} = -0.5 + 0.866025i$	$-0.5 - 0.866025i$
(d)	0	1	2	$p_7 = 2.260086$	
	3	4	5	$p_{14} = -0.198710 + 0.813313i$	$-0.198710 + 0.813313i$
	11	12	13	$p_{22} = -0.250237$	
	-9	-10	-11	$p_6 = -12.612430$	
(e)	0	1	2	$p_6 = 0.846743$	
	3	4	5	$p_{12} = -1.494349 + 1.744218i$	$-1.494349 - 1.744218i$
	-1	-2	-3	$p_7 = -3.358044$	
(f)	0	1	2	$p_6 = 2.069323$	
	-1	0	1	$p_5 = 0.861174$	
	-1	-2	-3	$p_8 = -1.465248 + 0.811672i$	$-1.465248 - 0.811672i$
(g)	0	1	2	$p_6 = 1.414214$	
	-2	-1	0	$p_7 = -0.732051$	
	0	-2	-1	$p_7 = -1.414214$	
	2	3	4	$p_6 = 2.732051$	
(h)	0	1	2	$p_8 = 3$	
	-1	0	1	$p_5 = 0.585786$	
	2.5	3.5	4	$p_6 = 3.414214$	

5. (a) The roots are 1.244, 8.847, and -1.091, and the critical points are 0 and 6.
   (b) The roots are 0.5798, 1.521, 2.332, and -2.432, and the critical points are 1, 2.001, and -1.5.
6. We get convergence to the root 0.27 with $p_0 = 0.28$. We need p_0 closer to 0.29 since $f'(0.28\overline{3}) = 0$.
7. The methods all find the solution 0.23235.
8. The width is approximately $W = 16.2121$ ft.
9. The minimal material is approximately 573.64895 cm^2.
10. Fibonacci's answer was 1.3688081078532, and Newton's Method gives 1.36880810782137 with a tolerance of 10^{-16}, so Fibonacci's answer is within 4×10^{-11}. This accuracy is amazing for the time.