

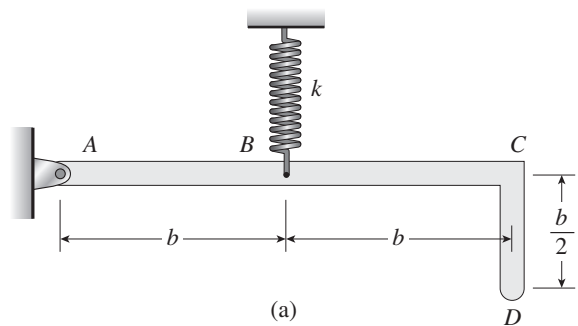
2

Axially Loaded Members

Changes in Lengths of Axially Loaded Members

Problem 2.2-1 The L-shaped arm $ABCD$ shown in the figure lies in a vertical plane and pivots about a horizontal pin at A . The arm has constant cross-sectional area and total weight W . A vertical spring of stiffness k supports the arm at point B .

- Obtain a formula for the elongation of the spring due to the weight of the arm.
- Repeat part (a) if the pin support at A is moved to D .

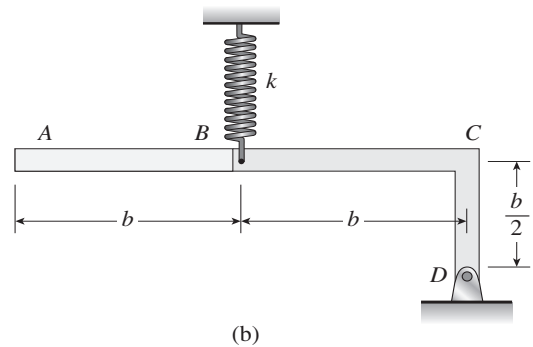


Solution 2.2-1

- SUM MOMENTS ABOUT A

$$\Sigma M_A = 0 \quad \frac{2b}{5}Wb + \frac{b}{2}W(2b)k\delta b$$

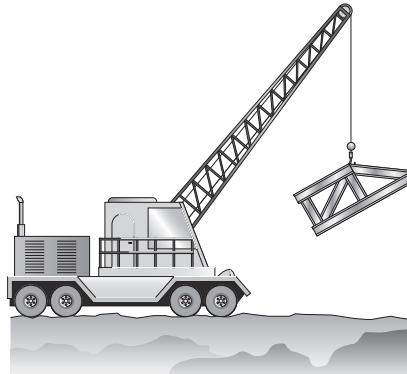
$$\delta = \frac{\frac{2b}{5}Wb + \frac{b}{2}W(2b)}{kb} = \frac{6W}{5k}$$



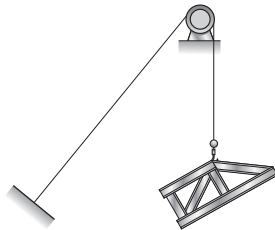
$$(b) \Sigma M_D = 0 \quad kb\delta = \frac{2b}{5}Wb = \frac{4Wb}{5} \quad \text{so} \quad \delta = \frac{\frac{2b}{5}Wb}{kb} = \frac{4W}{5k}$$

Problem 2.2-2 A steel cable with nominal diameter 25 mm (see Table 2-1) is used in a construction yard to lift a bridge section weighing 38 kN, as shown in the figure. The cable has an effective modulus of elasticity $E = 140$ GPa.

- If the cable is 14 m long, how much will it stretch when the load is picked up?
- If the cable is rated for a maximum load of 70 kN, what is the factor of safety with respect to failure of the cable?



Solution 2.2-2 Bridge section lifted by a cable



$A = 304 \text{ mm}^2$ (from
Table 2-1)
 $W = 38 \text{ kN}$
 $E = 140 \text{ GPa}$
 $L = 14 \text{ m}$

(b) FACTOR OF SAFETY

$P_{ULT} = 406 \text{ kN}$ (from Table 2-1)

$P_{max} = 70 \text{ kN}$

$$n = \frac{P_{ULT}}{P_{max}} = \frac{406 \text{ kN}}{70 \text{ kN}} = 5.8 \quad \leftarrow$$

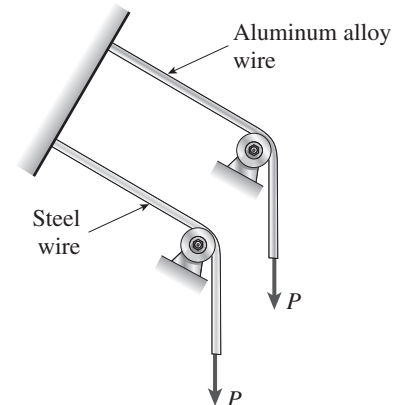
(a) STRETCH OF CABLE

$$\delta = \frac{WL}{EA} = \frac{(38 \text{ kN})(14 \text{ m})}{(140 \text{ GPa})(304 \text{ mm}^2)}$$

$$= 12.5 \text{ mm} \quad \leftarrow$$

Problem 2.2-3 A steel wire and an aluminum alloy wire have equal lengths and support equal loads P (see figure). The moduli of elasticity for the steel and aluminum alloy are $E_s = 30,000$ ksi and $E_a = 11,000$ ksi, respectively.

- If the wires have the same diameters, what is the ratio of the elongation of the aluminum alloy wire to the elongation of the steel wire?
- If the wires stretch the same amount, what is the ratio of the diameter of the aluminum alloy wire to the diameter of the steel wire?
- If the wires have the same diameters and same load P , what is the ratio of the initial length of the aluminum alloy wire to that of the steel wire if the aluminum alloy wire stretches 1.5 times that of the steel wire?
- If the wires have the same diameters, same initial length, and same load P , what is the material of the upper wire if it elongates 1.7 times that of the steel wire?



Solution 2.2-3

$$(a) \frac{\delta_a}{\delta_s} = \frac{\frac{PL}{E_a A}}{\left(\frac{PL}{E_s A}\right)} \rightarrow \frac{E_s}{E_a}$$

$$E_s = 30,000 \text{ ksi} \quad E_a = 11,000 \text{ ksi}$$

$$\boxed{\frac{E_s}{E_a} = 2.727} \quad \frac{30}{11} = 2.727$$

$$(b) \delta_a = \delta_s \quad \text{so} \quad \frac{PL}{E_a A_a} = \frac{PL}{E_s A_s} \quad \text{so} \quad \frac{A_a}{A_s} = \frac{E_s}{E_a} \quad \text{and} \quad \boxed{\frac{d_a}{d_s} = \sqrt{\frac{E_s}{E_a}} = 1.651}$$

- (c) SAME DIAMETER, SAME LOAD, FIND RATIO OF LENGTH OF ALUMINUM TO STEEL WIRE IF ELONGATION OF ALUMINUM IS 1.5 TIMES THAT OF STEEL WIRE

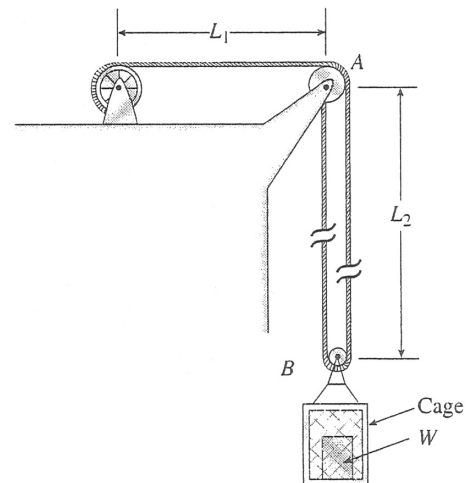
$$\frac{\delta_a}{\delta_s} = \frac{\frac{PL_a}{E_a A}}{\left(\frac{PL_s}{E_s A}\right)} = \frac{\frac{PL_a}{E_a A}}{\left(\frac{PL_s}{E_s A}\right)} = 1.5 \quad \boxed{\frac{L_a}{L_s} = 1.5 \frac{E_a}{E_s} = 0.55}$$

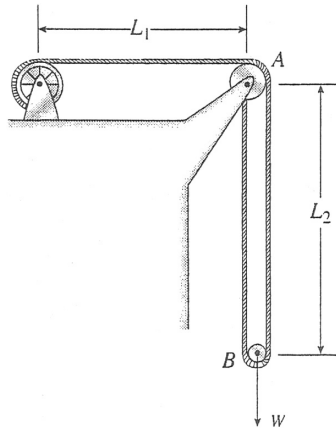
- (d) SAME DIAMETER, SAME LENGTH, SAME LOAD—BUT WIRE 1 ELONGATION 1.7 TIMES THE STEEL WIRE > WHAT IS WIRE 1 MATERIAL?

$$\frac{\delta_1}{\delta_s} = \frac{\frac{PL}{E_1 A}}{\left(\frac{PL}{E_s A}\right)} = \frac{\frac{PL}{E_1 A}}{\left(\frac{PL}{E_s A}\right)} = 1.7 \quad E_1 = \frac{E_s}{1.7} = 17,647 \text{ ksi} \quad \boxed{< \text{cast iron or copper alloy (see App. I)}}$$

Problem 2.2-4 By what distance h does the cage shown in the figure move downward when the weight W is placed inside it?

Consider only the effects of the stretching of the cable, which has axial rigidity $EA = 10,700 \text{ kN}$. The pulley at A has diameter $d_A = 300 \text{ mm}$ and the pulley at B has diameter $d_B = 150 \text{ mm}$. Also, the distance $L_1 = 4.6 \text{ m}$, the distance $L_2 = 10.5 \text{ m}$, and the weight $W = 22 \text{ kN}$. (Note: When calculating the length of the cable, include the parts of the cable that go around the pulleys at A and B .)



Solution 2.2-4 Cage supported by a cable

$$\begin{aligned}d_A &= 300 \text{ mm} \\d_B &= 150 \text{ mm} \\L_1 &= 4.6 \text{ m} \\L_2 &= 10.5 \text{ m} \\EA &= 10,700 \text{ kN} \\W &= 22 \text{ kN}\end{aligned}$$

TENSILE FORCE IN CABLE

$$T = \frac{W}{2} = 11 \text{ kN}$$

LENGTH OF CABLE

$$\begin{aligned}L &= L_1 + 2L_2 + \frac{1}{4}(\pi d_A) + \frac{1}{2}(\pi d_B) \\&= 4,600 \text{ mm} + 21,000 \text{ mm} + 236 \text{ mm} + 236 \text{ mm} \\&= 26,072 \text{ mm}\end{aligned}$$

ELONGATION OF CABLE

$$\delta = \frac{TL}{EA} = \frac{(11 \text{ kN})(26,072 \text{ mm})}{(10,700 \text{ kN})} = 26.8 \text{ mm}$$

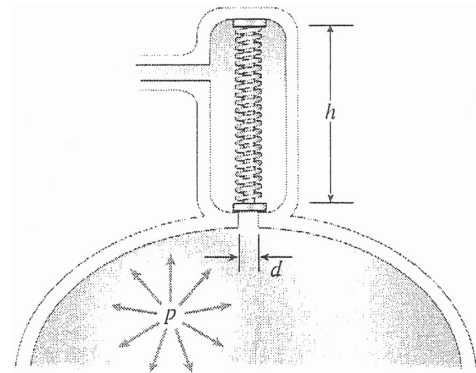
LOWERING OF THE CAGE

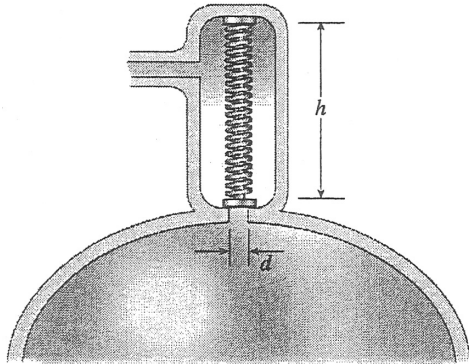
 h = distance the cage moves downward

$$h = \frac{1}{2}\delta = 13.4 \text{ mm} \quad \leftarrow$$

Problem 2.2-5 A safety valve on the top of a tank containing steam under pressure p has a discharge hole of diameter d (see figure). The valve is designed to release the steam when the pressure reaches the value p_{\max} .

If the natural length of the spring is L and its stiffness is k , what should be the dimension h of the valve? (Express your result as a formula for h .)



Solution 2.2-5 Safety valve

h = height of valve (compressed length of the spring)

d = diameter of discharge hole

p = pressure in tank

p_{\max} = pressure when valve opens

L = natural length of spring ($L > h$)

k = stiffness of spring

FORCE IN COMPRESSED SPRING

$$F = k(L - h) \text{ (From Eq. 2-1a)}$$

PRESSURE FORCE ON SPRING

$$P = p_{\max} \left(\frac{\pi d^2}{4} \right)$$

EQUATE FORCES AND SOLVE FOR h :

$$F = P \quad k(L - h) = \frac{\pi p_{\max} d^2}{4}$$

$$h = L - \frac{\pi p_{\max} d^2}{4k} \quad \leftarrow$$

Problem 2.2-6 The device shown in the figure consists of a prismatic rigid pointer ABC supported by a uniform translational spring of stiffness $k = 950 \text{ N/m}$. The spring is positioned at distance $b = 165 \text{ mm}$ from the pinned end A of the pointer. The device is adjusted so that when there is no load P , the pointer reads zero on the angular scale.

- If the load $P = 11 \text{ N}$, at what distance x should the load be placed so that the pointer will read $\theta = 2.5^\circ$ on the scale (see figure part a)?
- Repeat part (a) if a rotational spring $k_r = kb^2$ is added at A (see figure part b).
- Let $x = 7b/8$. What is P_{\max} (N) if θ cannot exceed 2° ? Include spring k_r in your analysis.
- Now, if the weight of the pointer ABC is known to be $W_p = 3 \text{ N}$ and the weight of the spring is $W_s = 2.75 \text{ N}$, what initial angular position (i.e., θ in degrees) of the pointer will result in a zero reading on the angular scale once the pointer is released from rest? Assume $P = k_r = 0$.
- If the pointer is rotated to a vertical position (see figure part c), find the required load P , applied at mid-height of the pointer, that will result in a pointer reading of $\theta = 2.5^\circ$ on the scale. Consider the weight of the pointer W_p in your analysis.

Solution 2.2-6

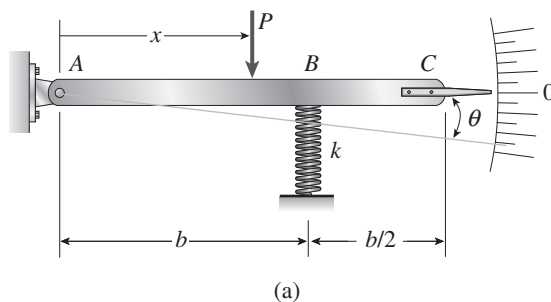
NUMERICAL DATA $k = 950 \text{ N/m}$ $b = 165 \text{ mm}$ $P = 11 \text{ N}$ $\theta = 2.5^\circ$ $\theta_{\max} = 2^\circ$

$$W_p = 3 \text{ N} \quad W_s = 2.75 \text{ N}$$

- (a) If the load $P = 11 \text{ N}$, at what distance x should the load be placed so that the pointer will read $\theta = 2.5^\circ$ on the scale (see Fig. a)?

Sum moments about A, then solve for x :

$$x = \frac{k \theta b^2}{P} = 102.6 \text{ mm} \quad \boxed{x = 102.6 \text{ mm}}$$

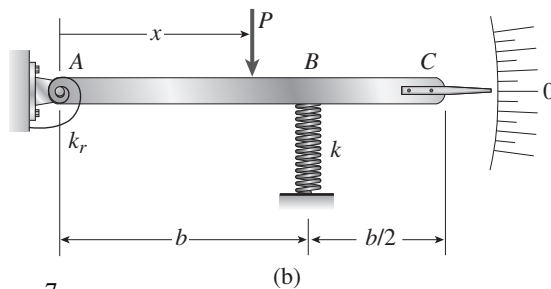


- (b) Repeat (a) if a rotational spring $k_r = kb^2$ is added at A (see Fig. b).

$$k_r = k b^2 = 25864 \text{ N}\cdot\text{mm}$$

Sum moments about A, then solve for x :

$$x = \frac{k \theta b^2 + k_r \theta}{P} = 205 \text{ mm} \quad \frac{x}{b} = 1.244 \quad \boxed{x = 205 \text{ mm}}$$



- (c) Now if $x = 7b/8$, what is P_{\max} (N) if θ cannot exceed 2° ? $x = \frac{7}{8} b = 144.375 \text{ mm}$

Sum moments about A, then solve for P :
$$P_{\max} = \frac{k \theta_{\max} b^2 + k_r \theta_{\max}}{\frac{7}{8} b} = 12.51 \text{ N} \quad \boxed{P_{\max} = 12.51 \text{ N}}$$

- (d) Now, if the weight of the pointer ABC is known to be $W_p = 3 \text{ N}$ and the weight of the spring is $W_s = 2.75 \text{ N}$, what initial angular position (i.e., θ in degrees) of the pointer will result in a zero reading on the angular scale once the pointer is released from rest? Assume $P = k_r = 0$.

Deflection at spring due to W_p :

$$\delta_{Bp} = \frac{W_p \left(\frac{3}{4} b \right)}{k b} = 2.368 \text{ mm}$$

Deflection at B due to self weight of spring:

$$\delta_{Bk} = \frac{W_s}{2k} = 1.447 \text{ mm}$$

$$\delta_B = \delta_{Bp} + \delta_{Bk} = 3.816 \text{ mm} \quad \theta_{\text{init}} = \frac{\delta_B}{b} = 1.325^\circ$$

$$\text{OR } \theta_{\text{init}} = \arctan \left(\frac{\delta_B}{b} \right) = 1.325^\circ \quad \boxed{\theta_{\text{init}} = 1.325^\circ}$$

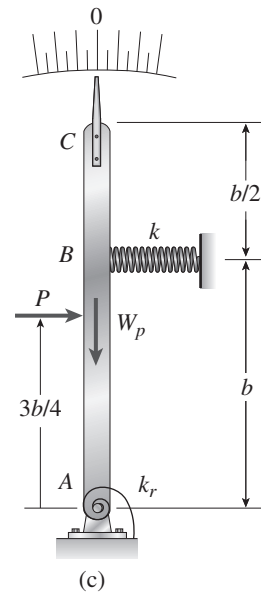
- (e) If the pointer is rotated to a vertical position (figure part c), find the required load P , applied at mid-height of the pointer that will result in a pointer reading of $\theta = 2.5^\circ$ on the scale. Consider the weight of the pointer, W_p , in your analysis.

$$k = 950 \text{ N/m} \quad b = 165 \text{ mm} \quad W_p = 3 \text{ N}$$

$$k_r = kb^2 = 25.864 \text{ N}\cdot\text{m} \quad \theta = 2.5^\circ$$

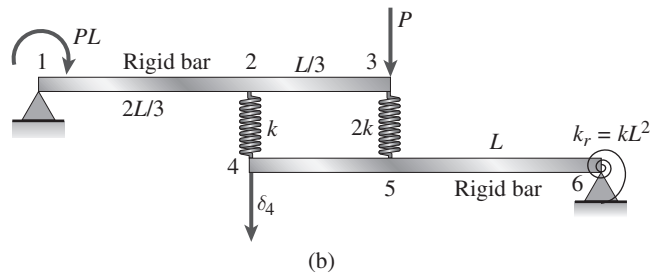
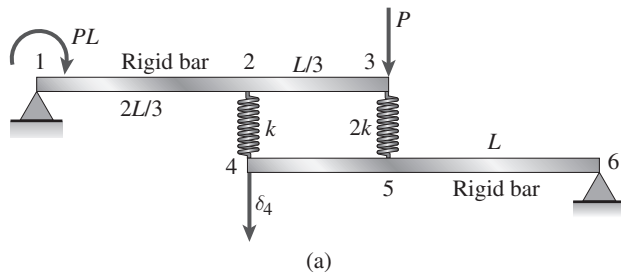
Sum moments about A to get P :

$$P = \frac{\theta}{\left(\frac{3b}{4}\right)} \left[k_r + k \left(\frac{5}{4}b^2 \right) - W_p \left(\frac{3b}{4} \right) \right] = 20.388 \text{ N} \quad \boxed{P = 20.4 \text{ N}}$$



Problem 2.2-7 Two rigid bars are connected to each other by two linearly elastic springs. Before loads are applied, the lengths of the springs are such that the bars are parallel and the springs are without stress.

- (a) Derive a formula for the displacement δ_4 at point 4 when the load P is applied at joint 3 and moment PL is applied at joint 1, as shown in the figure part a. (Assume that the bars rotate through very small angles under the action of the load P .)
- (b) Repeat part (a) if a rotational spring, $k_r = kL^2$, is now added at joint 6. What is the ratio of the deflection δ_4 in the figure part a to that in the figure part b?



Solution 2.2-7

- (a) Derive a formula for the displacement δ_4 at point 4 when the load P is applied at joint 3 and moment PL is applied at joint 1, as shown.

Cut horizontally through both springs to create upper and lower FBD's. Sum moments about joint 1 for upper FBD and also sum moments about joint 6 for lower FBD to get two equations of equilibrium; assume both springs are in tension.

Note that $\delta_2 = \frac{2}{3} \delta_3$ and $\delta_5 = \frac{3}{4} \delta_4$

Force in left spring: $k\left(\delta_4 - \frac{2}{3} \delta_3\right)$

Force in right spring: $2k\left(\frac{3}{4} \delta_4 - \delta_3\right)$

Summing moments about joint 1 (upper FBD) and about joint 6 (lower FBD) then dividing through by k gives

$$\begin{pmatrix} \frac{-22}{9} & \frac{13}{6} \\ \frac{-26}{9} & \frac{17}{6} \end{pmatrix} \begin{pmatrix} \delta_3 \\ \delta_4 \end{pmatrix} = \begin{pmatrix} \frac{-2P}{k} \\ 0 \end{pmatrix} \quad \begin{pmatrix} \delta_3 \\ \delta_4 \end{pmatrix} = \begin{pmatrix} \frac{-22}{9} & \frac{13}{6} \\ \frac{-26}{9} & \frac{17}{6} \end{pmatrix}^{-1} \begin{pmatrix} \frac{-2P}{k} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{17P}{2k} \\ \frac{26P}{3k} \end{pmatrix} \quad \begin{matrix} \frac{17}{2} = 8.5 \\ \frac{26}{3} = 8.667 \end{matrix} \quad \boxed{\delta_4 = \frac{26P}{3k}}$$

^ deltas are **positive downward**

- (b) Repeat part (a) if a rotational spring $k_r = kL^2$ is now added at joint 6. What is the ratio of the deflection δ_4 in part (a) to that in (b)?

Upper FBD—sum moments about joint 1:

$$k\left(\delta_4 - \frac{2}{3} \delta_3\right) \frac{2L}{3} + 2k\left(\frac{3}{4} \delta_4 - \delta_3\right)L = -2PL \quad \text{OR} \quad \left(\frac{22Lk}{9}\right)\delta_3 + \frac{13Lk}{6}\delta_4 = -2PL$$

Lower FBD—sum moments about joint 6:

$$k\left(\delta_4 - \frac{2}{3} \delta_3\right) \frac{4L}{3} + 2k\left(\frac{3}{4} \delta_4 - \delta_3\right)L - k_r\theta_6 = 0$$

$$\left[k\left(\delta_4 - \frac{2}{3} \delta_3\right) \frac{4L}{3} + 2k\left(\frac{3}{4} \delta_4 - \delta_3\right)L\right] + (kL^2)\left(\frac{\delta_4}{\frac{4}{3}L}\right) = 0 \quad \text{OR} \quad \left(\frac{26Lk}{9}\right)\delta_3 + \frac{43Lk}{12}\delta_4 = 0$$

Divide matrix equilibrium equations through by k to get the following displacement equations:

$$\begin{pmatrix} \frac{-22}{9} & \frac{13}{6} \\ \frac{-26}{9} & \frac{43}{6} \end{pmatrix} \begin{pmatrix} \delta_3 \\ \delta_4 \end{pmatrix} = \begin{pmatrix} \frac{-2P}{k} \\ 0 \end{pmatrix} \quad \begin{pmatrix} \delta_3 \\ \delta_4 \end{pmatrix} = \begin{pmatrix} \frac{-22}{9} & \frac{13}{6} \\ \frac{-26}{9} & \frac{43}{6} \end{pmatrix}^{-1} \begin{pmatrix} \frac{-2P}{k} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{43P}{15k} \\ \frac{104P}{45k} \end{pmatrix} \quad \begin{matrix} \frac{43}{15} = 2.867 \\ \frac{104}{45} = 2.311 \end{matrix} \quad \boxed{\delta_4 = \frac{104P}{45k}}$$

^ deltas are **positive downward**

$$\text{Ratio of the deflection } \delta_4 \text{ in part (a) to that in (b): } \frac{\frac{26}{104}}{\frac{3}{45}} = \frac{15}{4} \quad \boxed{\text{Ratio} = \frac{15}{4} = 3.75}$$

Problem 2.2-8 The three-bar truss ABC shown in figure part a has a span $L = 3$ m and is constructed of steel pipes having cross-sectional area $A = 3900 \text{ mm}^2$ and modulus of elasticity $E = 200 \text{ GPa}$. Identical loads P act both vertically and horizontally at joint C , as shown.

- If $P = 475 \text{ kN}$, what is the horizontal displacement of joint B ?
- What is the maximum permissible load value P_{\max} if the displacement of joint B is limited to 1.5 mm ?
- Repeat parts (a) and (b) if the plane truss is replaced by a space truss (see figure part b).

Solution 2.2-8

NUMERICAL DATA

$$A = 3900 \text{ mm}^2 \quad E = 200 \text{ GPa}$$

$$P = 475 \text{ kN} \quad L = 3000 \text{ mm}$$

$$\delta_{B\max} = 1.5 \text{ mm}$$

- (a) FIND HORIZONTAL DISPLACEMENT OF JOINT B

STATICS TO FIND SUPPORT REACTIONS AND THEN MEMBER FORCES:

$$\sum M_A = 0 \quad B_y = \frac{1}{L} \left(2P \frac{L}{2} \right)$$

$$B_y = P$$

$$\sum F_H = 0 \quad A_x = -P$$

$$\sum F_V = 0 \quad A_y = P - B_y \quad A_y = 0$$

METHOD OF JOINTS: $AC_V = A_Y \quad AC_V = 0 \quad \text{Force in } AC = 0$

$$AB = A_X$$

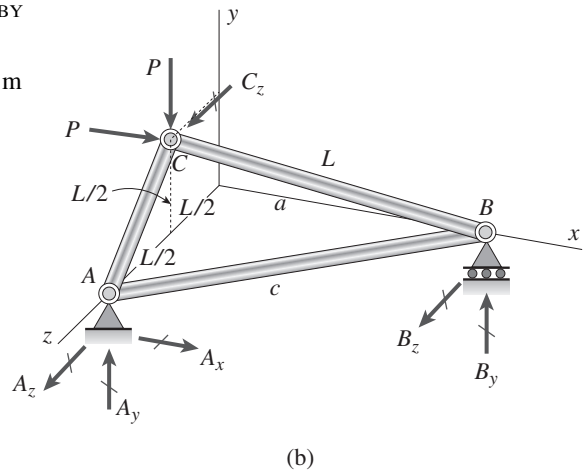
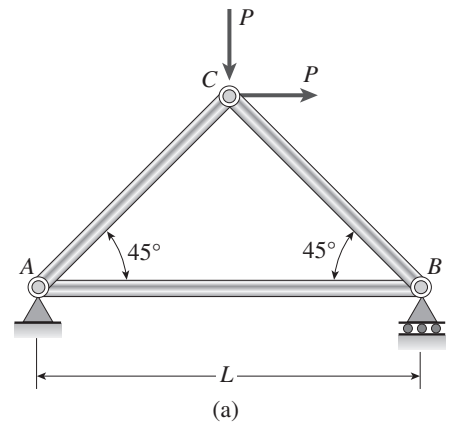
Force in AB is P (tension) so elongation of AB is the horizontal displacement of joint B .

$$\delta_B = \frac{F_{AB}L}{EA} \quad \delta_B = \frac{PL}{EA} \quad \delta_B = 1.82692 \text{ mm} \quad \boxed{\delta_B = 1.827 \text{ mm}}$$

- (b) FIND P_{\max} IF DISPLACEMENT OF JOINT $B = \delta_{B\max} = 1.5 \text{ mm}$ $P_{\max} = \frac{EA}{L} \delta_{B\max} \quad \boxed{P_{\max} = 390 \text{ kN}}$

- (c) REPEAT PARTS (a) AND (b) IF THE PLANE TRUSS IS REPLACED BY A SPACE TRUSS (SEE FIGURE PART b).

FIND MISSING DIMENSIONS a AND c : $P = 475 \text{ kN} \quad L = 3 \text{ m}$



$$a = \sqrt{L^2 - 2\left(\frac{L}{2}\right)^2} = 2.12132 \text{ m} \quad \frac{a}{L} = 0.707 \quad a = \frac{L}{\sqrt{2}} = 2.12132 \text{ m}$$

$$c = \sqrt{L^2 + a^2} = 3.67423 \text{ m} \quad c = \sqrt{L^2 + \left(\frac{L}{\sqrt{2}}\right)^2} = 3.67423 \text{ m} \quad c = L\sqrt{\frac{3}{2}} = 3.67423 \text{ m}$$

- (1) SUM MOMENTS ABOUT A LINE THRU
- A
- WHICH IS PARALLEL TO THE
- y
- AXIS

$$B_z = -P \frac{L}{a} = -671.751 \text{ kN}$$

- (2) SUM MOMENTS ABOUT THE
- z
- AXIS

$$B_y = \frac{P\left(\frac{L}{2}\right)}{a} = 335.876 \text{ kN} \quad \text{SO} \quad A_y = P - B_y = 139.124 \text{ kN}$$

- (3) SUM MOMENTS ABOUT THE
- x
- AXIS

$$C_z = \frac{A_y L - P \frac{L}{2}}{\frac{L}{2}} = -196.751 \text{ kN}$$

- (4) SUM FORCES IN THE
- x
- AND
- z
- DIRECTIONS
- $A_x = -P = -475 \text{ kN}$
- $A_z = -C_z - B_z = 868.503 \text{ kN}$

- (5) USE METHOD OF JOINTS TO FIND MEMBER FORCES

$$\text{Sum forces in } x\text{-direction at joint } A: \frac{a}{c} F_{AB} + A_x = 0 \quad F_{AB} = \frac{-c}{a} A_x = 823 \text{ kN}$$

$$\text{Sum forces in } y\text{-direction at joint } A: \frac{\frac{L}{2}}{\sqrt{2} \frac{L}{2}} F_{AC} + A_y = 0 \quad F_{AC} = \sqrt{2} (-A_y) = -196.8 \text{ kN}$$

$$\text{Sum forces in } y\text{-direction at joint } B: \frac{\frac{L}{2}}{L} F_{BC} + B_y = 0 \quad F_{BC} = -2B_y = -672 \text{ kN}$$

- (6) FIND DISPLACEMENT ALONG
- x
- AXIS AT JOINT
- B

Find change in length of member AB then find its projection along x axis:

$$\delta_{AB} = \frac{F_{ABC}}{EA} = 3.875 \text{ mm} \quad \beta = \arctan\left(\frac{L}{a}\right) = 54.736^\circ \quad \delta_{Bx} = \frac{\delta_{AB}}{\cos(\beta)} = 6.713 \text{ mm} \quad \boxed{\delta_{Bx} = 6.71 \text{ mm}}$$

- (7) FIND
- P_{\max}
- FOR SPACE TRUSS IF
- δ_{Bx}
- MUST BE LIMITED TO 1.5 mm

Displacements are linearly related to the loads for this linear elastic small displacement problem, so reduce load variable P from 475 kN to

$$\frac{1.5}{6.71254} 475 = 106.145 \text{ kN} \quad \boxed{P_{\max} = 106.1 \text{ kN}}$$

Repeat space truss analysis using vector operations $a = 2.121 \text{ m}$ $L = 3 \text{ m}$ $P = 475 \text{ kN}$

POSITION AND UNIT VECTORS:

$$r_{AB} = \begin{pmatrix} a \\ 0 \\ -L \end{pmatrix} \quad e_{AB} = \frac{r_{AB}}{|r_{AB}|} = \begin{pmatrix} 0.577 \\ 0 \\ -0.816 \end{pmatrix} \quad r_{AC} = \begin{pmatrix} 0 \\ \frac{L}{2} \\ -\frac{L}{2} \end{pmatrix} \quad e_{AC} = \frac{r_{AC}}{|r_{AC}|} = \begin{pmatrix} 0 \\ 0.707 \\ -0.707 \end{pmatrix}$$

FIND MOMENT AT A:

$$M_A = r_{AB} \times R_B + r_{AC} \times R_C$$

$$M_A = r_{AB} \times \begin{pmatrix} 0 \\ RB_y \\ RB_z \end{pmatrix} + r_{AC} \times \begin{pmatrix} 2P \\ -P \\ RC_z \end{pmatrix} = \begin{pmatrix} 3.0 \text{ m } RB_y + 1.5 \text{ m } RC_z - 712.5 \text{ kN}\cdot\text{m} \\ -2.1213 \text{ m } RB_z - 1425.0 \text{ kN}\cdot\text{m} \\ 2.1213 \text{ m } RB_y - 1425.0 \text{ kN}\cdot\text{m} \end{pmatrix}$$

FIND MOMENTS ABOUT LINES OR AXES:

$$M_A e_{AB} = -1.732 \text{ m } RB_y + 1.7321 \text{ m } RB_z + 0.86603 \text{ m } RC_z + 752.15 \text{ kN}\cdot\text{m}$$

$$RC_z = \frac{-244.12}{0.72169} = -338.262 \quad C_z = -196.751 \text{ kN}$$

$$M_A e_{AC} = -1.5 \text{ m } RB_y + -1.5 \text{ m } RB_z \quad \text{So} \quad RB_y = -RB_z$$

$$M_A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = -2.1213 \text{ m } RB_z + -1425.0 \text{ kN}\cdot\text{m} \quad \text{So} \quad RB_z = \frac{462.5}{-1.7678} = -261.625 \quad B_z = -671.75 \text{ kN}$$

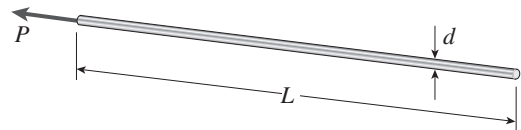
$$M_A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 2.1213 \text{ m } RB_y + -1425.0 \text{ kN}\cdot\text{m} \quad \text{So} \quad RB_y = -RB_z = 261.625 \quad B_y = -335.876 \text{ kN}$$

$$\sum F_y = 0 \quad A_y = P - B_y = 139.124 \text{ kN}$$

Reactions obtained using vector operations agree with those based on scalar operations.

Problem 2.2-9 An aluminum wire having a diameter $d = 1/10$ in. and length $L = 12$ ft is subjected to a tensile load P (see figure). The aluminum has modulus of elasticity $E = 10,600$ ksi

If the maximum permissible elongation of the wire is $1/8$ in. and the allowable stress in tension is 10 ksi, what is the allowable load P_{\max} ?



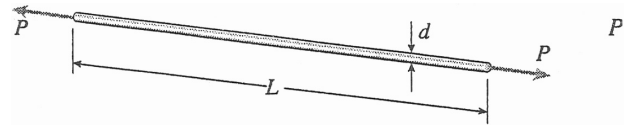
Solution 2.2-9

$$d = \frac{1}{10} \text{ in.} \quad L = 12(12) \text{ in.} \quad E = 10,600 \times (10^3) \text{ psi}$$

$$\delta_a = \frac{1}{8} \text{ in.} \quad \sigma_a = 10 \times (10^3) \text{ psi}$$

$$A = \frac{\pi d^2}{4} \quad A = 7.854 \times 10^{-3} \text{ in.}^2$$

$$EA = 8.325 \times 10^4 \text{ lb}$$



Maximum load based on elongation:

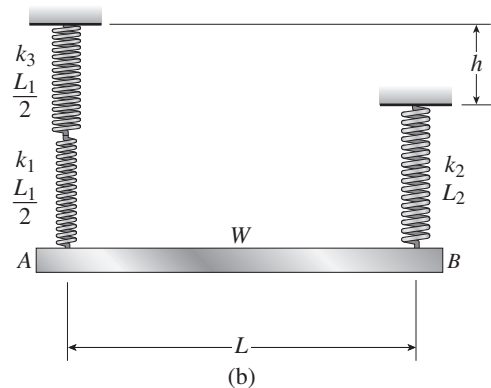
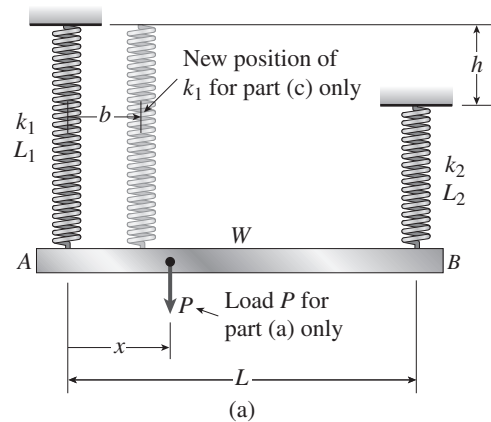
$$P_{\max 1} = \frac{EA}{L} \delta_a \quad P_{\max 1} = 72.3 \text{ lb} \quad \leftarrow \text{controls}$$

Maximum load based on stress:

$$P_{\max 2} = \sigma_a A \quad P_{\max 2} = 78.5 \text{ lb}$$

Problem 2.2-10 A uniform bar AB of weight $W = 25 \text{ N}$ is supported by two springs, as shown in the figure. The spring on the left has stiffness $k_1 = 300 \text{ N/m}$ and natural length $L_1 = 250 \text{ mm}$. The corresponding quantities for the spring on the right are $k_2 = 400 \text{ N/m}$ and $L_2 = 200 \text{ mm}$. The distance between the springs is $L = 350 \text{ mm}$, and the spring on the right is suspended from a support that is distance $h = 80 \text{ mm}$ below the point of support for the spring on the left. Neglect the weight of the springs.

- At what distance x from the left-hand spring (figure part a) should a load $P = 18 \text{ N}$ be placed in order to bring the bar to a horizontal position?
- If P is now removed, what new value of k_1 is required so that the bar (figure part a) will hang in a horizontal position under weight W ?
- If P is removed and $k_1 = 300 \text{ N/m}$, what distance b should spring k_1 be moved to the right so that the bar (figure part a) will hang in a horizontal position under weight W ?
- If the spring on the left is now replaced by two springs in series ($k_1 = 300 \text{ N/m}$, k_3) with overall natural length $L_1 = 250 \text{ mm}$ (see figure part b), what value of k_3 is required so that the bar will hang in a horizontal position under weight W ?



Solution 2.2-10

NUMERICAL DATA

$$W = 25 \text{ N} \quad k_1 = 0.300 \text{ N/mm} \quad L_1 = 250 \text{ mm}$$

$$k_2 = 0.400 \text{ N/mm} \quad L_2 = 200 \text{ mm}$$

$$L = 350 \text{ mm} \quad h = 80 \text{ mm} \quad P = 18 \text{ N}$$

- (a) LOCATION OF LOAD P TO BRING BAR TO HORIZONTAL POSITION

Use statics to get forces in both springs:

$$\sum M_A = 0 \quad F_2 = \frac{1}{L} \left(W \frac{L}{2} + Px \right)$$

$$F_2 = \frac{W}{2} + P \frac{x}{L}$$

$$\sum F_V = 0 \quad F_1 = W + P - F_2$$

$$F_1 = \frac{W}{2} + P \left(1 - \frac{x}{L} \right)$$

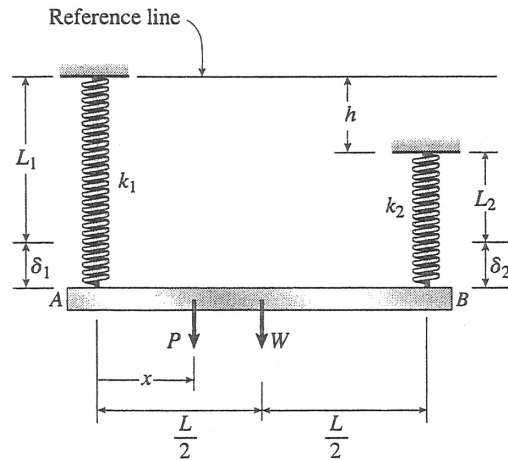
Use constraint equation to define horizontal position, then solve for location x :

$$L_1 + \frac{F_1}{k_1} = L_2 + h + \frac{F_2}{k_2}$$

Substitute expressions for F_1 and F_2 above into constraint equilibrium and solve for x :

$$x = \frac{-2L_1 L k_1 k_2 - k_2 W L - 2k_2 P L + 2L_2 L k_1 k_2 + 2h L k_1 k_2 + k_1 W L}{-2P(k_1 + k_2)}$$

$$x = 134.7 \text{ mm} \quad \leftarrow$$



- (b) NEXT REMOVE P AND FIND NEW VALUE OF SPRING CONSTANT k_1 SO THAT BAR IS HORIZONTAL UNDER WEIGHT W

$$\text{Now, } F_1 = \frac{W}{2} \quad F_2 = \frac{W}{2} \quad \text{since } P = 0$$

Same constraint equation as above but now $P = 0$:

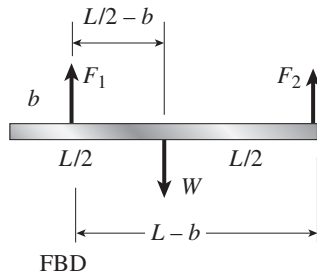
$$L_1 + \frac{W}{k_1} - (L_2 + h) - \frac{\left(\frac{W}{2}\right)}{k_2} = 0$$

Solve for k_1 :

$$k_1 = \frac{-Wk_2}{[2k_2[L_1 - (L_2 + h)]] - W}$$

$$k_1 = 0.204 \text{ N/mm} \quad \leftarrow$$

- (c) USE $k_1 = 0.300 \text{ N/mm}$ BUT RELOCATE SPRING k_1 ($x = b$) SO THAT BAR ENDS UP IN HORIZONTAL POSITION UNDER WEIGHT W



$$b = \frac{2L_1k_1k_2L + Wk_2 - 2L_2k_1k_2L - 2hk_1k_2L - Wk_1L}{(2L_1k_1k_2) - 2L_2k_1k_2 - 2hk_1k_2 - 2Wk_1}$$

- (d) REPLACE SPRING k_1 WITH SPRINGS IN SERIES: $k_1 = 0.3 \text{ N/mm}$, $L_1/2$, AND k_3 , $L_1/2$. FIND k_3 SO THAT BAR HANGS IN HORIZONTAL POSITION

$$\text{STATICS} \quad F_1 = \frac{W}{2} \quad F_2 = \frac{W}{2}$$

$$k_3 = \frac{Wk_1k_2}{-2L_1k_1k_2 - Wk_2 + 2L_2k_1k_2 + 2hk_1k_2 + Wk_1}$$

NOTE—equivalent spring constant for series springs:

$$k_e = \frac{k_1k_3}{k_1 + k_3}$$

PART (c)—CONTINUED

STATICS

$$\sum M_{k_1} = 0 \quad F_2 = \frac{w\left(\frac{L}{2} - b\right)}{L - b}$$

$$\sum F_V = 0$$

$$F_1 = W - F_2$$

$$F_1 = W - \frac{w\left(\frac{L}{2} - b\right)}{L - b}$$

$$F_1 = \frac{WL}{2(L - b)}$$

Constraint equation—substitute above expressions for F_1 and F_2 and solve for b :

$$L_1 + \frac{F_1}{k_1} - (L_2 + h) - \frac{F_2}{k_2} = 0$$

Use the following data:

$$k_1 = 0.300 \text{ N/mm} \quad k_2 = 0.4 \text{ N/mm} \quad L_1 = 250 \text{ mm}$$

$$L_2 = 200 \text{ mm} \quad L = 350 \text{ mm}$$

$$b = 74.1 \text{ mm} \quad \leftarrow$$

New constraint equation; solve for k_3 :

$$L_1 + \frac{F_1}{k_1} + \frac{F_1}{k_3} - (L_2 + h) - \frac{F_2}{k_2} = 0$$

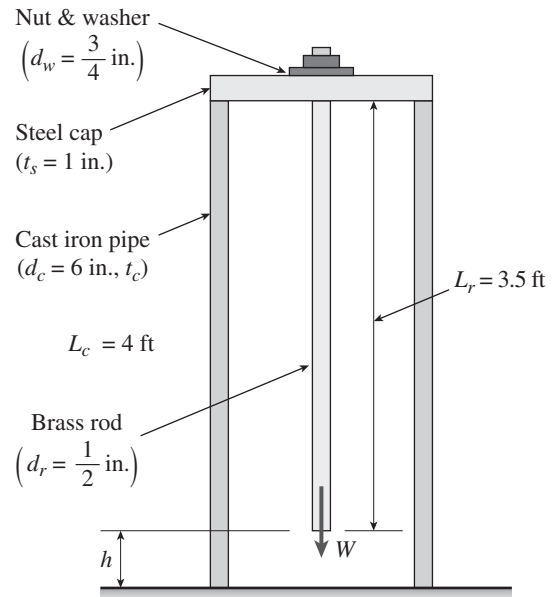
$$L_1 + \frac{W/2}{k_1} + \frac{W/2}{k_3} - (L_2 + h) - \frac{W/2}{k_2} = 0$$

$$k_3 = 0.638 \text{ N/mm} \quad \leftarrow$$

$$k_e = 0.204 \text{ N/mm} \quad \leftarrow \quad \text{checks—same as (b) above}$$

Problem 2.2-11 A hollow, circular, cast-iron pipe ($E_c = 12,000$ ksi) supports a brass rod ($E_b = 14,000$ ksi) and weight $W = 2$ kips, as shown. The outside diameter of the pipe is $d_c = 6$ in.

- If the allowable compressive stress in the pipe is 5000 psi and the allowable shortening of the pipe is 0.02 in., what is the minimum required wall thickness $t_{c,min}$? (Include the weights of the rod and steel cap in your calculations.)
- What is the elongation of the brass rod δ_r due to both load W and its own weight?
- What is the minimum required clearance h ?



Solution 2.2-11

The figure shows a section cut through the pipe, cap, and rod.

NUMERICAL DATA

$$E_c = 12000 \text{ ksi} \quad E_b = 14,000 \text{ ksi}$$

$$W = 2 \text{ k} \quad d_c = 6 \text{ in.} \quad d_r = \frac{1}{2} \text{ in.}$$

$$\sigma_a = 5 \text{ ksi} \quad \delta_a = 0.02 \text{ in.}$$

$$\text{Unit weights (see Table I-1): } \gamma_s = 2.836 \times 10^{-4} \text{ k/in.}^3$$

$$\gamma_b = 3.009 \times 10^{-4} \text{ k/in.}^3$$

$$L_c = 48 \text{ in.} \quad L_r = 42 \text{ in.}$$

$$t_s = 1 \text{ in.}$$

- (a) **MINIMUM REQUIRED WALL THICKNESS OF CAST IRON PIPE, $t_{c,min}$**

First check allowable stress then allowable shortening:

$$W_{\text{cap}} = \gamma_s \left(\frac{\pi}{4} d_c^2 t_s \right)$$

$$W_{\text{cap}} = 8.018 \times 10^{-3} \text{ k}$$

$$W_{\text{rod}} = \gamma_b \left(\frac{\pi}{4} d_r^2 L_r \right)$$

$$W_{\text{rod}} = 2.482 \times 10^{-3} \text{ k}$$

$$W_t = W + W_{\text{cap}} + W_{\text{rod}} \quad W_t = 2.01 \text{ k}$$

$$A_{\min} = \frac{W_t}{\sigma_a} \quad A_{\min} = 0.402 \text{ in.}^2$$

$$A_{\text{pipe}} = \frac{\pi}{4} [d_c^2 - (d_c - 2t_c)^2]$$

$$A_{\text{pipe}} = \pi t_c (d_c - t_c)$$

$$t_c (d_c - t_c) = \frac{W_t}{\pi \sigma_a}$$

$$\text{Let } \alpha = \frac{W_t}{\pi \sigma_a} \quad \alpha = 0.128:$$

$$t_c^2 - d_c t_c + \alpha = 0$$

$$t_c = \frac{d_c - \sqrt{d_c^2 - 4\alpha}}{2} \quad t_c = 0.021 \text{ in.}$$

^ minimum based
on σ_a

Now check allowable shortening requirement:

$$\delta_{\text{pipe}} = \frac{W_t L_c}{E_c A_{\text{min}}} \quad A_{\text{min}} = \frac{W_t L_c}{E_c \delta_a}$$

$$A_{\text{min}} = 0.447 \text{ in.}^2 < \text{larger than value based on } \sigma_a \text{ above}$$

$$\pi t_c (d_c - t_c) = \frac{W_t L_c}{E_c \delta_a}$$

$$t_c^2 - d_c t_c + \beta = 0 \quad \beta = \frac{W_t L_c}{\pi E_c \delta_a}$$

$$\beta = 0.142$$

$$t_c = \frac{d_c - \sqrt{d_c^2 - 4\beta}}{2}$$

$$t_c = 0.021 \text{ in.} \quad \leftarrow \text{minimum based on } \delta_a \text{ and } \sigma_a \text{ controls}$$

(b) ELONGATION OF ROD DUE TO SELF WEIGHT AND ALSO WEIGHT W

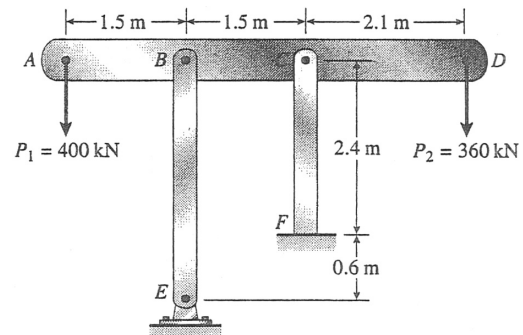
$$\delta_r = \frac{\left(W + \frac{W_{\text{rod}}}{2}\right) L_r}{E_b \left(\frac{\pi}{4} d_r^2\right)} \quad \delta_r = 0.031 \text{ in.} \quad \leftarrow$$

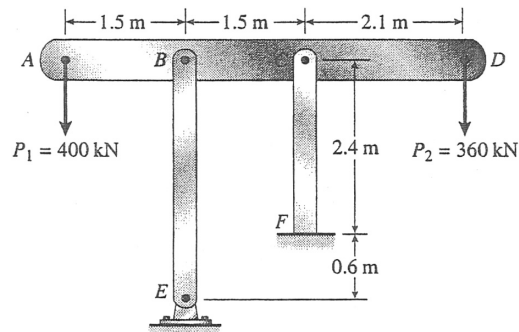
(c) MINIMUM CLEARANCE h

$$h_{\text{min}} = \delta_a + \delta_r \quad h_{\text{min}} = 0.051 \text{ in.} \quad \leftarrow$$

Problem 2.2-12 The horizontal rigid beam $ABCD$ is supported by vertical bars BE and CF and is loaded by vertical forces $P_1 = 400 \text{ kN}$ and $P_2 = 360 \text{ kN}$ acting at points A and D , respectively (see figure). Bars BE and CF are made of steel ($E = 200 \text{ GPa}$) and have cross-sectional areas $A_{BE} = 11,100 \text{ mm}^2$ and $A_{CF} = 9,280 \text{ mm}^2$. The distances between various points on the bars are shown in the figure.

Determine the vertical displacements δ_A and δ_D of points A and D , respectively.



Solution 2.2-12 Rigid beam supported by vertical bars

$$A_{BE} = 11,100 \text{ mm}^2$$

$$A_{CF} = 9,280 \text{ mm}^2$$

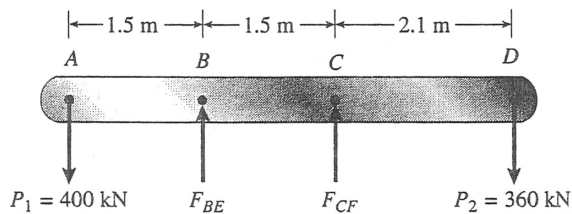
$$E = 200 \text{ GPa}$$

$$L_{BE} = 3.0 \text{ m}$$

$$L_{CF} = 2.4 \text{ m}$$

$$P_1 = 400 \text{ kN}; P_2 = 360 \text{ kN}$$

FREE-BODY DIAGRAM OF BAR *ABCD*



$$\Sigma M_B = 0 \quad \curvearrowright$$

$$(400 \text{ kN})(1.5 \text{ m}) + F_{CF}(1.5 \text{ m}) - (360 \text{ kN})(3.6 \text{ m}) = 0$$

$$F_{CF} = 464 \text{ kN}$$

$$\Sigma M_C = 0$$

$$(400 \text{ kN})(3.0 \text{ m}) - F_{BE}(1.5 \text{ m}) - (360 \text{ kN})(2.1 \text{ m}) = 0$$

$$F_{BE} = 296 \text{ kN}$$

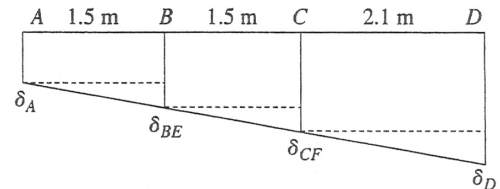
SHORTENING OF BAR *BE*

$$\delta_{BE} = \frac{F_{BE}L_{BE}}{EA_{BE}} = \frac{(296 \text{ kN})(3.0 \text{ m})}{(200 \text{ GPa})(11,100 \text{ mm}^2)} = 0.400 \text{ mm}$$

SHORTENING OF BAR *CF*

$$\delta_{CF} = \frac{F_{CF}L_{CF}}{EA_{CF}} = \frac{(464 \text{ kN})(2.4 \text{ m})}{(200 \text{ GPa})(9,280 \text{ mm}^2)} = 0.600 \text{ mm}$$

DISPLACEMENT DIAGRAM



$$\delta_{BE} - \delta_A = \delta_{CF} - \delta_{BE} \text{ or } \delta_A = 2\delta_{BE} - \delta_{CF}$$

$$\delta_A = 2(0.400 \text{ mm}) - 0.600 \text{ mm}$$

$$= 0.200 \text{ mm} \quad \leftarrow$$

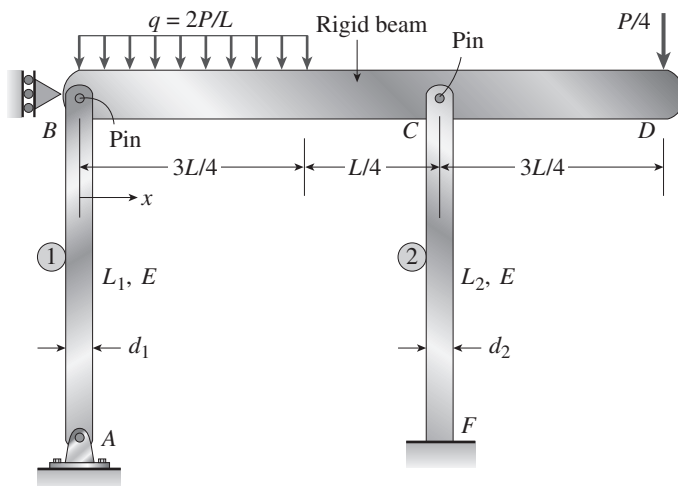
(Downward)

$$\delta_D - \delta_{CF} = \frac{2.1}{1.5}(\delta_{CF} - \delta_{BE})$$

$$\begin{aligned} \text{or } \delta_D &= \frac{12}{5}\delta_{CF} - \frac{7}{5}\delta_{BE} \\ &= \frac{12}{5}(0.600 \text{ mm}) - \frac{7}{5}(0.400 \text{ mm}) \\ &= 0.880 \text{ mm} \quad \leftarrow \\ &\quad \text{(Downward)} \end{aligned}$$

Problem 2.2-13 Two pipe columns (AB , FC) are pin-connected to a rigid beam (BCD) as shown in the figure. Each pipe column has modulus E , but heights (L_1 or L_2) and outer diameters (d_1 or d_2) are different for each column. Assume the inner diameter of each column is $3/4$ of outer diameter. Uniformly distributed downward load $q = 2P/L$ is applied over a distance of $3L/4$ along BC , and concentrated load $P/4$ is applied downward at D .

- Derive a formula for the displacement δ_D at point D in terms of P and column flexibilities f_1 and f_2 .
- If $d_1 = (9/8)d_2$, find the L_1/L_2 ratio so that beam BCD displaces downward to a horizontal position under the load system in (a).
- If $L_1 = 2L_2$, find the d_1/d_2 ratio so that beam BCD displaces downward to a horizontal position under the load system in (a).
- If $d_1 = (9/8)d_2$ and $L_1/L_2 = 1.5$, at what horizontal distance x from B should load $P/4$ be placed so that beam BCD displaces downward to a horizontal position under the load system in part (a)?



Solution 2.2-13

(a) DISPLACEMENT δ_D

Use FBD of beam BCD $\sum M_B = 0$ $R_C = \frac{1}{L} \left[\left(2 \frac{P}{L} \right) \left(\frac{3}{4} L \right) \left(\frac{3}{8} L \right) + \frac{P}{4} \left(L + \frac{3}{4} L \right) \right] = P$ <compression force in column CF

$$\sum F_V = 0 \quad R_B = \left(2 \frac{P}{L} \right) \left(\frac{3}{4} L \right) + \frac{P}{4} - R_C = \frac{3P}{4} \quad \text{<compression force in column } BA$$

Downward displacements at B and C : $\delta_B = R_B f_1 = \frac{3P f_1}{4}$ $\delta_C = R_C f_2 = P f_2$

Geometry: $\delta_D = \delta_B + (\delta_C - \delta_B) \left(\frac{L + \frac{3}{4}L}{L} \right) = \frac{7P f_2}{4} - \frac{9P f_1}{16}$ $\delta_D = \frac{7P f_2}{4} - \frac{9P f_1}{16} = \boxed{\frac{P}{16} (28 f_2 - 9 f_1)}$

(b) DISPLACEMENT TO HORIZONTAL POSITION, SO $\delta_C = \delta_B$ and $\frac{3P f_1}{4} = P f_2$ or $\frac{f_1}{f_2} = \frac{4}{3}$

$$\frac{\frac{L_1}{EA_1}}{\frac{L_2}{EA_2}} = \frac{4}{3} \quad \text{or} \quad \frac{L_1}{L_2} = \frac{4}{3} \left(\frac{A_1}{A_2} \right) \quad \frac{L_1}{L_2} = \frac{4}{3} \left(\frac{\frac{\pi}{4} d_1^2}{\frac{\pi}{4} d_2^2} \right) = \frac{4 d_1^2}{3 d_2^2} \quad \frac{L_1}{L_2} = \frac{4}{3} \left(\frac{d_1}{d_2} \right)^2 \quad \text{with} \quad \frac{d_1}{d_2} = \frac{9}{8}$$

$$\frac{L_1}{L_2} = \frac{4}{3} \left(\frac{9}{8} \right)^2 = \frac{27}{16} \quad \boxed{\frac{L_1}{L_2} = \frac{27}{16}}$$

- (c) If $L_1 = 2 L_2$, FIND THE d_1/d_2 RATIO SO THAT BEAM BCD DISPLACES DOWNWARD TO A HORIZONTAL POSITION

$$\frac{L_1}{L_2} = 2 \quad \text{and} \quad \delta_C = \delta_B \quad \text{from part (b).} \quad \left(\frac{d_1}{d_2}\right)^2 = \frac{3}{4} \left(\frac{L_1}{L_2}\right) \quad \text{so} \quad \boxed{\frac{d_1}{d_2} = \sqrt{\frac{3}{4}}(2) = 1.225}$$

- (d) If $d_1 = (9/8) d_2$ AND $L_1/L_2 = 1.5$, AT WHAT HORIZONTAL DISTANCE x FROM B SHOULD LOAD $P/4$ AT D BE PLACED?

$$\text{Given} \quad \frac{d_1}{d_2} = \frac{9}{8} \quad \text{and} \quad \frac{L_1}{L_2} = 1.5 \quad \text{or} \quad \frac{f_1}{f_2} = \frac{L_1}{L_2} \left(\frac{A_2}{A_1}\right) \quad \frac{f_1}{f_2} = \frac{L_1}{L_2} \left(\frac{d_2}{d_1}\right)^2 = \frac{3}{2} \left(\frac{8}{9}\right)^2 = \frac{32}{27}$$

Recompute column forces R_B and R_C but now with load $P/4$ positioned at distance x from B .

$$\text{Use FBD of beam } BCD: \quad \sum M_B = 0 \quad R_C = \frac{1}{L} \left[\left(2\frac{P}{L}\right) \left(\frac{3}{4}L\right) \left(\frac{3}{8}L\right) + \frac{P}{4}(x) \right] = \frac{\frac{9LP}{16} + \frac{Px}{4}}{L}$$

$$\sum F_V = 0 \quad R_B = \left(2\frac{P}{L}\right) \left(\frac{3}{4}L\right) + \frac{P}{4} - R_C = \frac{7P}{4} - \frac{\frac{9LP}{16} + \frac{Px}{4}}{L}$$

Horizontal displaced position under load q and load $P/4$ so $\delta_C = \delta_B$ or $R_C f_2 = R_B f_1$.

$$\left(\frac{\frac{9LP}{16} + \frac{Px}{4}}{L}\right) f_2 = \left(\frac{7P}{4} - \frac{\frac{9LP}{16} + \frac{Px}{4}}{L}\right) f_1 \quad \text{solve, } x = -\frac{9Lf_2 - 19Lf_1}{4f_1 + 4f_2} = -\frac{L(9f_2 - 19f_1)}{4(f_1 + f_2)}$$

$$x = -\frac{L(9f_2 - 19f_1)}{4(f_1 + f_2)} \quad \text{or} \quad x = L \left[\frac{19\frac{f_1}{f_2} - 9}{4\left(\frac{f_1}{f_2} + 1\right)} \right]$$

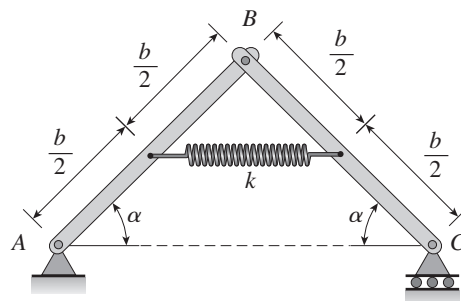
Now substitute f_1/f_2 ratio from above:

$$\boxed{x = L \left[\frac{19\frac{32}{27} - 9}{4\left(\frac{32}{27} + 1\right)} \right] = \frac{365L}{236}} \quad \frac{365}{236} = 1.547$$

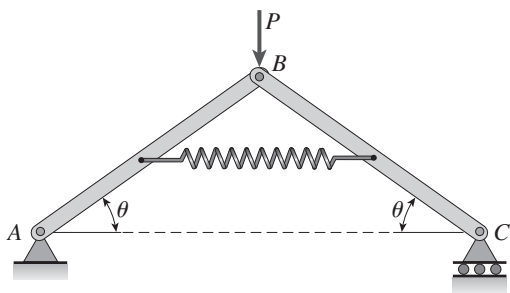
Problem 2.2-14 A framework ABC consists of two rigid bars AB and BC , each having a length b (see the first part of the figure part a). The bars have pin connections at A, B , and C and are joined by a spring of stiffness k . The spring is attached at the midpoints of the bars. The framework has a pin support at A and a roller support at C , and the bars are at an angle α to the horizontal.

When a vertical load P is applied at joint B (see the second part of the figure part a) the roller support C moves to the right, the spring is stretched, and the angle of the bars decreases from α to the angle θ .

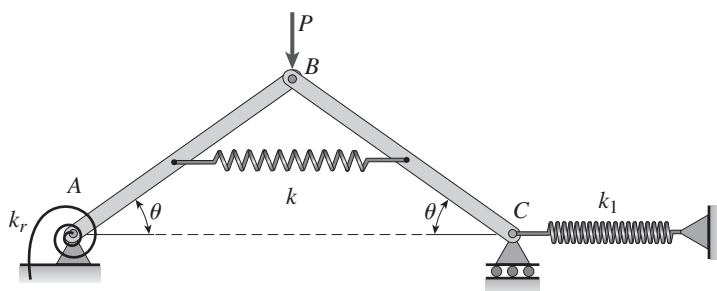
- Determine the angle θ and the increase δ in the distance between points A and C . Also find reactions at A and C . (Use the following data: $b = 200$ mm, $k = 3.2$ kN/m, $\alpha = 45^\circ$, and $P = 50$ N.)
- Repeat part (a) if a translational spring $k_1 = k/2$ is added at C and a rotational spring $k_r = kb^2/2$ is added at A (see figure part b).



(a) Initial position of structure



(a) - cont'd: displaced position of structure



(b) Displaced structure

Solution 2.2-14

Apply the laws of statics to the structure in its displaced position; also use FBD's of the left and right bars alone (referred to as LHFB and RHFB below).

$$\text{OVERALL FBD: } \sum F_H = 0 \quad H_A - k_1 \delta = 0 \quad \text{so} \quad H_A = k_1 \delta$$

$$\sum F_V = 0 \quad R_A + R_C = P$$

$$\sum M_A = 0 \quad k_r(\alpha - \theta) - P \frac{L_2}{2} + R_C L_2 = 0 \quad R_C = \frac{1}{L_2} \left[P \frac{L_2}{2} - k_r(\alpha - \theta) \right]$$

$$\text{LHFB: } \sum M_B = 0 \quad H_A h + k \frac{\delta}{2} \left(\frac{h}{2} \right) - R_A \left(\frac{L_2}{2} \right) + k_r(\alpha - \theta) = 0$$

$$R_A = \frac{2}{L_2} \left[k_1 \delta h + k \frac{\delta}{2} \left(\frac{h}{2} \right) + k_r(\alpha - \theta) \right]$$

$$\text{RHFB: } \sum M_B = 0 \quad -k \frac{\delta}{2} \left(\frac{h}{2} \right) - k_1 \delta h + R_C \frac{L_2}{2} = 0 \quad R_C = \frac{2}{L_2} \left[k \frac{\delta}{2} \left(\frac{h}{2} \right) + k_1 \delta h \right]$$

Equate the two expressions for R_C then substitute expressions for L_2 , k_r , k_1 , h and δ

$$\frac{1}{L_2} \left[P \frac{L_2}{2} - k_r(\alpha - \theta) \right] = \frac{2}{L_2} \left[k \frac{\delta}{2} \left(\frac{h}{2} \right) + k_1 \delta h \right] \quad \text{OR}$$

$$\frac{1}{L_2} \left[P \frac{L_2}{2} - k_r(\alpha - \theta) \right] - \left[\frac{2}{L_2} \left[k \frac{2b(\cos(\theta) - \cos(\alpha))}{2} \frac{b \sin(\theta)}{2} + k_1 [2b(\cos(\theta) - \cos(\alpha))] (b \sin(\theta)) \right] \right] = 0$$

(a) SUBSTITUTE NUMERICAL VALUES, THEN SOLVE NUMERICALLY FOR ANGLE θ AND DISTANCE INCREASE δ

$$b = 200 \text{ mm} \quad k = 3.2 \text{ kN/m} \quad \alpha = 45^\circ \quad P = 50 \text{ N} \quad k_1 = 0 \quad k_r = 0$$

$$L_2 = 2b \cos(\theta) \quad L_1 = 2b \cos(\alpha) \quad \delta = L_2 - L_1 \quad \delta = 2b(\cos(\theta) - \cos(\alpha)) \quad h = b \sin(\theta)$$

$$\frac{1}{L_2} \left[P \frac{L_2}{2} - k_r(\alpha - \theta) \right] - \left[\frac{1}{L_2} \left[k \frac{2b(\cos(\theta) - \cos(\alpha))}{2} \frac{b \sin(\theta)}{2} + k_1 [2b(\cos(\theta) - \cos(\alpha))] (b \sin(\theta)) \right] \right] = 0$$

$$\text{Solving above equation numerically gives } \boxed{\theta = 35.1^\circ} \quad \boxed{\delta = 44.6 \text{ mm}}$$

COMPUTE REACTIONS

$$R_C = \frac{2}{L_2} \left[k \frac{\delta}{2} \left(\frac{h}{2} \right) + k_1 \delta h \right] = 25 \text{ N} \quad R_C = \frac{1}{L_C} \left[P \frac{L_2}{2} - k_r(\alpha - \theta) \right] = 25 \text{ N}$$

$$R_A = \frac{2}{L_2} \left[k_1 \delta h + k \frac{\delta}{2} \left(\frac{h}{2} \right) + k_r(\alpha - \theta) \right] = 25 \text{ N} \quad M_A = k_r(\alpha - \theta) = 0$$

$$R_A + R_C = 50 \text{ N} < \text{check}$$

$$\boxed{R_A = 25 \text{ N}}$$

$$\boxed{R_C = 25 \text{ N}}$$

(b) SUBSTITUTE NUMERICAL VALUES, THEN SOLVE NUMERICALLY FOR ANGLE θ AND DISTANCE INCREASE δ

$$b = 200 \text{ mm} \quad k = 3.2 \text{ kN/m} \quad \alpha = 45^\circ \quad P = 50 \text{ N} \quad k_1 = \frac{k}{2} \quad k_r = \frac{k}{2} b^2$$

$$L_2 = 2b \cos(\theta) \quad L_1 = 2b \cos(\alpha) \quad \delta = L_2 - L_1 \quad \delta = 2b(\cos(\theta) - \cos(\alpha)) \quad h = b \sin(\theta)$$

$$\frac{1}{L_2} \left[P \frac{L_2}{2} - k_r(\alpha - \theta) \right] - \left[\frac{2}{L_2} \left[k \frac{2b(\cos(\theta) - \cos(\alpha))}{2} \frac{b \sin(\theta)}{2} + k_1 [2b(\cos(\theta) - \cos(\alpha))] (b \sin(\theta)) \right] \right] = 0$$

$$\text{Solving above equation numerically gives } \boxed{\theta = 43.3^\circ} \quad \boxed{\delta = 8.19 \text{ mm}}$$

COMPUTE REACTIONS

$$R_C = \frac{2}{L_2} \left[k \frac{\delta}{2} \left(\frac{h}{2} \right) + k_1 \delta h \right] = 18.5 \text{ N} \quad R_2 = \frac{1}{L_2} \left[P \frac{L_2}{2} - k_r(\alpha - \theta) \right] = 18.5 \text{ N}$$

$$R_A = \frac{2}{L_2} \left[k_1 \delta h + k \frac{\delta}{2} \left(\frac{h}{2} \right) + k_r(\alpha - \theta) \right] = 31.5 \text{ N} \quad M_A = k_r(\alpha - \theta) = 1.882 \text{ N}\cdot\text{m}$$

$$R_A + R_C = 50 \text{ N} < \text{check}$$

$$\boxed{R_A = 31.5 \text{ N}}$$

$$\boxed{R_C = 18.5 \text{ N}}$$

$$\boxed{M_A = 1.882 \text{ N}\cdot\text{m}}$$

Problem 2.2-15 Solve the preceding problem for the following data:

$$b = 8.0 \text{ in.}, k = 16 \text{ lb/in.}, \alpha = 45^\circ, \text{ and } P = 10 \text{ lb.}$$

Solution 2.2-15

Apply the laws of statics to the structure in its displaced position; also use FBD's of the left and right bars alone (referred to as LHFB and RHFB below)

$$\text{OVERALL FBD} \quad \sum F_H = 0 \quad H_A - k_1 \delta = 0 \quad \text{so} \quad H_A = k_1 \delta$$

$$\sum F_V = 0 \quad R_A + R_C = P$$

$$\sum M_A = 0 \quad k_r(\alpha - \theta) - P \frac{L_2}{2} + R_C L_2 = 0 \quad R_C = \frac{1}{L_2} \left[P \frac{L_2}{2} - k_r(\alpha - \theta) \right]$$

$$\text{LHFB} \quad \sum M_B = 0 \quad H_A h + k \frac{\delta}{2} \left(\frac{h}{2} \right) - R_A \frac{L_2}{2} + k_r(\alpha - \theta) = 0$$

$$R_A = \frac{2}{L_2} \left[k_1 \delta h + k \frac{\delta}{2} \left(\frac{h}{2} \right) + k_r(\alpha - \theta) \right]$$

$$\text{RHFB} \quad \sum M_B = 0 \quad -k \frac{\delta}{2} \left(\frac{h}{2} \right) - k_1 \delta h + R_C \frac{L_2}{2} = 0 \quad R_C = \frac{2}{L_2} \left[k \frac{\delta}{2} \left(\frac{h}{2} \right) + k_1 \delta h \right]$$

Equate the two expressions above for R_C then substitute expressions for L_2 , k_r , k_1 , h , and δ

$$\frac{1}{L_2} \left[P \frac{L_2}{2} - k_r(\alpha - \theta) \right] = \frac{2}{L_2} \left[k \frac{\delta}{2} \left(\frac{h}{2} \right) + k_1 \delta h \right] \quad \text{OR}$$

$$\frac{1}{L_2} \left[P \frac{L_2}{2} - k_r(\alpha - \theta) \right] - \left[\frac{2}{L_2} \left[k \frac{2b(\cos(\theta) - \cos(\alpha))}{2} \frac{b \sin(\theta)}{2} + k_1 [2b(\cos(\theta) - \cos(\alpha))] (b \sin(\theta)) \right] \right] = 0$$

(a) SUBSTITUTE NUMERICAL VALUES, THEN SOLVE NUMERICALLY FOR ANGLE θ AND DISTANCE INCREASE δ

$$b = 8 \text{ in.} \quad k = 16 \text{ lb/in.} \quad \alpha = 45^\circ \quad P = 10 \text{ lb} \quad k_1 = 0 \quad k_r = 0$$

$$L_2 = 2b \cos(\theta) \quad L_1 = 2b \cos(\alpha) \quad \delta = L_2 - L_1 \quad \delta = 2b(\cos(\theta) - \cos(\alpha)) \quad h = b \sin(\theta)$$

$$\frac{1}{L_2} \left[P \frac{L_2}{2} - k_r(\alpha - \theta) \right] - \left[\frac{2}{L_2} \left[k \frac{2b(\cos(\theta) - \cos(\alpha))}{2} \frac{b \sin(\theta)}{2} + k_1 [2b(\cos(\theta) - \cos(\alpha))] (b \sin(\theta)) \right] \right] = 0$$

$$\text{Solving above equation numerically gives } \boxed{\theta = 35.1^\circ} \quad \boxed{\delta = 1.782 \text{ in.}}$$

COMPUTE REACTIONS

$$R_C = \frac{2}{L_2} \left[k \frac{\delta}{2} \left(\frac{h}{2} \right) + k_1 \delta h \right] = 5 \text{ lb} \quad R_C = \frac{1}{L_2} \left[P \frac{L_2}{2} - k_r(\alpha - \theta) \right] = 5 \text{ lb}$$

$$R_A = \frac{2}{L_2} \left[k_1 \delta h + k \frac{\delta}{2} \left(\frac{h}{2} \right) + k_r(\alpha - \theta) \right] = 5 \text{ lb} \quad M_A = k_r(\alpha - \theta) = 0$$

$$R_A + R_C = 10 \text{ lb} \quad < \text{check} \quad \boxed{R_A = 5 \text{ lb}} \quad \boxed{R_C = 5 \text{ lb}}$$

- (b) SUBSTITUTE NUMERICAL VALUES, THEN SOLVE NUMERICALLY FOR ANGLE
- θ
- AND DISTANCE INCREASE
- δ

$$b = 8 \text{ in.} \quad k = 16 \text{ lb/in.} \quad \alpha = 45^\circ \quad P = 101b \quad k_1 = \frac{k}{2} \quad k_r = \frac{k}{2} b^2$$

$$L_2 = 2b \cos(\theta) \quad L_1 = 2b \cos(\alpha) \quad \delta = L_2 - L_1 \quad \delta = 2b(\cos(\theta) - \cos(\alpha)) \quad h = b \sin(\theta)$$

$$\frac{1}{L_2} \left[P \frac{L_2}{2} - k_r(\alpha - \theta) \right] - \left[\frac{2}{L_2} \left[k \frac{2b(\cos(\theta) - \cos(\alpha))}{2} \frac{b \sin(\theta)}{2} + k_1 [2b(\cos(\theta) - \cos(\alpha))] (b \sin(\theta)) \right] \right] = 0$$

Solving above equation numerically gives $\theta = 43.3^\circ$ $\delta = 0.327 \text{ in.}$

COMPUTE REACTIONS

$$R_C = \frac{2}{L_2} \left[k \frac{\delta}{2} \left(\frac{h}{2} \right) + k_1 \delta h \right] = 3.71 \text{ lb} \quad R_C = \frac{1}{L_2} \left[P \frac{L_2}{2} - k_r(\alpha - \theta) \right] = 3.71 \text{ lb}$$

$$R_A = \frac{2}{L_2} \left[k_1 \delta h + k \frac{\delta}{2} \left(\frac{h}{2} \right) + k_r(\alpha - \theta) \right] = 6.3 \text{ lb} \quad M_A = k_r(\alpha - \theta) = 1.252 \text{ ft}\cdot\text{lb}$$

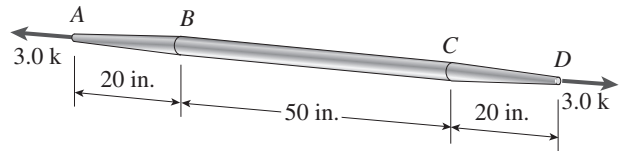
$$R_A + R_C = 10.01 \text{ lb} < \text{check} \quad R_A = 6.3 \text{ lb} \quad R_C = 3.71 \text{ lb} \quad M_A = 1.252 \text{ lb}\cdot\text{ft}$$

Changes in Lengths under Nonuniform Conditions

Problem 2.3-1

- (a) Calculate the elongation of a copper bar of solid circular cross section with tapered ends when it is stretched by axial loads of magnitude 3.0 k (see figure).

(The length of the end segments is 20 in. and the length of the prismatic middle segment is 50 in. Also, the diameters at cross sections A, B, C, and D are 0.5, 1.0, 1.0, and 0.5 in., respectively, and the modulus of elasticity is 18,000 ksi. (Hint: Use the result of Example 2-4.)



- (b) If the total elongation of the bar cannot exceed 0.025 in., what are the required diameters at B and C? Assume that diameters at A and D remain at 0.5 in.

Solution 2.3-1

NUMERICAL DATA

$$P = 3 \text{ k} \quad L_1 = 20 \text{ in.} \quad L_2 = 50 \text{ in.} \quad d_A = 0.5 \text{ in.} \quad d_B = 1 \text{ in.} \quad E = 18000 \text{ ksi}$$

- (a) TOTAL ELONGATION

$$\delta_1 = \frac{4PL_1}{\pi E d_A d_B} = 0.00849 \text{ in.} \quad \delta_2 = \frac{PL_2}{E \frac{\pi}{4} d_B^2} = 0.01061 \text{ in.}$$

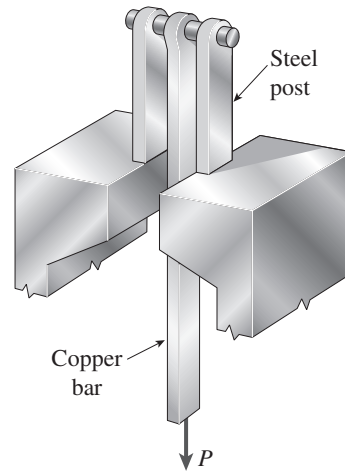
$$\delta = 2\delta_1 + \delta_2 = 0.0276 \text{ in.} \quad \delta = 0.0276 \text{ in.}$$

- (b) FIND NEW DIAMETERS AT B AND C IF TOTAL ELONGATION CANNOT EXCEED 0.025 in.

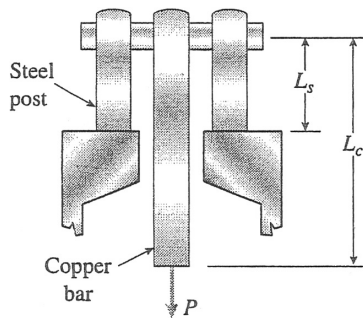
$$2 \left(\frac{4PL_1}{\pi E d_A d_B} \right) + \frac{PL_2}{E \frac{\pi}{4} d_B^2} = 0.025 \text{ in.} \quad \text{Solving for } d_B: \quad d_B = 1.074 \text{ in.}$$

Problem 2.3-2 A long, rectangular copper bar under a tensile load P hangs from a pin that is supported by two steel posts (see figure). The copper bar has a length of 2.0 m, a cross-sectional area of 4800 mm^2 , and a modulus of elasticity $E_c = 120 \text{ GPa}$. Each steel post has a height of 0.5 m, a cross-sectional area of 4500 mm^2 , and a modulus of elasticity $E_s = 200 \text{ GPa}$.

- Determine the downward displacement δ of the lower end of the copper bar due to a load $P = 180 \text{ kN}$.
- What is the maximum permissible load P_{\max} if the displacement δ is limited to 1.0 mm?



Solution 2.3-2 Copper bar with a tensile load



$$\begin{aligned} L_c &= 2.0 \text{ m} \\ A_c &= 4800 \text{ mm}^2 \\ E_c &= 120 \text{ GPa} \\ L_s &= 0.5 \text{ m} \\ A_s &= 4500 \text{ mm}^2 \\ E_s &= 200 \text{ GPa} \end{aligned}$$

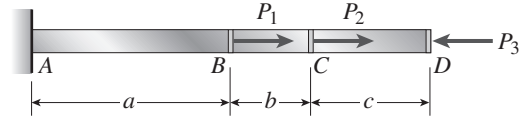
- (a) DOWNWARD DISPLACEMENT δ ($P = 180 \text{ kN}$)

$$\begin{aligned} \delta_c &= \frac{PL_c}{E_c A_c} = \frac{(180 \text{ kN})(2.0 \text{ m})}{(120 \text{ GPa})(4800 \text{ mm}^2)} \\ &= 0.625 \text{ mm} \\ \delta_s &= \frac{(P/2)L_s}{E_s A_s} = \frac{(90 \text{ kN})(0.5 \text{ m})}{(200 \text{ GPa})(4500 \text{ mm}^2)} \\ &= 0.050 \text{ mm} \\ \delta &= \delta_c + \delta_s = 0.625 \text{ mm} + 0.050 \text{ mm} \\ &= 0.675 \text{ mm} \quad \leftarrow \end{aligned}$$

- (b) MAXIMUM LOAD P_{\max} ($\delta_{\max} = 1.0 \text{ mm}$)

$$\begin{aligned} \frac{P_{\max}}{P} &= \frac{\delta_{\max}}{\delta} \quad P_{\max} = P \left(\frac{\delta_{\max}}{\delta} \right) \\ P_{\max} &= (180 \text{ kN}) \left(\frac{1.0 \text{ mm}}{0.675 \text{ mm}} \right) = 267 \text{ kN} \quad \leftarrow \end{aligned}$$

Problem 2.3-3 An aluminum bar AD (see figure) has a cross-sectional area of 0.40 in.^2 and is loaded by forces $P_1 = 1700 \text{ lb}$, $P_2 = 1200 \text{ lb}$, and $P_3 = 1300 \text{ lb}$. The lengths of the segments of the bar are $a = 60 \text{ in.}$, $b = 24 \text{ in.}$, and $c = 36 \text{ in.}$



- Assuming that the modulus of elasticity $E = 10.4 \times 10^6 \text{ psi}$, calculate the change in length of the bar. Does the bar elongate or shorten?
- By what amount P should the load P_3 be increased so that the bar does not change in length when the three loads are applied?
- If P_3 remains at 1300 lb , what revised cross-sectional area for segment AB will result in no change of length when all three loads are applied?

Solution 2.3-3

NUMERICAL DATA

$$A = 0.40 \text{ in.}^2 \quad P_1 = 1700 \text{ lb}$$

$$P_2 = 1200 \text{ lb} \quad P_3 = 1300 \text{ lb}$$

$$E = 10.4 (10^6) \text{ psi}$$

$$a = 60 \text{ in.} \quad b = 24 \text{ in.} \quad c = 36 \text{ in.}$$

(a) TOTAL ELONGATION

$$\delta = \frac{1}{EA} [(P_1 + P_2 - P_3)a + (P_2 - P_3)b + (-P_3)c] = 0.01125 \text{ in.} \quad \boxed{\delta = 0.01125 \text{ in.}} \quad (\text{elongation})$$

(b) INCREASE P_3 SO THAT BAR DOES NOT CHANGE LENGTH

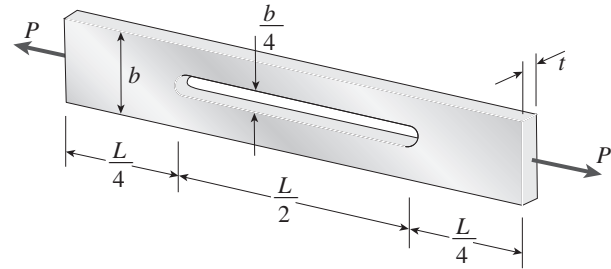
$$\frac{1}{EA} [(P_1 + P_2 - P_3)a + (P_2 - P_3)b + (-P_3)c] = 0 \text{ solve, } P_3 = 1690 \text{ lb} \quad \boxed{\text{So new value of } P_3 \text{ is } 1690 \text{ lb, an increase of } 390 \text{ lb.}}$$

(c) NOW CHANGE CROSS-SECTIONAL AREA OF AB SO THAT BAR DOES NOT CHANGE LENGTH $P_3 = 1300 \text{ lb}$

$$\frac{1}{E} \left[(P_1 + P_2 - P_3) \frac{a}{A_{AB}} + (P_2 - P_3) \frac{b}{A} + (-P_3) \frac{c}{A} \right] = 0$$

$$\text{Solving for } A_{AB}: \quad \boxed{A_{AB} = 0.78 \text{ in.}^2} \quad \frac{A_{AB}}{A} = 1.951$$

Problem 2.3-4 A rectangular bar of length L has a slot in the middle half of its length (see figure). The bar has width b , thickness t , and modulus of elasticity E . The slot has width $b/4$.



- Obtain a formula for the elongation δ of the bar due to the axial loads P .
- Calculate the elongation of the bar if the material is high-strength steel, the axial stress in the middle region is 160 MPa, the length is 750 mm, and the modulus of elasticity is 210 GPa.
- If the total elongation of the bar is limited to $\delta_{\max} = 0.475$ mm, what is the maximum length of the slotted region? Assume that the axial stress in the middle region remains at 160 MPa.

Solution 2.3-4

$$(a) \quad \delta = \frac{P}{E} \left(\frac{2 \frac{L}{4}}{bt} + \frac{\frac{L}{2}}{\frac{3}{4}bt} \right) = \frac{7LP}{6Ebt} \quad \boxed{\delta = \frac{7PL}{6Ebt}}$$

$$(b) \quad \text{NUMERICAL DATA} \quad E = 210 \text{ GPa} \quad L = 750 \text{ mm} \quad \sigma_{\text{mid}} = 160 \text{ MPa}$$

$$\text{so } \sigma_{\text{mid}} = \frac{P}{\frac{3}{4}bt} \quad \text{and} \quad \frac{P}{bt} = \frac{3}{4}\sigma_{\text{mid}}$$

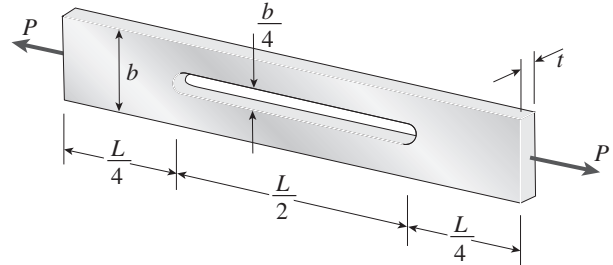
$$\delta = \frac{7LP}{6Ebt} \quad \text{or} \quad \delta = \frac{7L}{6E} \left(\frac{3}{4}\sigma_{\text{mid}} \right) = 0.5 \text{ mm} \quad \boxed{\delta = 0.5 \text{ mm}}$$

$$(c) \quad \delta_{\max} = \frac{P}{E} \left(\frac{L - L_{\text{slot}}}{bt} + \frac{L_{\text{slot}}}{\frac{3}{4}bt} \right) \quad \text{or} \quad \delta_{\max} = \left(\frac{P}{bt} \right) \left(\frac{1}{E} \right) \left(L - L_{\text{slot}} + \frac{4}{3}L_{\text{slot}} \right)$$

$$\text{or } \delta_{\max} = \left(\frac{3}{4}\sigma_{\text{mid}} \right) \left(\frac{1}{E} \right) \left(L + \frac{L_{\text{slot}}}{3} \right) \quad \text{Solving for } L_{\text{slot}} \text{ with } \delta_{\max} = 0.475 \text{ mm}$$

$$L_{\text{slot}} = \frac{4E\delta_{\max} - 3L\sigma_{\text{mid}}}{\sigma_{\text{mid}}} = 244 \text{ mm} \quad \boxed{L_{\text{slot}} = 244 \text{ mm}} \quad \frac{L_{\text{slot}}}{L} = 0.325$$

Problem 2.3-5 Solve the preceding problem if the axial stress in the middle region is 24,000 psi, the length is 30 in., and the modulus of elasticity is 30×10^6 psi. In part (c), assume that $\delta_{\max} = 0.02$ in.



Solution 2.3-5

$$(a) \quad \delta = \frac{P}{E} \left(\frac{2 \frac{L}{4}}{bt} + \frac{\frac{L}{2}}{\frac{3}{4}bt} \right) = \frac{7LP}{6Ebt}$$

$$(b) \quad E = 30,000 \text{ ksi} \quad L = 30 \text{ in.} \quad \sigma_{\text{mid}} = 24 \text{ ksi}$$

$$\text{So} \quad \sigma_{\text{mid}} = \frac{P}{\frac{3}{4}bt} \quad \text{and} \quad \frac{P}{bt} = \frac{3}{4} \sigma_{\text{mid}}$$

$$\delta = \frac{7LP}{6Ebt} \quad \text{or} \quad \delta = \frac{7L}{6E} \left(\frac{3}{4} \sigma_{\text{mid}} \right) = 0.021 \text{ in.} \quad \boxed{\delta = 0.021 \text{ in.}}$$

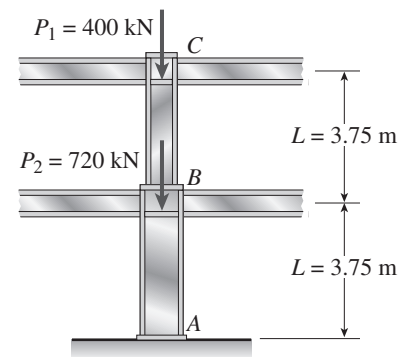
$$(c) \quad \delta_{\max} = \frac{P}{E} \left(\frac{L - L_{\text{slot}}}{bt} + \frac{L_{\text{slot}}}{\frac{3}{4}bt} \right) \quad \text{or} \quad \delta_{\max} = \left(\frac{P}{bt} \right) \left(\frac{1}{E} \right) \left(L - L_{\text{slot}} + \frac{4}{3} L_{\text{slot}} \right)$$

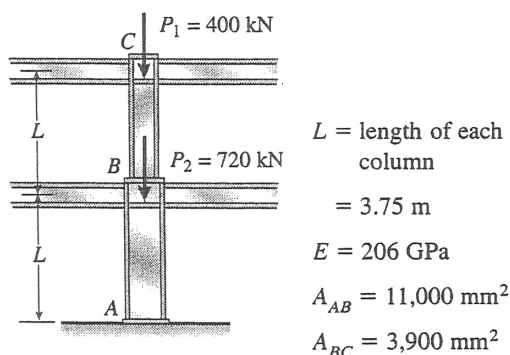
$$\text{or} \quad \delta_{\max} = \left(\frac{3}{4} \sigma_{\text{mid}} \right) \left(\frac{1}{E} \right) \left(L + \frac{L_{\text{slot}}}{3} \right) \quad \text{Solving for } L_{\text{slot}} \text{ with } \delta_{\max} = 0.02 \text{ in.:$$

$$L_{\text{slot}} = \frac{4E\delta_{\max} - 3L\sigma_{\text{mid}}}{\sigma_{\text{mid}}} = 10 \text{ in.} \quad \boxed{L_{\text{slot}} = 10 \text{ in.}} \quad \frac{L_{\text{slot}}}{L} = 0.333$$

Problem 2.3-6 A two-story building has steel columns AB in the first floor and BC in the second floor, as shown in the figure. The roof load P_1 equals 400 kN and the second-floor load P_2 equals 720 kN. Each column has length $L = 3.75$ m. The cross-sectional areas of the first- and second-floor columns are $11,000 \text{ mm}^2$ and $3,900 \text{ mm}^2$, respectively.

- Assuming that $E = 206 \text{ GPa}$, determine the total shortening δ_{AC} of the two columns due to the combined action of the loads P_1 and P_2 .
- How much additional load P_0 can be placed at the top of the column (point C) if the total shortening δ_{AC} is not to exceed 4.0 mm?



Solution 2.3-6 Steel columns in a building(a) SHORTENING δ_{AC} OF THE TWO COLUMNS

$$\begin{aligned}
 \delta_{AC} &= \sum \frac{N_i L_i}{E_i A_i} = \frac{N_{AB} L}{E A_{AB}} + \frac{N_{BC} L}{E A_{BC}} \\
 &= \frac{(1120 \text{ kN})(3.75 \text{ m})}{(206 \text{ GPa})(11,000 \text{ mm}^2)} \\
 &\quad + \frac{(400 \text{ kN})(3.75 \text{ m})}{(206 \text{ GPa})(3,900 \text{ mm}^2)} \\
 &= 1.8535 \text{ mm} + 1.8671 \text{ mm} = 3.7206 \text{ mm} \\
 \delta_{AC} &= 3.72 \text{ mm} \quad \leftarrow
 \end{aligned}$$

(b) ADDITIONAL LOAD P_0 AT POINT C

$$(\delta_{AC})_{\max} = 4.0 \text{ mm}$$

 $\delta_0 = \text{additional shortening of the two columns due to the load } P_0$

$$\begin{aligned}
 \delta_0 &= (\delta_{AC})_{\max} - \delta_{AC} = 4.0 \text{ mm} - 3.7206 \text{ mm} \\
 &= 0.2794 \text{ mm}
 \end{aligned}$$

$$\text{Also, } \delta_0 = \frac{P_0 L}{E A_{AB}} + \frac{P_0 L}{E A_{BC}} = \frac{P_0 L}{E} \left(\frac{1}{A_{AB}} + \frac{1}{A_{BC}} \right)$$

Solve for P_0 :

$$P_0 = \frac{E \delta_0}{L} \left(\frac{A_{AB} A_{BC}}{A_{AB} + A_{BC}} \right)$$

SUBSTITUTE NUMERICAL VALUES:

$$E = 206 \times 10^9 \text{ N/m}^2 \quad \delta_0 = 0.2794 \times 10^{-3} \text{ m}$$

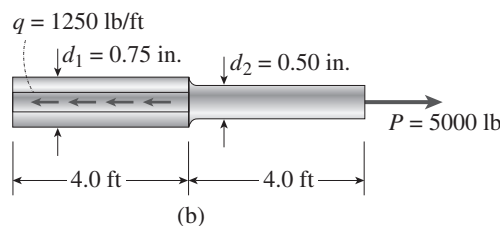
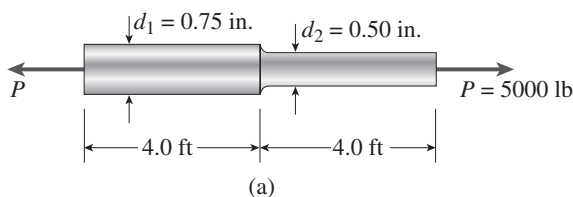
$$L = 3.75 \text{ m} \quad A_{AB} = 11,000 \times 10^{-6} \text{ m}^2$$

$$A_{BC} = 3,900 \times 10^{-6} \text{ m}^2$$

$$P_0 = 44,200 \text{ N} = 44.2 \text{ kN} \quad \leftarrow$$

Problem 2.3-7 A steel bar 8.0 ft long has a circular cross section of diameter $d_1 = 0.75$ in. over one-half of its length and diameter $d_2 = 0.5$ in. over the other half (see figure part a). The modulus of elasticity $E = 30 \times 10^6$ psi.

- How much will the bar elongate under a tensile load $P = 5000$ lb?
- If the same volume of material is made into a bar of constant diameter d and length 8.0 ft, what will be the elongation under the same load P ?
- If the uniform axial centroidal load $q = 1250$ lb/ft is applied to the left over segment 1 (see figure part b), find the ratio of the total elongation of the bar to that in parts (a) and (b).



Solution 2.3-7

NUMERICAL DATA $E = 30(10^6)$ psi $P = 5000$ lb $L = 4$ ft $d_1 = 0.75$ in. $d_2 = 0.5$ in.

$$(a) \delta_a = \frac{PL}{E} \left(\frac{1}{\frac{\pi}{4}d_1^2} + \frac{1}{\frac{\pi}{4}d_2^2} \right) = 0.0589 \text{ in.} \quad \boxed{\delta_a = 0.0589 \text{ in.}}$$

$$(b) V_a = \left(\frac{\pi}{4}d_1^2 + \frac{\pi}{4}d_2^2 \right)L = 30.631 \text{ in.}^3 \quad d = \sqrt{\frac{V_a}{\frac{\pi}{4}(2L)}} = 0.637 \text{ in.} \quad A = \frac{\pi}{4}d^2 = 0.31907 \text{ in.}^2$$

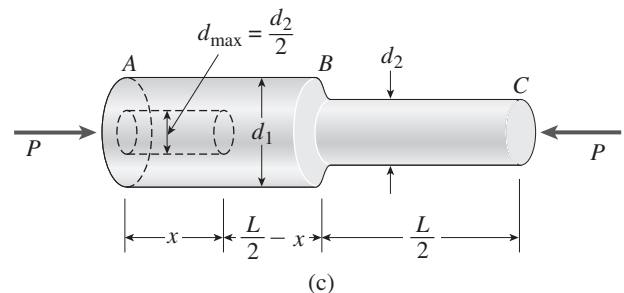
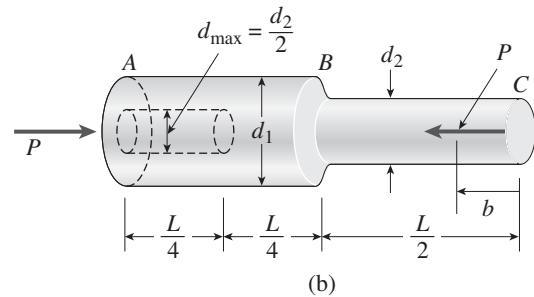
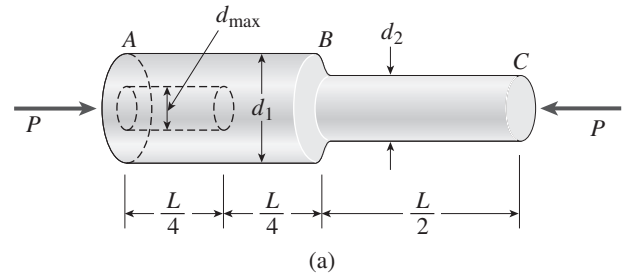
$$\delta_b = \frac{P(2L)}{EA} = 0.0501 \text{ in.} \quad \boxed{\delta_b = 0.0501 \text{ in.}}$$

$$(c) q = 1250 \text{ lb/ft} \quad L = 4 \text{ ft}$$

$$\delta_c = \frac{qL^2}{2E\left(\frac{\pi}{4}d_1^2\right)} + \frac{PL}{EA} = 0.0341 \text{ in.} \quad \boxed{\frac{\delta_c}{\delta_a} = 0.58} \quad \boxed{\frac{\delta_c}{\delta_b} = 0.681}$$

Problem 2.3-8 A bar ABC of length L consists of two parts of equal lengths but different diameters. Segment AB has diameter $d_1 = 100$ mm, and segment BC has diameter $d_2 = 60$ mm. Both segments have length $L/2 = 0.6$ m. A longitudinal hole of diameter d is drilled through segment AB for one-half of its length (distance $L/4 = 0.3$ m). The bar is made of plastic having modulus of elasticity $E = 4.0$ GPa. Compressive loads $P = 110$ kN act at the ends of the bar.

- If the shortening of the bar is limited to 8.0 mm, what is the maximum allowable diameter d_{\max} of the hole? (See figure part a.)
- Now, if d_{\max} is instead set at $d_2/2$, at what distance b from end C should load P be applied to limit the bar shortening to 8.0 mm? (See figure part b.)
- Finally, if loads P are applied at the ends and $d_{\max} = d_2/2$, what is the permissible length x of the hole if shortening is to be limited to 8.0 mm? (See figure part c.)



Solution 2.3-8

NUMERICAL DATA

$$d_1 = 100 \text{ mm} \quad d_2 = 60 \text{ mm}$$

$$L = 1200 \text{ mm} \quad E = 4.0 \text{ GPa} \quad P = 110 \text{ kN}$$

$$\delta_a = 8.0 \text{ mm}$$

(a) FIND d_{\max} IF SHORTENING IS LIMITED TO δ_a

$$A_1 = \frac{\pi}{4} d_1^2 \quad A_2 = \frac{\pi}{4} d_2^2$$

$$\delta = \frac{P}{E} \left[\frac{\frac{L}{4}}{\frac{\pi}{4} (d_1^2 - d_{\max}^2)} + \frac{\frac{L}{4}}{A_1} + \frac{\frac{L}{2}}{A_2} \right]$$

Set δ to δ_a , and solve for d_{\max} :

$$d_{\max} = d_1 \sqrt{\frac{E\delta_a \pi d_1^2 d_2^2 - 2PLd_2^2 - 2PLd_1^2}{E\delta_a \pi d_1^2 d_2^2 - PLd_2^2 - 2PLd_1^2}}$$

$$d_{\max} = 23.9 \text{ mm} \quad \leftarrow$$

(b) NOW, IF d_{\max} IS INSTEAD SET AT $d_2/2$, AT WHAT DISTANCE b FROM END C SHOULD LOAD P BE APPLIED TO LIMIT THE BAR SHORTENING TO $\delta_a = 8.0 \text{ mm}$?

$$A_0 = \frac{\pi}{4} \left[d_1^2 - \left(\frac{d_2}{2} \right)^2 \right]$$

$$A_1 = \frac{\pi}{4} d_1^2 \quad A_2 = \frac{\pi}{4} d_2^2$$

$$\delta = \frac{P}{E} \left[\frac{L}{4A_0} + \frac{L}{4A_1} + \frac{\left(\frac{L}{2} - b \right)}{A_2} \right]$$

No axial force in segment at end of length b ; set $\delta = \delta_a$ and solve for b :

$$b = \left[\frac{L}{2} - A_2 \left[\frac{E\delta_a}{P} - \left(\frac{L}{4A_0} + \frac{L}{4A_1} \right) \right] \right]$$

$$b = 4.16 \text{ mm} \quad \leftarrow$$

(c) FINALLY IF LOADS P ARE APPLIED AT THE ENDS AND $d_{\max} = d_2/2$, WHAT IS THE PERMISSIBLE LENGTH x OF THE HOLE IF SHORTENING IS TO BE LIMITED TO $\delta_a = 8.0 \text{ mm}$?

$$\delta = \frac{P}{E} \left[\frac{x}{A_0} + \frac{\left(\frac{L}{2} - x \right)}{A_1} + \frac{\left(\frac{L}{2} \right)}{A_2} \right]$$

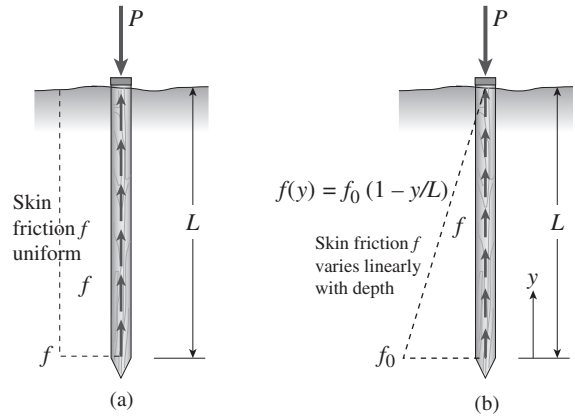
Set $\delta = \delta_a$ and solve for x :

$$x = \frac{\left[A_0 A_1 \left(\frac{E\delta_a}{P} - \frac{L}{2A_2} \right) \right] - \frac{1}{2} A_0 L}{A_1 - A_0}$$

$$x = 183.3 \text{ mm} \quad \leftarrow$$

Problem 2.3-9 A wood pile, driven into the earth, supports a load P entirely by friction along its sides (see figure part a). The friction force f per unit length of pile is assumed to be uniformly distributed over the surface of the pile. The pile has length L , cross-sectional area A , and modulus of elasticity E .

- Derive a formula for the shortening δ of the pile in terms of P , L , E , and A .
- Draw a diagram showing how the compressive stress σ_c varies throughout the length of the pile.
- Repeat parts (a) and (b) if skin friction f varies linearly with depth (see figure part b).



Solution 2.3-9

AFD LINEAR

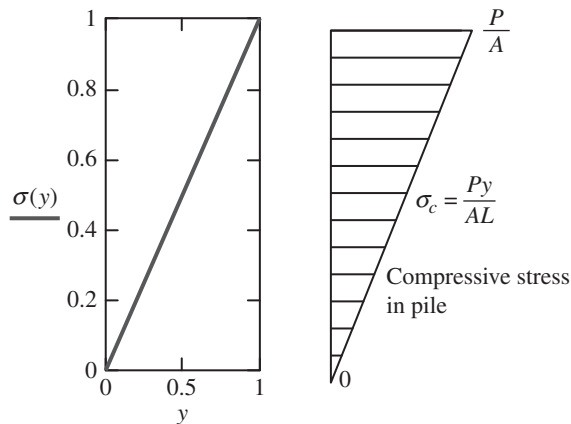
$$(a) \quad N(y) = fy \quad \delta = \int_0^L \frac{f(y)}{EA} dy = \frac{L^2 f}{2AE} \quad \boxed{\delta = \frac{PL}{2EA}}$$

$$(b) \quad \sigma(y) = \frac{N(y)}{A} \quad \sigma(y) = \frac{fy}{A} \quad \sigma(L) = \frac{fL}{A} = \frac{P}{A}$$

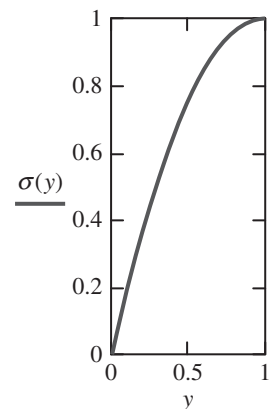
$$\sigma(0) = 0$$

So linear variation, zero at bottom, P/A at top (i.e., at ground surface)

$$N(L) = f \quad \boxed{\sigma(y) = \frac{P}{A} \left(\frac{y}{L} \right)}$$



$f(y)$ is constant
and AFD is linear



$f(y)$ is linear and
AFD quadratic

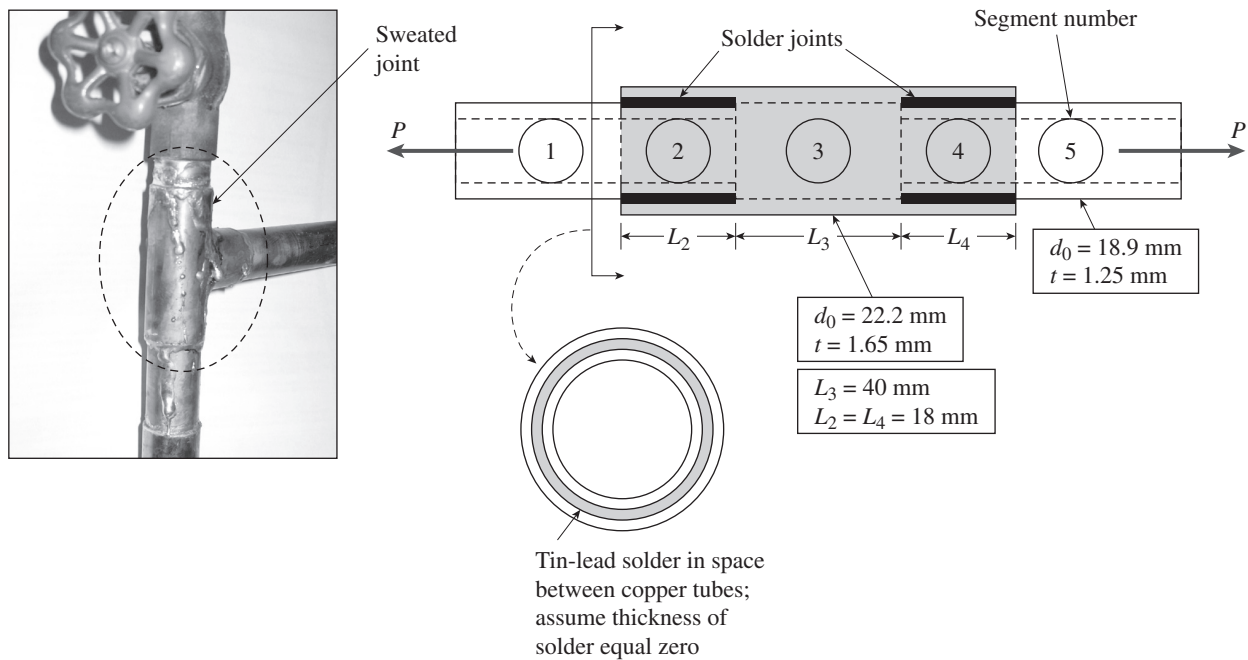
(c) $N(y) = f(y)y$

$$N(y) = \int_0^y f_0 \left(1 - \frac{\xi}{L}\right) d\xi = \frac{f_0 y (y - 2)}{2} \quad N(L) = \frac{f_0}{2} \quad N(0) = 0$$

$$\delta = \frac{\left(\frac{f_0 L}{2}\right)}{\frac{3}{2}EA} \quad P = \frac{1}{2}f_0 L \quad \boxed{\delta = \frac{PL}{EA} \left(\frac{2}{3}\right)} \quad \boxed{\sigma(y) = \frac{P}{A} \left[\frac{y}{L} \left(2 - \frac{y}{L}\right)\right]} \quad \sigma(0) = 0 \quad \sigma(L) = \frac{f_0}{2} = P/A$$

Problem 2.3-10 Consider the copper tubes joined below using a “sweated” joint. Use the properties and dimensions given.

- Find the total elongation of segment 2-3-4 (δ_{2-4}) for an applied tensile force of $P = 5$ kN. Use $E_c = 120$ GPa.
- If the yield strength in shear of the tin-lead solder is $\tau_y = 30$ MPa and the tensile yield strength of the copper is $\sigma_y = 200$ MPa, what is the maximum load P_{\max} that can be applied to the joint if the desired factor of safety in shear is $FS_\tau = 2$ and in tension is $FS_\sigma = 1.7$?
- Find the value of L_2 at which tube and solder capacities are equal.



Solution 2.3-10

NUMERICAL DATA

$$\begin{aligned}
 P &= 5 \text{ kN} & E_c &= 120 \text{ GPa} \\
 L_2 &= 18 \text{ mm} & L_4 &= L_2 \\
 L_3 &= 40 \text{ mm} \\
 d_{o3} &= 22.2 \text{ mm} & t_3 &= 1.65 \text{ mm} \\
 d_{o5} &= 18.9 \text{ mm} & t_5 &= 1.25 \text{ mm} \\
 \tau_Y &= 30 \text{ MPa} & \sigma_Y &= 200 \text{ MPa} \\
 \text{FS}_\tau &= 2 & \text{FS}_\sigma &= 1.7
 \end{aligned}$$

$$\tau_a = \frac{\tau_Y}{\text{FS}_\tau} \quad \tau_a = 15 \text{ MPa}$$

$$\sigma_a = \frac{\sigma_Y}{\text{FS}_\sigma} \quad \sigma_a = 117.6 \text{ MPa}$$

(a) ELONGATION OF SEGMENT 2-3-4

$$A_2 = \frac{\pi}{4} [d_{o3}^2 - (d_{o5} - 2t_5)^2]$$

$$A_3 = \frac{\pi}{4} [d_{o3}^2 - (d_{o3} - 2t_3)^2]$$

$$A_2 = 175.835 \text{ mm}^2 \quad A_3 = 106.524 \text{ mm}^2$$

$$\delta_{24} = \frac{P}{E_c} \left(\frac{L_2 + L_4}{A_2} + \frac{L_3}{A_3} \right)$$

$$\delta_{24} = 0.024 \text{ mm} \quad \leftarrow$$

(b) MAXIMUM LOAD P_{\max} THAT CAN BE APPLIED TO THE JOINT

First check normal stress:

$$A_1 = \frac{\pi}{4} [d_{o5}^2 - (d_{o5} - 2t_5)^2]$$

$$A_1 = 69.311 \text{ mm}^2 < \text{smallest cross-sectional area controls normal stress}$$

$$P_{\max\sigma} = \sigma_a A_1 \quad P_{\max\sigma} = 8.15 \text{ kN} \quad \leftarrow \text{smaller than } P_{\max} \text{ based on shear below so normal stress controls}$$

Next check shear stress in solder joint:

$$A_{\text{sh}} = \pi d_{o5} L_2 \quad A_{\text{sh}} = 1.069 \times 10^3 \text{ mm}^2$$

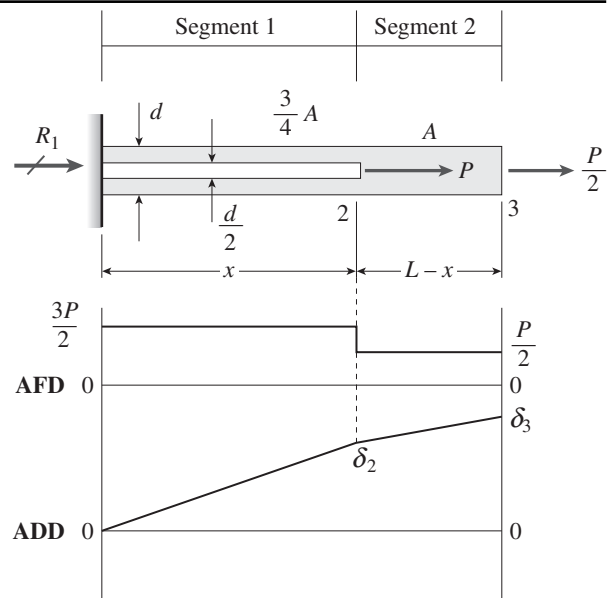
$$P_{\max\tau} = \tau_a A_{\text{sh}} \quad P_{\max\tau} = 16.03 \text{ kN}$$

(c) FIND THE VALUE OF L_2 AT WHICH TUBE AND SOLDER CAPACITIES ARE EQUALSet P_{\max} based on shear strength equal to P_{\max} based on tensile strength and solve for L_2 :

$$L_2 = \frac{\sigma_a A_1}{\tau_a (\pi d_{o5})} \quad L_2 = 9.16 \text{ mm} \quad \leftarrow$$

Problem 2.3-11 The nonprismatic cantilever circular bar shown has an internal cylindrical hole of diameter $d/2$ from 0 to x , so the net area of the cross section for Segment 1 is $(3/4)A$. Load P is applied at x , and load $P/2$ is applied at $x = L$. Assume that E is constant.

- Find reaction force R_1 .
- Find internal axial forces N_i in segments 1 and 2.
- Find x required to obtain axial displacement at joint 3 of $\delta_3 = PL/EA$.
- In (c), what is the displacement at joint 2, δ_2 ?
- If P acts at $x = 2L/3$ and $P/2$ at joint 3 is replaced by βP , find β so that $\delta_3 = PL/EA$.
- Draw the axial force (AFD: $N(x)$, $0 \leq x \leq L$) and axial displacement (ADD: $\delta(x)$, $0 \leq x \leq L$) diagrams using results from (b) through (d) above.



Solution 2.3-11

(a) STATICS $\sum F_H = 0$ $R_1 = -P - \frac{P}{2}$

$$R_1 = \frac{-3}{2}P \quad \leftarrow$$

(b) DRAW FBD'S CUTTING THROUGH SEGMENT 1 AND AGAIN THROUGH SEGMENT 2

$$N_1 = \frac{3P}{2} < \text{tension} \quad N_2 = \frac{P}{2} < \text{tension}$$

(c) FIND x REQUIRED TO OBTAIN AXIAL DISPLACEMENT AT JOINT 3 OF $\delta_3 = PL/EA$

Add axial deformations of segments 1 and 2, then set to δ_3 ; solve for x :

$$\frac{N_1 x}{E \frac{3}{4}A} + \frac{N_2 (L - x)}{EA} = \frac{PL}{EA}$$

$$\frac{\frac{3P}{2}x}{E \frac{3}{4}A} + \frac{\frac{P}{2}(L - x)}{EA} = \frac{PL}{EA}$$

$$\frac{3}{2}x = \frac{L}{2} \quad x = \frac{L}{3} \quad \leftarrow$$

(d) WHAT IS THE DISPLACEMENT AT JOINT 2, δ_2 ?

$$\delta_2 = \frac{N_1 x}{E \frac{3}{4}A} \quad \delta_2 = \frac{\left(\frac{3P}{2}\right)\frac{L}{3}}{E \frac{3}{4}A}$$

$$\delta_2 = \frac{2}{3} \frac{PL}{EA}$$

(e) IF $x = 2L/3$ AND $P/2$ AT JOINT 3 IS REPLACED BY βP , FIND β SO THAT $\delta_3 = PL/EA$

$$N_1 = (1 + \beta)P \quad N_2 = \beta P \quad x = \frac{2L}{3}$$

substitute in axial deformation expression above and solve for β

$$\frac{[(1 + \beta)P] \frac{2L}{3}}{E \frac{3}{4}A} + \frac{\beta P \left(L - \frac{2L}{3}\right)}{EA} = \frac{PL}{EA}$$

$$\frac{1}{9} PL \frac{8 + 11\beta}{EA} = \frac{PL}{EA}$$

$$(8 + 11\beta) = 9$$

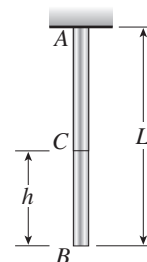
$$\beta = \frac{1}{11} \quad \leftarrow$$

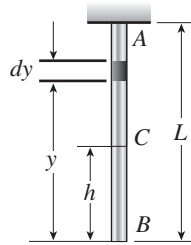
$$\beta = 0.091$$

(f) Draw AFD, ADD—see plots for $x = \frac{L}{3}$

Problem 2.3-12 A prismatic bar AB of length L , cross-sectional area A , modulus of elasticity E , and weight W hangs vertically under its own weight (see figure).

- Derive a formula for the downward displacement δ_C of point C , located at distance h from the lower end of the bar.
- What is the elongation δ_B of the entire bar?
- What is the ratio β of the elongation of the upper half of the bar to the elongation of the lower half of the bar?
- If bar AB is a riser pipe hanging from a drill rig at sea, what is the total elongation of the pipe? Let $L = 1500$ m, $A = 0.0157$ m², $E = 210$ GPa. See Appendix I for weight densities of steel and sea water. (See Problems 1.4-2 and 1.7-11 for additional figures).



Solution 2.3-12 Prismatic bar hanging vertically $W =$ Weight of bar

(a) DOWNWARD DISPLACEMENT δ_C
Consider an element at distance y from the lower end.

$$N(y) = \frac{Wy}{L} \quad d\delta = \frac{N(y)dy}{EA} = \frac{Wydy}{EAL}$$

$$\delta_C = \int_h^L d\delta = \int_h^L \frac{Wydy}{EAL} = \frac{W}{2EAL} (L^2 - h^2)$$

$$\delta_C = \frac{W}{2EAL} (L^2 - h^2) \quad \leftarrow$$

(b) ELONGATION OF BAR ($h = 0$)

$$\delta_B = \frac{WL}{2EA} \quad \leftarrow$$

(c) RATIO OF ELONGATIONS

Elongation of upper half of bar ($h = \frac{L}{2}$):

$$\delta_{\text{upper}} = \frac{3WL}{8EA}$$

Elongation of lower half of bar:

$$\delta_{\text{lower}} = \delta_B - \delta_{\text{upper}} = \frac{WL}{2EA} - \frac{3WL}{8EA} = \frac{WL}{8EA}$$

$$\beta = \frac{\delta_{\text{upper}}}{\delta_{\text{lower}}} = \frac{3/8}{1/8} = 3 \quad \leftarrow$$

(d) NUMERICAL DATA

$$\gamma_s = 77 \text{ kN/m}^3$$

$$\gamma_w = 10 \text{ kN/m}^3$$

$$L = 1500 \text{ m}$$

$$A = 0.0157 \text{ m}^2$$

$$E = 210 \text{ GPa}$$

In sea water:

$$W = (\gamma_s - \gamma_w)AL = 1577.85 \text{ kN}$$

$$\delta = \frac{WL}{2EA} = 359 \text{ mm}$$

$$\frac{\delta}{L} = 2.393 \times 10^{-4}$$

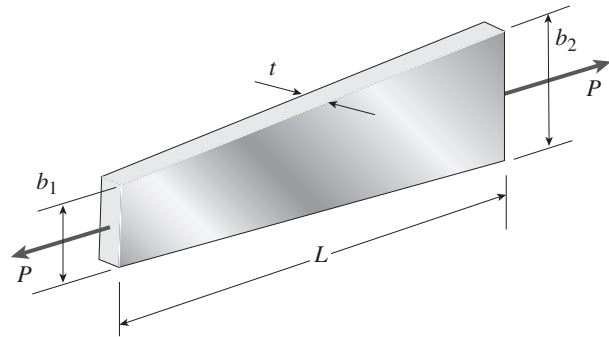
In air:

$$W = (\gamma_s)AL = 1813.35 \text{ kN}$$

$$\delta = \frac{WL}{2EA} = 412 \text{ mm}$$

$$\frac{\delta}{L} = 2.75 \times 10^{-4}$$

Problem 2.3-13 A flat bar of rectangular cross section, length L , and constant thickness t is subjected to tension by forces P (see figure). The width of the bar varies linearly from b_1 at the smaller end to b_2 at the larger end. Assume that the angle of taper is small.

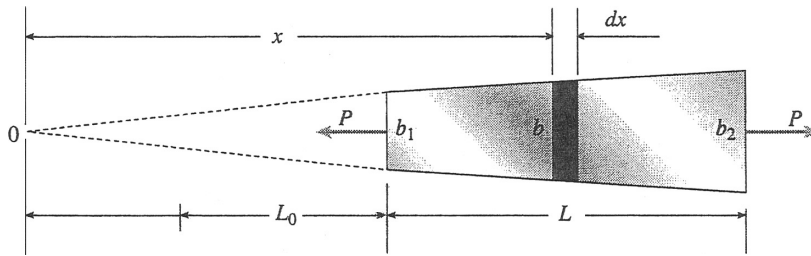


- (a) Derive the following formula for the elongation of the bar:

$$\delta = \frac{PL}{Et(b_2 - b_1)} \ln \frac{b_2}{b_1}$$

- (b) Calculate the elongation, assuming $L = 5$ ft, $t = 1.0$ in., $P = 25$ k, $b_1 = 4.0$ in., $b_2 = 6.0$ in., and $E = 30 \times 10^6$ psi.

Solution 2.3-13 Tapered bar (rectangular cross section)



t = thickness (constant)

$$b = b_1 \left(\frac{x}{L_0} \right) \quad b_2 = b_1 \left(\frac{L_0 + L}{L_0} \right) \quad (\text{Eq. 1})$$

$$A(x) = bt = b_1 t \left(\frac{x}{L_0} \right)$$

(a) ELONGATION OF THE BAR

$$d\delta = \frac{Pdx}{EA(x)} = \frac{PL_0 dx}{Eb_1 tx}$$

$$\begin{aligned} \delta &= \int_{L_0}^{L_0+L} d\delta = \frac{PL_0}{Eb_1 t} \int_{L_0}^{L_0+L} \frac{dx}{x} \\ &= \frac{PL_0}{Eb_1 t} \ln x \Big|_{L_0}^{L_0+L} = \frac{PL_0}{Eb_1 t} \ln \frac{L_0 + L}{L_0} \quad (\text{Eq. 2}) \end{aligned}$$

$$\text{From Eq. (1): } \frac{L_0 + L}{L_0} = \frac{b_2}{b_1} \quad (\text{Eq. 3})$$

$$\text{Solve Eq. (3) for } L_0: L_0 = L \left(\frac{b_1}{b_2 - b_1} \right) \quad (\text{Eq. 4})$$

Substitute Eqs. (3) and (4) into Eq. (2):

$$\delta = \frac{PL}{Et(b_2 - b_1)} \ln \frac{b_2}{b_1} \quad (\text{Eq. 5})$$

(b) SUBSTITUTE NUMERICAL VALUES:

$$L = 5 \text{ ft} = 60 \text{ in.} \quad t = 1.0 \text{ in.}$$

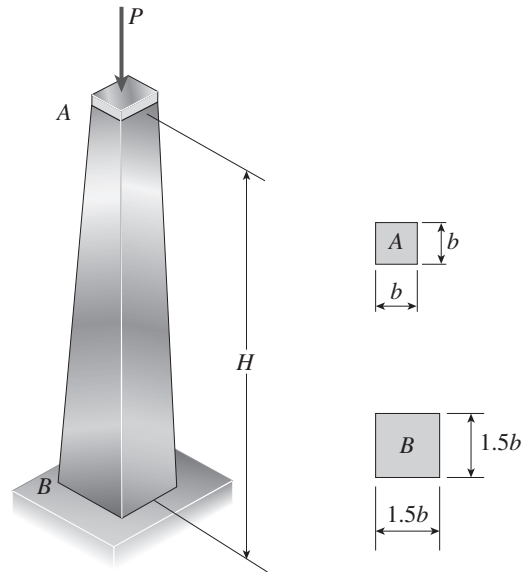
$$P = 25 \text{ k} \quad b_1 = 4.0 \text{ in.}$$

$$b_2 = 6.0 \text{ in.} \quad E = 30 \times 10^6 \text{ psi}$$

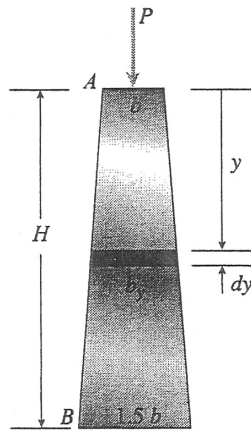
$$\text{From Eq. (5): } \delta = 0.010 \text{ in.} \quad \leftarrow$$

Problem 2.3-14 A post AB supporting equipment in a laboratory is tapered uniformly throughout its height H (see figure). The cross sections of the post are square, with dimensions $b \times b$ at the top and $1.5b \times 1.5b$ at the base.

Derive a formula for the shortening δ of the post due to the compressive load P acting at the top. (Assume that the angle of taper is small and disregard the weight of the post itself.)



Solution 2.3-14 Tapered post



Square cross sections:

$$b = \text{width at } A$$

$$1.5b = \text{width at } B$$

$$b_y = \text{width at distance } y$$

$$= b + (1.5b - b) \frac{y}{H}$$

$$= \frac{b}{H}(H + 0.5y)$$

A_y = cross-sectional area at distance y

$$= (b_y)^2 = \frac{b^2}{H^2}(H + 0.5y)^2$$

SHORTENING OF ELEMENT dy

$$d\delta = \frac{Pdy}{EA_y} = \frac{Pdy}{E\left(\frac{b^2}{H^2}\right)(H + 0.5y)^2}$$

SHORTENING OF ENTIRE POST

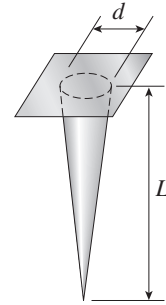
$$\delta = \int d\delta = \frac{PH^2}{Eb^2} \int_0^H \frac{dy}{(H + 0.5y)^2}$$

From Appendix D: $\int \frac{dx}{(a + bx)^2} = -\frac{1}{b(a + bx)}$

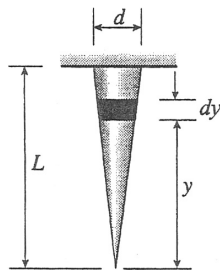
$$\begin{aligned} \delta &= \frac{PH^2}{Eb^2} \left[-\frac{1}{(0.5)(H + 0.5y)} \right]_0^H \\ &= \frac{PH^2}{Eb^2} \left[-\frac{1}{(0.5)(1.5H)} + \frac{1}{0.5H} \right] \\ &= \frac{2PH}{3Eb^2} \quad \leftarrow \end{aligned}$$

Problem 2.3-15 A long, slender bar in the shape of a right circular cone with length L and base diameter d hangs vertically under the action of its own weight (see figure). The weight of the cone is W and the modulus of elasticity of the material is E .

Derive a formula for the increase δ in the length of the bar due to its own weight. (Assume that the angle of taper of the cone is small.)



Solution 2.3-15 Conical bar hanging vertically



TERMINOLOGY

N_y = axial force acting on element dy

A_y = cross-sectional area at element dy

A_B = cross-sectional area at base of cone

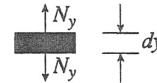
$$= \frac{\pi d^2}{4} \quad V = \text{volume of cone}$$

$$= \frac{1}{3} A_B L \quad V_y = \text{volume of cone below element } dy$$

$$= \frac{1}{3} A_y y \quad W_y = \text{weight of cone below element } dy$$

$$= \frac{V_y}{V} (W) = \frac{A_y y W}{A_B L} \quad N_y = W_y$$

ELEMENT OF BAR



W = weight of cone

ELONGATION OF ELEMENT dy

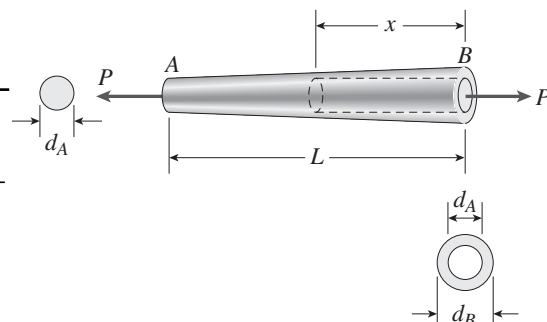
$$d\delta = \frac{N_y dy}{E A_y} = \frac{W_y dy}{E A_B L} = \frac{4W}{\pi d^2 E L} y dy$$

ELONGATION OF CONICAL BAR

$$\delta = \int d\delta = \frac{4W}{\pi d^2 E L} \int_0^L y dy = \frac{2WL}{\pi d^2 E} \quad \leftarrow$$

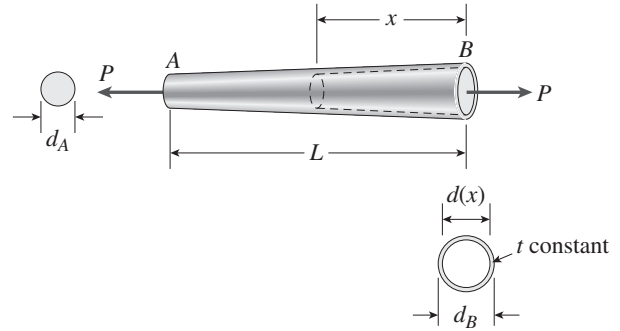
Problem 2.3-16 A uniformly tapered plastic tube AB of circular cross section and length L is shown in the figure. The average diameters at the ends are d_A and $d_B = 2d_A$. Assume E is constant. Find the elongation δ of the tube when it is subjected to loads P acting at the ends. Use the following numerical data: $d_A = 35$ mm, $L = 300$ mm, $E = 2.1$ GPa, $P = 25$ kN. Consider two cases as follows:

- (a) A hole of constant diameter d_A is drilled from B toward A to form a hollow section of length $x = L/2$ (see figure part a).



(a)

- (b) A hole of *variable* diameter $d(x)$ is drilled from B toward A to form a hollow section of length $x = L/2$ and constant thickness t (see figure part b). (Assume that $t = d_A/20$.)



(b)

Solution 2.3-16

- (a) ELONGATION δ FOR CASE OF CONSTANT DIAMETER HOLE

$$d(\zeta) = d_A \left(1 + \frac{\zeta}{L} \right) \quad A(\zeta) = \frac{\pi}{4} d(\zeta)^2 \quad < \text{solid portion of length } L - x$$

$$A(\zeta) = \frac{\pi}{4} (d(\zeta)^2 - d_A^2) \quad < \text{hollow portion of length } x$$

$$\delta = \frac{P}{E} \left(\int \frac{1}{A(\zeta)} d\zeta \right) \quad \delta = \frac{P}{E} \left[\int_0^{L-x} \frac{4}{\pi d(\zeta)^2} d\zeta + \int_{L-x}^L \frac{4}{\pi (d(\zeta)^2 - d_A^2)} d\zeta \right]$$

$$\delta = \frac{P}{E} \left[\int_0^{L-x} \frac{1}{\left[\frac{\pi}{4} \left[d_A \left(1 + \frac{\zeta}{L} \right) \right]^2 \right]} d\zeta + \int_{L-x}^L \frac{1}{\left[\frac{\pi}{4} \left[d_A \left(1 + \frac{\zeta}{L} \right) \right]^2 - d_A^2 \right]} d\zeta \right]$$

$$\delta = \frac{P}{E} \left[4 \frac{L^2}{(-2+x)\pi d_A^2} + \left[4 \frac{L}{\pi d_A^2} + \int_{L-x}^L \frac{1}{\left[\frac{\pi}{4} \left[d_A \left(1 + \frac{\zeta}{L} \right) \right]^2 - d_A^2 \right]} d\zeta \right] \right]$$

$$\delta = \frac{P}{E} \left[4 \frac{L^2}{(-2+x)\pi d_A^2} + \left(4 \frac{L}{\pi d_A^2} - 2L \frac{\ln(3)}{\pi d_A^2} + 2L \frac{-\ln(L-x) + \ln(3L-x)}{\pi d_A^2} \right) \right]$$

$$\text{if } x = L/2 \quad \delta = \frac{P}{E} \left(\frac{4}{3} \frac{L}{\pi d_A^2} - 2L \frac{\ln(3)}{\pi d_A^2} + 2L \frac{-\ln\left(\frac{1}{2}L\right) + \ln\left(\frac{5}{2}L\right)}{\pi d_A^2} \right)$$

Substitute numerical data:

$$\delta = 2.18 \text{ mm} \quad \leftarrow$$

(b) ELONGATION δ FOR CASE OF VARIABLE DIAMETER HOLE BUT CONSTANT WALL THICKNESS $t = d_A/20$ OVER SEGMENT x

$$d(\zeta) = d_A \left(1 + \frac{\zeta}{L} \right) \quad A(\zeta) = \frac{\pi}{4} d(\zeta)^2 \quad \text{solid portion of length } L - x$$

$$A(\zeta) = \frac{\pi}{4} \left[d(\zeta)^2 - \left(d(\zeta) - 2 \frac{d_A}{20} \right)^2 \right] \quad \text{hollow portion of length } x$$

$$\delta = \frac{P}{E} \left(\int \frac{1}{A(\zeta)} d\zeta \right) \quad \delta = \frac{P}{E} \left[\int_0^{L-x} \frac{4}{\pi d(\zeta)^2} d\zeta + \int_{L-x}^L \frac{4}{\pi \left[d(\zeta)^2 - \left(d(\zeta) - 2 \frac{d_A}{20} \right)^2 \right]} d\zeta \right]$$

$$\delta = \frac{P}{E} \left[\int_0^{L-x} \frac{4}{\pi \left[d_A \left(1 + \frac{\zeta}{L} \right) \right]^2} d\zeta + \int_{L-x}^L \frac{4}{\pi \left[\left[d_A \left(1 + \frac{\zeta}{L} \right) \right]^2 - \left[d_A \left(1 + \frac{\zeta}{L} \right) - 2 \frac{d_A}{20} \right]^2 \right]} d\zeta \right]$$

$$\delta = \frac{P}{E} \left[4 \frac{L^2}{(-2L + x) \pi d_A^2} + 4 \frac{L}{\pi d_A^2} + 20L \frac{\ln(3) + \ln(13) + 2\ln(d_A) + \ln(L)}{\pi d_A^2} - 20L \frac{2\ln(d_A) + \ln(39L - 20x)}{\pi d_A^2} \right]$$

if $x = L/2$

$$\delta = \frac{P}{E} \left(\frac{4}{3} \frac{L}{\pi d_A^2} + 20L \frac{\ln(3) + \ln(13) + 2\ln(d_A) + \ln(L)}{\pi d_A^2} - 20L \frac{2\ln(d_A) + \ln(29L)}{\pi d_A^2} \right)$$

Substitute numerical data:

$$\delta = 6.74 \text{ mm} \quad \leftarrow$$

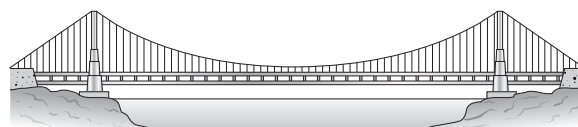
Problem 2.3-17 The main cables of a suspension bridge [see part (a) of the figure] follow a curve that is nearly parabolic because the primary load on the cables is the weight of the bridge deck, which is uniform in intensity along the horizontal. Therefore, let us represent the central region AOB of one of the main cables [see part (b) of the figure] as a parabolic cable supported at points A and B and carrying a uniform load of intensity q along the horizontal. The span of the cable is L , the sag is h , the axial rigidity is EA , and the origin of coordinates is at midspan.

- (a) Derive the following formula for the elongation of cable AOB shown in part (b) of the figure:

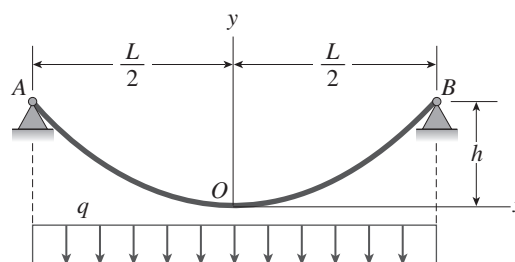
$$\delta = \frac{qL^3}{8hEA} \left(1 + \frac{16h^2}{3L^2} \right)$$

- (b) Calculate the elongation δ of the central span of one of the main cables of the Golden Gate Bridge, for which the dimensions and properties are $L = 4200$ ft, $h = 470$ ft, $q = 12,700$ lb/ft, and $E = 28,800,000$ psi. The cable consists of 27,572 parallel wires of diameter 0.196 in.

Hint: Determine the tensile force T at any point in the cable from a free-body diagram of part of the cable; then determine the elongation of an element of the cable of length ds ; finally, integrate along the curve of the cable to obtain an equation for the elongation δ .

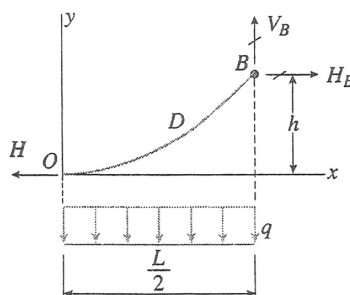
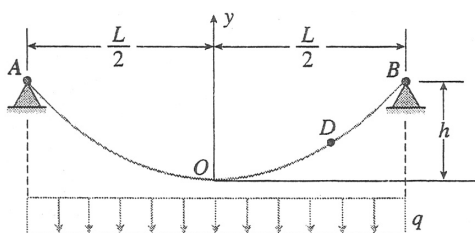


(a)



(b)

Solution 2.3-17 Cable of a suspension bridge



Equation of parabolic curve:

$$y = \frac{4hx^2}{L^2}$$

$$\frac{dy}{dx} = \frac{8hx}{L^2}$$

FREE-BODY DIAGRAM OF HALF OF CABLE

$$\Sigma M_B = 0 \quad \curvearrowright$$

$$-Hh + \frac{qL}{2} \left(\frac{L}{4} \right) = 0$$

$$H = \frac{qL^2}{8h}$$

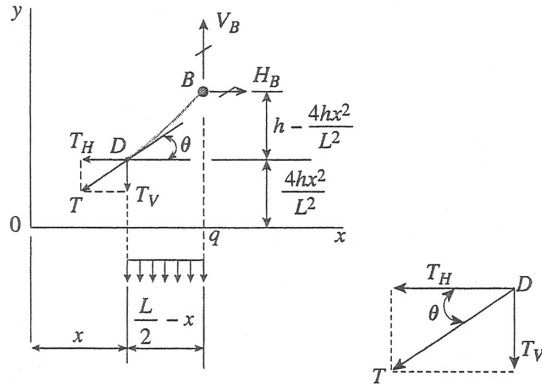
$$\Sigma F_{\text{horizontal}} = 0$$

$$H_B = H = \frac{qL^2}{8h} \quad (\text{Eq. 1})$$

$$\Sigma F_{\text{vertical}} = 0$$

$$V_B = \frac{qL}{2} \quad (\text{Eq. 2})$$

FREE-BODY DIAGRAM OF SEGMENT *DB* OF CABLE



$$\Sigma F_{\text{horiz}} = 0 \quad T_H = H_B = \frac{qL^2}{8h} \quad (\text{Eq. 3})$$

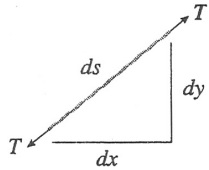
$$\Sigma F_{\text{vert}} = 0 \quad V_B - T_V - q\left(\frac{L}{2} - x\right) = 0$$

$$T_V = V_B - q\left(\frac{L}{2} - x\right) = \frac{qL}{2} - \frac{qL}{2} + qx = qx \quad (\text{Eq. 4})$$

TENSILE FORCE *T* IN CABLE

$$T = \sqrt{T_H^2 + T_V^2} = \sqrt{\left(\frac{qL^2}{8h}\right)^2 + (qx)^2} = \frac{qL^2}{8h} \sqrt{1 + \frac{64h^2x^2}{L^4}} \quad (\text{Eq. 5})$$

ELONGATION $d\delta$ OF AN ELEMENT OF LENGTH ds



$$\begin{aligned} d\delta &= \frac{Tds}{EA} \\ ds &= \sqrt{(dx)^2 + (dy)^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \\ &= dx\sqrt{1 + \left(\frac{8hx}{L^2}\right)^2} \\ &= dx\sqrt{1 + \frac{64h^2x^2}{L^4}} \end{aligned} \quad (\text{Eq. 6})$$

(a) ELONGATION δ OF CABLE *AOB*

$$\delta = \int d\delta = \int \frac{Tds}{EA}$$

Substitute for *T* from Eq. (5) and for *ds* from Eq. (6):

$$\delta = \frac{1}{EA} \int \frac{qL^2}{8h} \left(1 + \frac{64h^2x^2}{L^4}\right) dx$$

For both halves of cable:

$$\begin{aligned} \delta &= \frac{2}{EA} \int_0^{L/2} \frac{qL^2}{8h} \left(1 + \frac{64h^2x^2}{L^4}\right) dx \\ \delta &= \frac{qL^3}{8hEA} \left(1 + \frac{16h^2}{3L^4}\right) \leftarrow \end{aligned} \quad (\text{Eq. 7})$$

(b) GOLDEN GATE BRIDGE CABLE

$$L = 4200 \text{ ft} \quad h = 470 \text{ ft}$$

$$q = 12,700 \text{ lb/ft} \quad E = 28,800,000 \text{ psi}$$

$$27,572 \text{ wires of diameter } d = 0.196 \text{ in.}$$

$$A = (27,572) \left(\frac{\pi}{4}\right) (0.196 \text{ in.})^2 = 831.90 \text{ in.}^2$$

Substitute into Eq. (7):

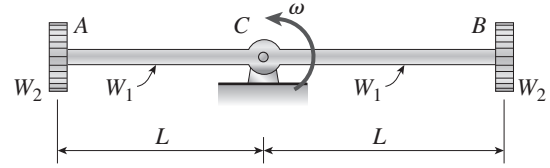
$$\delta = 133.7 \text{ in} = 11.14 \text{ ft} \leftarrow$$

Problem 2.3-18 A bar ABC revolves in a horizontal plane about a vertical axis at the midpoint C (see figure). The bar, which has length $2L$ and cross-sectional area A , revolves at constant angular speed ω . Each half of the bar (AC and BC) has weight W_1 and supports a weight W_2 at its end.

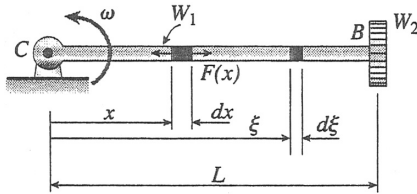
Derive the following formula for the elongation of one-half of the bar (that is, the elongation of either AC or BC):

$$\delta = \frac{L^2 \omega^2}{3gEA} (W_1 + 3W_2)$$

in which E is the modulus of elasticity of the material of the bar and g is the acceleration of gravity.



Solution 2.3-18 Rotating bar



ω = angular speed

A = cross-sectional area

E = modulus of elasticity

g = acceleration of gravity

$F(x)$ = axial force in bar at distance x from point C

Consider an element of length dx at distance x from point C .

To find the force $F(x)$ acting on this element, we must find the inertia force of the part of the bar from distance x to distance L , plus the inertia force of the weight W_2 .

Since the inertia force varies with distance from point C , we now must consider an element of length $d\xi$ at distance ξ , where ξ varies from x to L .

$$\text{Mass of element } d = \frac{d}{L} \left(\frac{W_1}{g} \right)$$

$$\text{Acceleration of element} = \xi \omega^2$$

Centrifugal force produced by element

$$= (\text{mass})(\text{acceleration}) = \frac{W_1 \omega^2}{gL} d$$

Centrifugal force produced by weight W_2

$$= \left(\frac{W_2}{g} \right) (L \omega^2)$$

AXIAL FORCE $F(x)$

$$\begin{aligned} F(x) &= \int_x^L \frac{W_1 \omega^2}{gL} d + \frac{W_2 L \omega^2}{g} \\ &= \frac{W_1 \omega^2}{2gL} (L^2 - x^2) + \frac{W_2 L \omega^2}{g} \end{aligned}$$

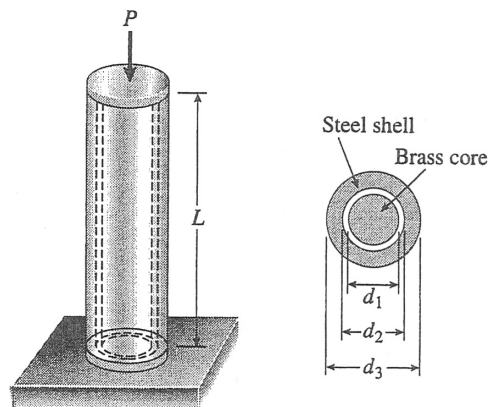
ELONGATION OF BAR BC

$$\begin{aligned} \delta &= \int_0^L \frac{F(x) dx}{EA} \\ &= \int_0^L \frac{W_1 \omega^2}{2gL} (L^2 - x^2) dx + \int_0^L \frac{W_2 L \omega^2}{gEA} dx \\ &= \frac{W_1 L \omega^2}{2gLEA} \left[\int_0^L L^2 dx - \int_0^L x^2 dx \right] + \frac{W_2 L \omega^2}{gEA} \int_0^L dx \\ &= \frac{W_1 L^2 \omega^2}{3gEA} + \frac{W_2 L^2 \omega^2}{gEA} \\ &= \frac{L^2 \omega^2}{3gEA} + (W_1 + 3W_2) \quad \leftarrow \end{aligned}$$

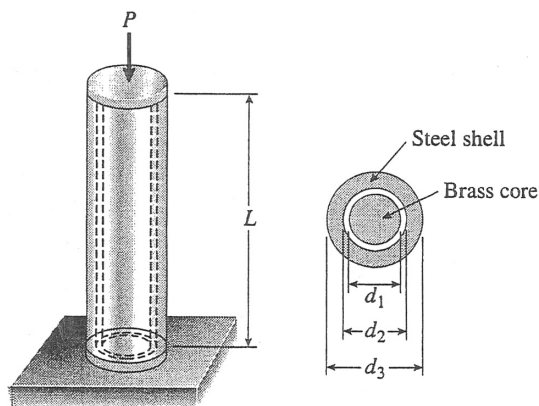
Statically Indeterminate Structures

Problem 2.4-1 The assembly shown in the figure consists of a brass core (diameter $d_1 = 0.25$ in.) surrounded by a steel shell (inner diameter $d_2 = 0.28$ in., outer diameter $d_3 = 0.35$ in.). A load P compresses the core and shell, which have length $L = 4.0$ in. The moduli of elasticity of the brass and steel are $E_b = 15 \times 10^6$ psi and $E_s = 30 \times 10^6$ psi, respectively.

- What load P will compress the assembly by 0.003 in.?
- If the allowable stress in the steel is 22 ksi and the allowable stress in the brass is 16 ksi, what is the allowable compressive load P_{allow} ? (Suggestion: Use the equations derived in Example 2-6.)



Solution 2.4-1 Cylindrical assembly in compression



$$d_1 = 0.25 \text{ in.} \quad E_b = 15 \times 10^6 \text{ psi}$$

$$d_2 = 0.28 \text{ in.} \quad E_s = 30 \times 10^6 \text{ psi}$$

$$d_3 = 0.35 \text{ in.} \quad A_s = \frac{\pi}{4}(d_3^2 - d_2^2) = 0.03464 \text{ in.}^2$$

$$L = 4.0 \text{ in.} \quad A_b = \frac{\pi}{4}d_1^2 = 0.04909 \text{ in.}^2$$

- (a) DECREASE IN LENGTH ($\delta = 0.003$ in.)

Use Eq. (2-18) of Example 2-6.

$$\delta = \frac{PL}{E_s A_s + E_b A_b} \quad \text{or}$$

$$P = (E_s A_s + E_b A_b) \left(\frac{\delta}{L} \right)$$

Substitute numerical values:

$$\begin{aligned} E_s A_s + E_b A_b &= (30 \times 10^6 \text{ psi})(0.03464 \text{ in.}^2) \\ &\quad + (15 \times 10^6 \text{ psi})(0.04909 \text{ in.}^2) \\ &= 1.776 \times 10^6 \text{ lb} \end{aligned}$$

$$\begin{aligned} P &= (1.776 \times 10^6 \text{ lb}) \left(\frac{0.003 \text{ in.}}{4.0 \text{ in.}} \right) \\ &= 1330 \text{ lb} \quad \leftarrow \end{aligned}$$

- (b) ALLOWABLE LOAD

$$\sigma_s = 22 \text{ ksi} \quad \sigma_b = 16 \text{ ksi}$$

Use Eqs. (2-17a and b) of Example 2-6.

For steel:

$$\begin{aligned} \sigma_s &= \frac{PE_s}{E_s A_s + E_b A_b} \quad P_s = (E_s A_s + E_b A_b) \frac{\sigma_s}{E_s} \\ P_s &= (1.776 \times 10^6 \text{ lb}) \left(\frac{22 \text{ ksi}}{30 \times 10^6 \text{ psi}} \right) = 1300 \text{ lb} \end{aligned}$$

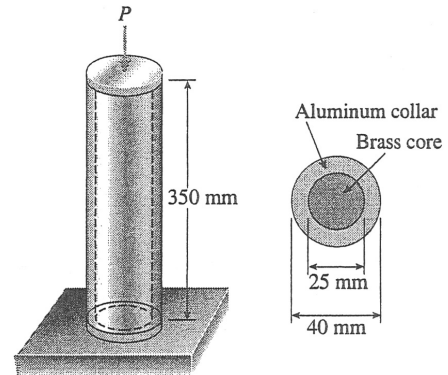
For brass:

$$\begin{aligned} \sigma_b &= \frac{PE_b}{E_s A_s + E_b A_b} \quad P_b = (E_s A_s + E_b A_b) \frac{\sigma_b}{E_b} \\ P_b &= (1.776 \times 10^6 \text{ lb}) \left(\frac{16 \text{ ksi}}{15 \times 10^6 \text{ psi}} \right) = 1890 \text{ lb} \end{aligned}$$

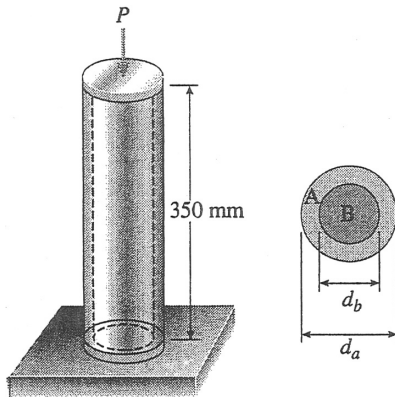
Steel governs. $P_{\text{allow}} = 1300 \text{ lb} \quad \leftarrow$

Problem 2.4-2 A cylindrical assembly consisting of a brass core and an aluminum collar is compressed by a load P (see figure). The length of the aluminum collar and brass core is 350 mm, the diameter of the core is 25 mm, and the outside diameter of the collar is 40 mm. Also, the moduli of elasticity of the aluminum and brass are 72 GPa and 100 GPa, respectively.

- If the length of the assembly decreases by 0.1% when the load P is applied, what is the magnitude of the load?
- What is the maximum permissible load P_{\max} if the allowable stresses in the aluminum and brass are 80 MPa and 120 MPa, respectively? (*Suggestion:* Use the equations derived in Example 2-6.)



Solution 2.4-2 Cylindrical assembly in compression



A = aluminum

B = brass

$L = 350$ mm

$d_a = 40$ mm

$d_b = 25$ mm

$$A_a = \frac{\pi}{4}(d_a^2 - d_b^2)$$

$$= 765.8 \text{ mm}^2$$

$$E_a = 72 \text{ GPa} \quad E_b = 100 \text{ GPa} \quad A_b = \frac{\pi}{4}d_b^2$$

$$= 490.9 \text{ mm}^2$$

(a) DECREASE IN LENGTH

$$(\delta = 0.1\% \text{ of } L = 0.350 \text{ mm})$$

Use Eq. (2-18) of Example 2-6.

$$\delta = \frac{PL}{E_a A_a + E_b A_b} \quad \text{or}$$

$$P = (E_a A_a + E_b A_b) \left(\frac{\delta}{L} \right)$$

Substitute numerical values:

$$\begin{aligned} E_a A_a + E_b A_b &= (72 \text{ GPa})(765.8 \text{ mm}^2) \\ &\quad + (100 \text{ GPa})(490.9 \text{ mm}^2) \\ &= 55.135 \text{ MN} + 49.090 \text{ MN} \\ &= 104.23 \text{ MN} \end{aligned}$$

$$\begin{aligned} P &= (104.23 \text{ MN}) \left(\frac{0.350 \text{ mm}}{350 \text{ mm}} \right) \\ &= 104.2 \text{ kN} \quad \leftarrow \end{aligned}$$

(b) ALLOWABLE LOAD

$$\sigma_a = 80 \text{ MPa} \quad \sigma_b = 120 \text{ MPa}$$

Use Eqs. (2-17a and b) of Example 2-6.

For aluminum:

$$\sigma_a = \frac{PE_a}{E_a A_a + E_b A_b} \quad P_a = (E_a A_a + E_b A_b) \left(\frac{\sigma_a}{E_a} \right)$$

$$P_a = (104.23 \text{ MN}) \left(\frac{80 \text{ MPa}}{72 \text{ GPa}} \right) = 115.8 \text{ kN}$$

For brass:

$$\sigma_b = \frac{PE_b}{E_a A_a + E_b A_b} \quad P_b = (E_a A_a + E_b A_b) \left(\frac{\sigma_b}{E_b} \right)$$

$$P_b = (104.23 \text{ MN}) \left(\frac{120 \text{ MPa}}{100 \text{ GPa}} \right) = 125.1 \text{ kN}$$

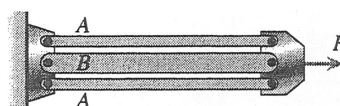
Aluminum governs. $P_{\max} = 116 \text{ kN} \quad \leftarrow$

Problem 2.4-3 Three prismatic bars, two of material *A* and one of material *B*, transmit a tensile load *P* (see figure). The two outer bars (material *A*) are identical. The cross-sectional area of the middle bar (material *B*) is 50% larger than the cross-sectional area of one of the outer bars. Also, the modulus of elasticity of material *A* is twice that of material *B*.

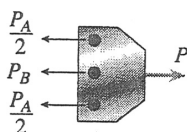


- What fraction of the load *P* is transmitted by the middle bar?
- What is the ratio of the stress in the middle bar to the stress in the outer bars?
- What is the ratio of the strain in the middle bar to the strain in the outer bars?

Solution 2.4-3 Prismatic bars in tension



FREE-BODY DIAGRAM OF END PLATE



EQUATION OF EQUILIBRIUM

$$\Sigma F_{\text{horiz}} = 0 \quad P_A + P_B - P = 0 \quad (1)$$

EQUATION OF COMPATIBILITY

$$\delta_A = \delta_B \quad (2)$$

FORCE-DISPLACEMENT RELATIONS

A_A = total area of both outer bars

$$\delta_A = \frac{P_A L}{E_A A_A} \quad \delta_B = \frac{P_B L}{E_B A_B} \quad (3)$$

Substitute into Eq. (2):

$$\frac{P_A L}{E_A A_A} = \frac{P_B L}{E_B A_B} \quad (4)$$

SOLUTION OF THE EQUATIONS

Solve simultaneously Eqs. (1) and (4):

$$P_A = \frac{E_A A_A P}{E_A A_A + E_B A_B} \quad P_B = \frac{E_B A_B P}{E_A A_A + E_B A_B} \quad (5)$$

Substitute into Eq. (3):

$$\delta = \delta_A = \delta_B = \frac{PL}{E_A A_A + E_B A_B} \quad (6)$$

STRESSES:

$$\sigma_A = \frac{P_A}{A_A} = \frac{E_A P}{E_A A_A + E_B A_B}$$

$$\sigma_B = \frac{P_B}{A_B} = \frac{E_B P}{E_A A_A + E_B A_B} \quad (7)$$

(a) LOAD IN MIDDLE BAR

$$\frac{P_B}{P} = \frac{E_B A_B}{E_A A_A + E_B A_B} = \frac{1}{\frac{E_A A_A}{E_B A_B} + 1}$$

$$\text{Given: } \frac{E_A}{E_B} = 2 \quad \frac{A_A}{A_B} = \frac{1 + 1}{1.5} = \frac{4}{3}$$

$$\therefore \frac{P_B}{P} = \frac{1}{\left(\frac{E_A}{E_B}\right)\left(\frac{A_A}{A_B}\right) + 1} = \frac{1}{\frac{8}{3} + 1} = \frac{3}{11} \quad \leftarrow$$

(b) RATIO OF STRESSES

$$\frac{\sigma_B}{\sigma_A} = \frac{E_B}{E_A} = \frac{1}{2} \quad \leftarrow$$

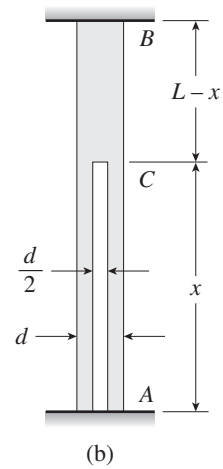
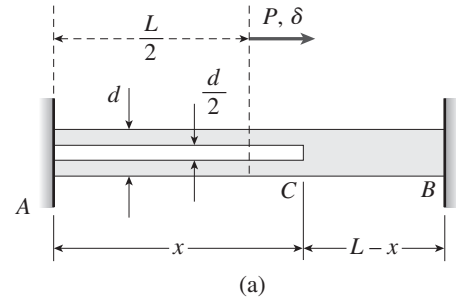
(c) RATIO OF STRAINS

All bars have the same strain

$$\text{Ratio} = 1 \quad \leftarrow$$

Problem 2.4-4 A circular bar ACB of diameter d having a cylindrical hole of length x and diameter $d/2$ from A to C is held between rigid supports at A and B . A load P acts at $L/2$ from ends A and B . Assume E is constant.

- Obtain formulas for the reactions R_A and R_B at supports A and B , respectively, due to the load P (see figure part a).
- Obtain a formula for the displacement δ at the point of load application (see figure part a).
- For what value of x is $R_B = (6/5) R_A$? (See figure part a.)
- Repeat part (a) if the bar is now rotated to a vertical position, load P is removed, and the bar is hanging under its own weight (assume mass density $= \rho$). (See figure part b.) Assume that $x = L/2$.



Solution 2.4-4

(a) REACTIONS AT A AND B DUE TO LOAD P AT $L/2$

$$A_{AC} = \frac{\pi}{4} \left[d^2 - \left(\frac{d}{2} \right)^2 \right] \quad A_{AC} = \frac{3}{16} \pi d^2$$

$$A_{CB} = \frac{\pi}{4} d^2$$

Select R_B as the redundant; use superposition and a compatibility equation at B :

$$\text{if } x \leq L/2 \quad \delta_{B1a} = \frac{Px}{EA_{AC}} + \frac{P\left(\frac{L}{2} - x\right)}{EA_{CB}} \quad \delta_{B1a} = \frac{P}{E} \left(\frac{x}{\frac{3}{16} \pi d^2} + \frac{\frac{L}{2} - x}{\frac{\pi}{4} d^2} \right)$$

$$\delta_{B1a} = \frac{2}{3} P \frac{2x + 3L}{E \pi d^2}$$

$$\text{if } x \geq L/2 \quad \delta_{B1b} = \frac{P \frac{L}{2}}{EA_{AC}} \quad \delta_{B1b} = \frac{P \frac{L}{2}}{E \left(\frac{3}{16} \pi d^2 \right)} \quad \delta_{B1b} = \frac{8}{3} \frac{PL}{E \pi d^2}$$

The following expression for δ_{B2} is good for all x :

$$\delta_{B2} = \frac{R_B}{E} \left(\frac{x}{A_{AC}} + \frac{L-x}{A_{CB}} \right) \quad \delta_{B2} = \frac{R_B}{E} \left(\frac{x}{\frac{3}{16}\pi d^2} + \frac{L-x}{\frac{\pi}{4}d^2} \right)$$

$$\delta_{B2} = \frac{R_B}{E} \left(\frac{16}{3} \frac{x}{\pi d^2} + 4 \frac{L-x}{\pi d^2} \right)$$

Solve for R_B and R_A assuming that $x \leq L/2$:

$$\text{Compatibility:} \quad \delta_{B1a} + \delta_{B2} = 0 \quad R_{Ba} = \frac{-\left(\frac{2}{3}P \frac{2x+3L}{\pi d^2}\right)}{\left(\frac{16}{3} \frac{x}{\pi d^2} + 4 \frac{L-x}{\pi d^2}\right)} \quad R_{Ba} = \frac{-1}{2} P \frac{2x+3L}{x+3L} \quad \leftarrow$$

^ check—if $x = 0$, $R_B = -P/2$

$$\text{Statics:} \quad R_{Aa} = -P - R_{Ba} \quad R_{Aa} = -P - \frac{-1}{2} P \frac{2x+3L}{x+3L} \quad R_{Aa} = \frac{-3}{2} P \frac{L}{x+3L} \quad \leftarrow$$

^ check—if $x = 0$, $R_{Aa} = -P/2$

Solve for R_B and R_A assuming that $x \geq L/2$:

$$\text{Compatibility:} \quad \delta_{B1b} + \delta_{B2} = 0 \quad R_{Bb} = \frac{\frac{-8}{3} \frac{PL}{\pi d^2}}{\left(\frac{16}{3} \frac{x}{\pi d^2} + 4 \frac{L-x}{\pi d^2}\right)} \quad R_{Bb} = \frac{-2PL}{x+3L} \quad \leftarrow$$

^ check—if $x = L$, $R_B = -P/2$

$$\text{Statics:} \quad R_{Ab} = -P - R_{Bb} \quad R_{Ab} = -P - \left(\frac{-2PL}{x+3L}\right) \quad R_{Ab} = -P \frac{x+L}{x+3L} \quad \leftarrow$$

(b) FIND δ AT POINT OF LOAD APPLICATION; AXIAL FORCE FOR SEGMENT 0 TO $L/2 = -R_A$ AND $\delta =$ ELONGATION OF THIS SEGMENT

Assume that $x \leq L/2$:

$$\delta_a = \frac{-R_{Aa}}{E} \left(\frac{x}{A_{AC}} + \frac{\frac{L}{2} - x}{A_{CB}} \right) \quad \delta_a = \frac{-\left(\frac{-3}{2} P \frac{L}{x+3L}\right)}{E} \left(\frac{x}{\frac{3}{16}\pi d^2} + \frac{\frac{L}{2} - x}{\frac{\pi}{4}d^2} \right)$$

$$\delta_a = PL \frac{2x+3L}{(x+3L)E\pi d^2}$$

$$\text{For } x = L/2, \quad \delta_a = \frac{8}{7} L \frac{P}{E\pi d^2} \quad \leftarrow$$

ASSUME THAT $x \geq L/2$:

$$\delta_b = \frac{(-R_{Ab})\frac{L}{2}}{EA_{AC}} \quad \delta_b = \frac{\left(P \frac{x+L}{x+3L}\right)\frac{L}{2}}{E\left(\frac{3}{16}\pi d^2\right)} \quad \delta_b = \frac{8}{3}P\left(\frac{x+L}{x+3L}\right)\frac{L}{E\pi d^2} \quad \leftarrow$$

$$\text{for } x = L/2 \quad \delta_b = \frac{8}{7}P\frac{L}{E\pi d^2} < \text{same as } \delta_a \text{ above (OK)}$$

(c) FOR WHAT VALUE OF x IS $R_B = (6/5)R_A$?

Guess that $x < L/2$ here and use R_{Ba} expression above to find x :

$$\frac{-1}{2}P\frac{2x+3L}{x+3L} - \frac{6}{5}\left(\frac{-3}{2}P\frac{L}{x+3L}\right) = 0 \quad \frac{-1}{10}P\frac{10x-3L}{x+3L} = 0 \quad x = \frac{3L}{10} \quad \leftarrow$$

Now try $R_{Bb} = (6/5)R_{Ab}$, assuming that $x > L/2$

$$\frac{-2PL}{x+3L} - \frac{6}{5}\left(-P\frac{x+L}{x+3L}\right) = 0 \quad \frac{2}{5}P\frac{-2L+3x}{x+3L} = 0 \quad x = \frac{2}{3}L \quad \leftarrow$$

So, there are two solutions for x .

(d) FIND REACTIONS IF THE BAR IS NOW ROTATED TO A VERTICAL POSITION, LOAD P IS REMOVED, AND THE BAR IS HANGING UNDER ITS OWN WEIGHT (ASSUME MASS DENSITY = ρ). ASSUME THAT $x = L/2$.

$$A_{AC} = \frac{3}{16}\pi d^2 \quad A_{CB} = \frac{\pi}{4}d^2$$

Select R_B as the redundant; use superposition and a compatibility equation at B

from (a) above. compatibility: $\delta_{B1} + \delta_{B2} = 0$

$$\delta_{B2} = \frac{R_B}{E}\left(\frac{x}{A_{AC}} + \frac{L-x}{A_{CB}}\right) \quad \text{For } x = L/2, \delta_{B2} = \frac{R_B}{E}\left(\frac{14}{3}\frac{L}{\pi d^2}\right)$$

$$\delta_{B1} = \int_0^{\frac{L}{2}} \frac{N_{AC}}{EA_{AC}} d\zeta + \int_{\frac{L}{2}}^L \frac{N_{CB}}{EA_{CB}} d\zeta$$

Where axial forces in bar due to self weight are $W_{AC} = \rho g A_{AC} \frac{L}{2}$ $W_{CB} = \rho g A_{CB} \frac{L}{2}$
(assume ζ is measured upward from A):

$$N_{AC} = -\left[\rho g A_{CB} \frac{L}{2} + \rho g A_{AC} \left(\frac{L}{2} - \zeta\right)\right] \quad A_{AC} = \frac{3}{16}\pi d^2 \quad A_{CB} = \frac{\pi}{4}d^2$$

$$N_{CB} = -[\rho g A_{CB}(L - \zeta)]$$

$$N_{AC} = \frac{-1}{8}\rho g \pi d^2 L - \frac{3}{16}\rho g \pi d^2 \left(\frac{1}{2}L - \zeta\right) \quad N_{CB} = -\left[\frac{1}{4}\rho g \pi d^2 (L - \zeta)\right]$$

$$\delta_{B1} = \int_0^{\frac{L}{2}} \frac{-\frac{1}{8} \rho g \pi d^2 L - \frac{3}{16} \rho g \pi d^2 \left(\frac{1}{2} L - \zeta\right)}{E \left(\frac{3}{16} \pi d^2\right)} d\zeta + \int_{\frac{L}{2}}^L \frac{-\left[\frac{1}{4} \rho g \pi d^2 (L - \zeta)\right]}{E \left(\frac{\pi}{4} d^2\right)} d\zeta$$

$$\delta_{B1} = \left(-\frac{11}{24} \rho g \frac{L^2}{E} + \frac{-1}{8} \rho g \frac{L^2}{E} \right) \quad \delta_{B1} = \frac{-7}{12} \rho g \frac{L^2}{E} \quad \frac{7}{12} = 0.583$$

Compatibility: $\delta_{B1} + \delta_{B2} = 0$

$$R_B = \frac{-\left(\frac{-7}{12} \rho g \frac{L^2}{E}\right)}{\left(\frac{14}{3} \frac{L}{E \pi d^2}\right)} \quad R_B = \frac{1}{8} \rho g \pi d^2 L \quad \leftarrow$$

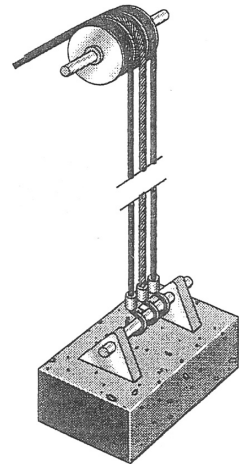
Statics: $R_A = (W_{AC} + W_{CB}) - R_B$

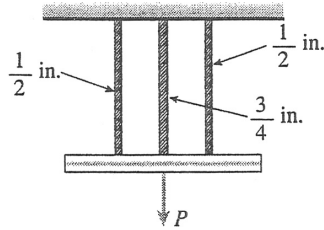
$$R_A = \left[\left[\rho g \left(\frac{3}{16} \pi d^2 \right) \frac{L}{2} + \rho g \left(\frac{\pi}{4} d^2 \right) \frac{L}{2} \right] - \frac{1}{8} \rho g \pi d^2 L \right]$$

$$R_A = \frac{3}{32} \rho g \pi d^2 L \quad \leftarrow$$

Problem 2.4-5 Three steel cables jointly support a load of 12 k (see figure). The diameter of the middle cable is $\frac{3}{4}$ in. and the diameter of each outer cable is $\frac{1}{2}$ in. The tensions in the cables are adjusted so that each cable carries one-third of the load (i.e., 4 k). Later, the load is increased by 9 k to a total load of 21 k.

- What percent of the total load is now carried by the middle cable?
 - What are the stresses σ_M and σ_O in the middle and outer cables, respectively?
- (NOTE: See Table 2-1 in Section 2.2 for properties of cables.)



Solution 2.4-5 Three cables in tension

AREAS OF CABLES (from Table 2-1)

Middle cable: $A_M = 0.268 \text{ in.}^2$

Outer cables: $A_O = 0.119 \text{ in.}^2$

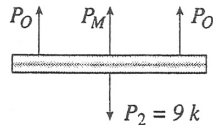
(for each cable)

FIRST LOADING

$$P_1 = 12 \text{ k} \left(\text{Each cable carries } \frac{P_1}{3} \text{ or } 4 \text{ k.} \right)$$

SECOND LOADING

$P_2 = 9 \text{ k}$ (additional load)



EQUATION OF EQUILIBRIUM

$$\Sigma F_{\text{vert}} = 0 \quad 2P_O + P_M - P_2 = 0 \quad (1)$$

EQUATION OF COMPATIBILITY

$$\delta_M = \delta_O \quad (2)$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_M = \frac{P_M L}{EA_M} \quad \delta_O = \frac{P_O L}{EA_O} \quad (3, 4)$$

SUBSTITUTE INTO COMPATIBILITY EQUATION:

$$\frac{P_M L}{EA_M} = \frac{P_O L}{EA_O} \quad \frac{P_M}{A_M} = \frac{P_O}{A_O} \quad (5)$$

SOLVE SIMULTANEOUSLY EQS. (1) AND (5):

$$P_M = P_2 \left(\frac{A_M}{A_M + 2A_O} \right) = (9 \text{ k}) \left(\frac{0.268 \text{ in.}^2}{0.506 \text{ in.}^2} \right) = 4.767 \text{ k}$$

$$P_O = P_2 \left(\frac{A_O}{A_M + 2A_O} \right) = (9 \text{ k}) \left(\frac{0.119 \text{ in.}^2}{0.506 \text{ in.}^2} \right) = 2.117 \text{ k}$$

FORCES IN CABLES

Middle cable: Force = $4 \text{ k} + 4.767 \text{ k} = 8.767 \text{ k}$

Outer cables: Force = $4 \text{ k} + 2.117 \text{ k} = 6.117 \text{ k}$

(for each cable)

(a) PERCENT OF TOTAL LOAD CARRIED BY MIDDLE CABLE

$$\text{Percent} = \frac{8.767 \text{ k}}{21 \text{ k}} (100\%) = 41.7\% \quad \leftarrow$$

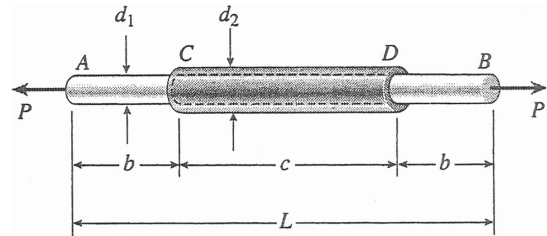
(b) STRESSES IN CABLES ($\sigma = P/A$)

$$\text{Middle cable: } \sigma_M = \frac{8.767 \text{ k}}{0.268 \text{ in.}^2} = 32.7 \text{ ksi} \quad \leftarrow$$

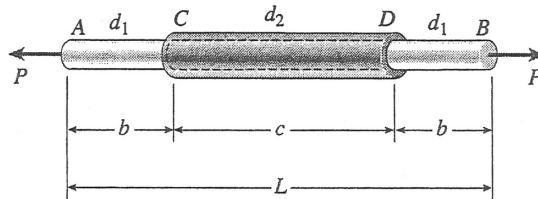
$$\text{Outer cables: } \sigma_O = \frac{6.117 \text{ k}}{0.119 \text{ in.}^2} = 51.4 \text{ ksi} \quad \leftarrow$$

Problem 2.4-6 A plastic rod AB of length $L = 0.5$ m has a diameter $d_1 = 30$ mm (see figure). A plastic sleeve CD of length $c = 0.3$ m and outer diameter $d_2 = 45$ mm is securely bonded to the rod so that no slippage can occur between the rod and the sleeve. The rod is made of an acrylic with modulus of elasticity $E_1 = 3.1$ GPa and the sleeve is made of a polyamide with $E_2 = 2.5$ GPa.

- Calculate the elongation δ of the rod when it is pulled by axial forces $P = 12$ kN.
- If the sleeve is extended for the full length of the rod, what is the elongation?
- If the sleeve is removed, what is the elongation?



Solution 2.4-6 Plastic rod with sleeve



$$P = 12 \text{ kN} \quad d_1 = 30 \text{ mm} \quad b = 100 \text{ mm}$$

$$L = 500 \text{ mm} \quad d_2 = 45 \text{ mm} \quad c = 300 \text{ mm}$$

Rod: $E_1 = 3.1$ GPa

Sleeve: $E_2 = 2.5$ GPa

$$\text{Rod: } A_1 = \frac{\pi d_1^2}{4} = 706.86 \text{ mm}^2$$

$$\text{Sleeve: } A_2 = \frac{\pi}{4}(d_2^2 - d_1^2) = 883.57 \text{ mm}^2$$

$$E_1 A_1 + E_2 A_2 = 4.400 \text{ MN}$$

(a) ELONGATION OF ROD

$$\text{Part AC: } \delta_{AC} = \frac{Pb}{E_1 A_1} = 0.5476 \text{ mm}$$

$$\text{Part CD: } \delta_{CD} = \frac{Pc}{E_1 A_1 + E_2 A_2}$$

$$= 0.81815 \text{ mm}$$

(From Eq. 2-18 of Example 2-6)

$$\delta = 2\delta_{AC} + \delta_{CD} = 1.91 \text{ mm} \quad \leftarrow$$

(b) SLEEVE AT FULL LENGTH

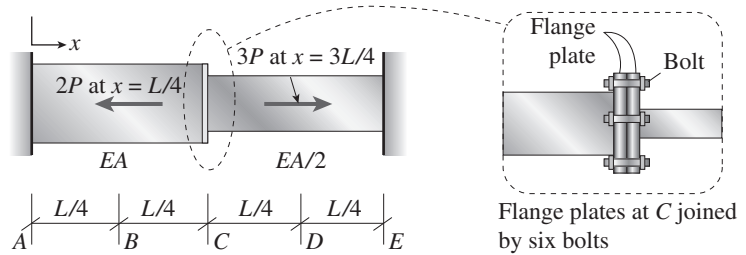
$$\delta = \delta_{CD} \left(\frac{L}{c} \right) = (0.81815 \text{ mm}) \left(\frac{500 \text{ mm}}{300 \text{ mm}} \right)$$

$$= 1.36 \text{ mm} \quad \leftarrow$$

(c) SLEEVE REMOVED

$$\delta = \frac{PL}{E_1 A_1} = 2.74 \text{ mm} \quad \leftarrow$$

Problem 2.4-7 A tube structure is acted on by loads at B and D , as shown in the figure. The tubes are joined using two flange plates at C , which are bolted together using six 0.5 in. diameter bolts.



- Derive formulas for the reactions R_A and R_E at the ends of the bar.
- Determine the axial displacement δ_B , δ_C , and δ_D at points B , C , and D , respectively.
- Draw an axial-displacement diagram (ADD) in which the abscissa is the distance x from support A to any point on the bar and the ordinate is the horizontal displacement δ at that point.
- Find the maximum value of the load variable P if allowable normal stress in the bolts is 14 ksi.

Solution 2.4-7

NUMERICAL DATA

$$n = 6 \quad d_b = 0.5 \text{ in.} \quad \sigma_a = 14 \text{ ksi} \quad A_b = \frac{\pi}{4} d_b^2 = 0.196 \text{ in.}^2$$

- (a) FORMULAS FOR REACTIONS F

$$\text{Segment } ABC \text{ flexibility: } f_1 = \frac{2\left(\frac{L}{4}\right)}{EA} = \frac{L}{2EA}$$

$$\text{Segment } CDE \text{ flexibility: } f_2 = \frac{2\left(\frac{L}{4}\right)}{\frac{1}{2}EA} = \frac{L}{EA}$$

Loads at points B and D :

$$P_B = -2P \quad P_D = 3P$$

- (1) Select R_E as the redundant; find axial displacement δ_1 = displacement at E due to loads P_B and P_D :

$$\delta_1 = \frac{(P_B + P_D)\frac{L}{4}}{EA} + \frac{P_D\frac{L}{4}}{EA} + \frac{P_D\frac{L}{4}}{\frac{1}{2}EA} = \frac{5LP}{2EA}$$

- (2) Next apply redundant R_E and find axial displacement δ_2 = displacement at E due to redundant R_E :

$$\delta_2 = R_E(f_1 + f_2) = \frac{3LR_E}{2EA}$$

- (3) Use compatibility equation to find redundant R_E then use statics to find R_A :

$$\delta_1 + \delta_2 = 0 \text{ solve, } R_E = -\frac{5P}{3} \quad R_E = \frac{-5}{3}P$$

$$R_A = -R_E - P_B - P_D = \frac{2P}{3} \quad R_A = \frac{2P}{3} \quad \boxed{R_A = \frac{2P}{3}} \quad \boxed{R_E = \frac{5P}{3}}$$

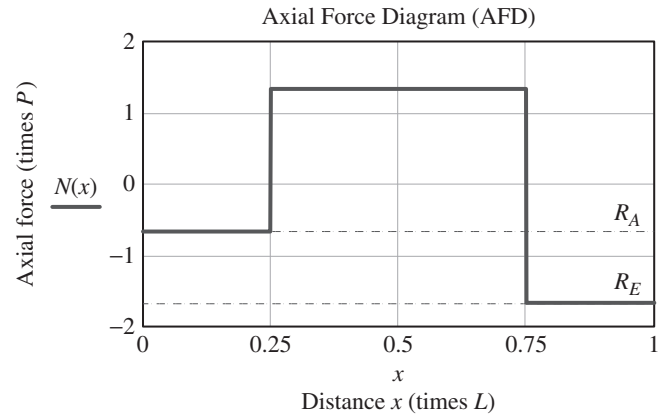
- (b) DETERMINE THE AXIAL DISPLACEMENTS δ_B , δ_C , AND δ_D AT POINTS B , C , AND D , RESPECTIVELY.

$$\delta_B = \frac{\left(\frac{-2P}{3}\right)\left(\frac{L}{4}\right)}{EA} = \frac{LP}{6EA} \quad \text{leftward} \quad \delta_C = \delta_B + \frac{\left(2P - \frac{2P}{3}\right)\left(\frac{L}{4}\right)}{EA} = \frac{LP}{6EA} \quad \text{to the right} \quad \delta_D = \frac{\left(\frac{5P}{3}\right)\left(\frac{L}{4}\right)}{\frac{EA}{2}} = \frac{5LP}{6EA} \quad \text{to the right}$$

- (c) DRAW AN AXIAL-DISPLACEMENT DIAGRAM (ADD) IN WHICH THE ABSCISSA IS THE DISTANCE x FROM SUPPORT A TO ANY POINT ON THE BAR AND THE ORDINATE IS THE HORIZONTAL DISPLACEMENT δ AT THAT POINT.

AFD for use below in Part (d)

AFD is composed of 4 constant segments, so ADD is linear with zero displacements at supports A and E .

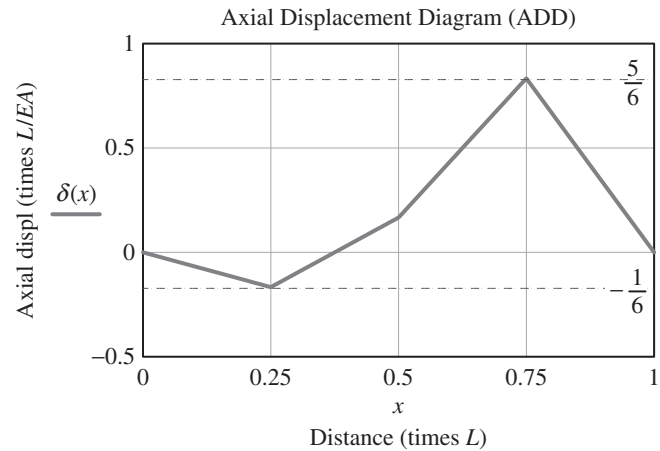


Plot displacements δ_B , δ_C , and δ_D from part (b) above, then connect points using straight lines showing linear variation of axial displacement between points

$$\delta_{\max} = \delta_D \quad \delta_{\max} = \frac{5LP}{6EA} \quad \text{to the right}$$

Boundary conditions at supports:

$$\delta_A = \delta_E = 0$$



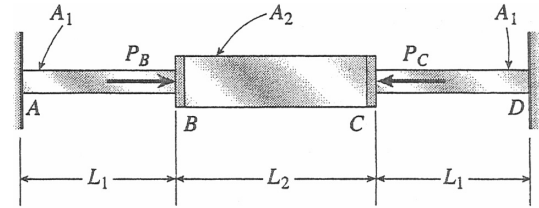
- (d) MAXIMUM PERMISSIBLE VALUE OF LOAD VARIABLE P BASED ON ALLOWABLE NORMAL STRESS IN FLANGE BOLTS
FROM AFD, FORCE AT $L/2$:

$$F_{\max} = \frac{4}{3}P \quad \text{and} \quad F_{\max} = n\sigma_a A_b = 16.493 \text{ k}$$

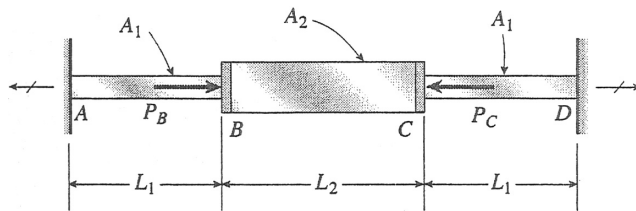
$$P_{\max} = \frac{3}{4}F_{\max} = 12.37 \text{ k} \quad \boxed{P_{\max} = 12.37 \text{ k}}$$

Problem 2.4-8 The fixed-end bar $ABCD$ consists of three prismatic segments, as shown in the figure. The end segments have cross-sectional area $A_1 = 840 \text{ mm}^2$ and length $L_1 = 200 \text{ mm}$. The middle segment has cross-sectional area $A_2 = 1260 \text{ mm}^2$ and length $L_2 = 250 \text{ mm}$. Loads P_B and P_C are equal to 25.5 kN and 17.0 kN , respectively.

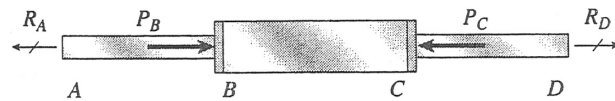
- Determine the reactions R_A and R_D at the fixed supports.
- Determine the compressive axial force F_{BC} in the middle segment of the bar.



Solution 2.4-8 Bar with three segments



FREE-BODY DIAGRAM



EQUATION OF EQUILIBRIUM

$$\Sigma F_{\text{horiz}} = 0 \rightarrow \leftarrow$$

$$P_B + R_D - P_C - R_A = 0 \text{ or}$$

$$R_A - R_D = P_B - P_C = 8.5 \text{ kN} \quad (\text{Eq. 1})$$

EQUATION OF COMPATIBILITY

$$\delta_{AD} = \text{elongation of entire bar}$$

$$\delta_{AD} = \delta_{AB} + \delta_{BC} + \delta_{CD} = 0 \quad (\text{Eq. 2})$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_{AB} = \frac{R_A L_1}{EA_1} = \frac{R_A}{E} \left(238.05 \frac{1}{\text{m}} \right) \quad (\text{Eq. 3})$$

$$\begin{aligned} \delta_{BC} &= \frac{(R_A - P_B)L_2}{EA_2} \\ &= \frac{R_A}{E} \left(198.413 \frac{1}{\text{m}} \right) - \frac{P_B}{E} \left(198.413 \frac{1}{\text{m}} \right) \end{aligned} \quad (\text{Eq. 4})$$

$$\delta_{CD} = \frac{R_D L_1}{EA_1} = \frac{R_D}{E} \left(238.095 \frac{1}{\text{m}} \right) \quad (\text{Eq. 5})$$

$$\begin{aligned} P_B &= 25.5 \text{ kN} & P_C &= 17.0 \text{ kN} \\ L_1 &= 200 \text{ mm} & L_2 &= 250 \text{ mm} \\ A_1 &= 840 \text{ mm}^2 & A_2 &= 1260 \text{ mm}^2 \\ m &= \text{meter} \end{aligned}$$

SOLUTION OF EQUATIONS

Substitute Eqs. (3), (4), and (5) into Eq. (2):

$$\begin{aligned} \frac{R_A}{E} \left(238.095 \frac{1}{\text{m}} \right) + \frac{R_A}{E} \left(198.413 \frac{1}{\text{m}} \right) \\ - \frac{P_B}{E} \left(198.413 \frac{1}{\text{m}} \right) + \frac{R_D}{E} \left(238.095 \frac{1}{\text{m}} \right) = 0 \end{aligned}$$

Simplify and substitute $P_B = 25.5 \text{ kN}$:

$$\begin{aligned} R_A \left(436.508 \frac{1}{\text{m}} \right) + R_D \left(238.095 \frac{1}{\text{m}} \right) \\ = 5,059.53 \text{ kN/m} \end{aligned} \quad (\text{Eq. 6})$$

- REACTIONS R_A AND R_D

Solve simultaneously Eqs. (1) and (6).

$$\text{From (1): } R_D = R_A - 8.5 \text{ kN}$$

Substitute into (6) and solve for R_A :

$$R_A \left(674.603 \frac{1}{\text{m}} \right) = 7083.34 \text{ kN/m}$$

$$R_A = 10.5 \text{ kN} \quad \leftarrow$$

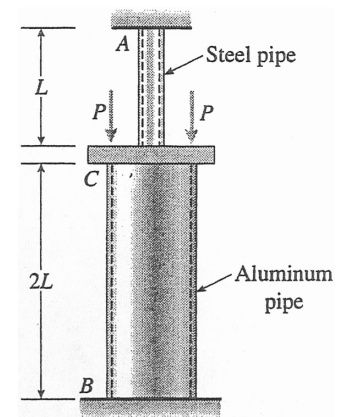
$$R_D = R_A - 8.5 \text{ kN} = 2.0 \text{ kN} \quad \leftarrow$$

- COMPRESSIVE AXIAL FORCE F_{BC}

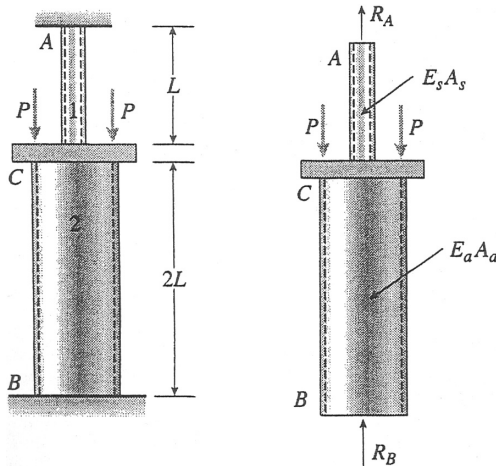
$$F_{BC} = P_B - R_A = P_C - R_D = 15.0 \text{ kN} \quad \leftarrow$$

Problem 2.4-9 The aluminum and steel pipes shown in the figure are fastened to rigid supports at ends A and B and to a rigid plate C at their junction. The aluminum pipe is twice as long as the steel pipe. Two equal and symmetrically placed loads P act on the plate at C .

- Obtain formulas for the axial stresses σ_a and σ_s in the aluminum and steel pipes, respectively.
- Calculate the stresses for the following data: $P = 12$ k, cross-sectional area of aluminum pipe $A_a = 8.92$ in.², cross-sectional area of steel pipe $A_s = 1.03$ in.², modulus of elasticity of aluminum $E_a = 10 \times 10^6$ psi, and modulus of elasticity of steel $E_s = 29 \times 10^6$ psi.



Solution 2.4-9 Pipes with intermediate loads



Pipe 1 is steel.
Pipe 2 is aluminum.

EQUATION OF EQUILIBRIUM

$$\Sigma F_{\text{vert}} = 0 \quad R_A + R_B = 2P \quad (\text{Eq. 1})$$

EQUATION OF COMPATIBILITY

$$\delta_{AB} = \delta_{AC} + \delta_{CB} = 0 \quad (\text{Eq. 2})$$

(A positive value of δ means elongation.)

FORCE-DISPLACEMENT RELATIONS

$$\delta_{AC} = \frac{R_A L}{E_s A_s} \quad \delta_{BC} = -\frac{R_B (2L)}{E_a A_a} \quad (\text{Eqs. 3, 4})$$

SOLUTION OF EQUATIONS

Substitute Eqs. (3) and (4) into Eq. (2):

$$\frac{R_A L}{E_s A_s} - \frac{R_B (2L)}{E_a A_a} = 0 \quad (\text{Eq. 5})$$

Solve simultaneously Eqs. (1) and (5):

$$R_A = \frac{4E_s A_s P}{E_a A_a + 2E_s A_s} \quad R_B = \frac{2E_a A_a P}{E_a A_a + 2E_s A_s} \quad (\text{Eqs. 6, 7})$$

(a) AXIAL STRESSES

$$\text{Aluminum: } \sigma_a = \frac{R_B}{A_a} = \frac{2E_a P}{E_a A_a + 2E_s A_s} \quad \leftarrow$$

(compression) (Eq. 8)

$$\text{Steel: } \sigma_s = \frac{R_A}{A_s} = \frac{4E_s P}{E_a A_a + 2E_s A_s} \quad \leftarrow$$

(tension) (Eq. 9)

(b) NUMERICAL RESULTS

$$P = 12 \text{ k} \quad A_a = 8.92 \text{ in.}^2 \quad A_s = 1.03 \text{ in.}^2$$

$$E_a = 10 \times 10^6 \text{ psi} \quad E_s = 29 \times 10^6 \text{ psi}$$

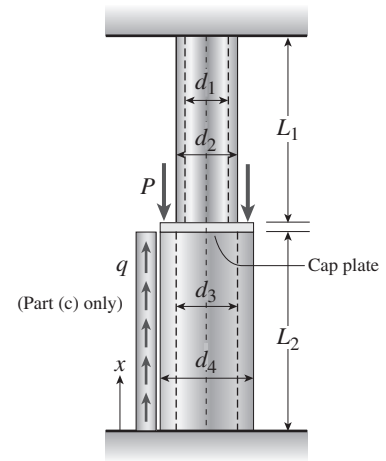
Substitute into Eqs. (8) and (9):

$$\sigma_a = 1,610 \text{ psi (compression)} \quad \leftarrow$$

$$\sigma_s = 9,350 \text{ psi (tension)} \quad \leftarrow$$

Problem 2.4-10 A hollow circular pipe (see figure) supports a load P which is uniformly distributed around a cap plate at the top of the lower pipe. The inner and outer diameters of the upper and lower parts of the pipe are $d_1 = 50$ mm, $d_2 = 60$ mm, $d_3 = 57$ mm, and $d_4 = 64$ mm, respectively. Pipe lengths are $L_1 = 2$ m and $L_2 = 3$ m. Neglect the self-weight of the pipes. Assume that cap plate thickness is small compared to L_1 and L_2 . Let $E = 110$ MPa.

- If the tensile stress in the upper part is $\sigma_1 = 10.5$ MPa, what is load P ? Also, what are reactions R_1 at the upper support and R_2 at the lower support. What is the stress σ_2 MPa in the lower part?
- Find displacement δ (mm) at the cap plate. Plot the Axial Force Diagram, AFD [$N(x)$] and Axial Displacement Diagram, ADD [$\delta(x)$].
- Add the uniformly distributed load q along the centroidal axis of pipe segment 2. Find q (kN/m) so that $R_2 = 0$. Assume that load P from part (a) is also applied.



Solution 2.4-10

- (a) STRESSES AND REACTIONS: SELECT R_1 AS REDUNDANT AND DO SUPERPOSITION ANALYSIS (HERE $q = 0$; DEFLECTION POSITION UPWARD)

$$d_1 = 50 \text{ mm} \quad d_2 = 60 \text{ mm} \quad d_3 = 57 \text{ mm} \quad d_4 = 64 \text{ mm} \quad A_1 = \frac{\pi}{4} (d_2^2 - d_1^2) = 863.938 \text{ mm}^2$$

$$E = 110 \text{ MPa}$$

$$A_2 = \frac{\pi}{4} (d_4^2 - d_3^2) = 665.232 \text{ mm}^2$$

$$\text{SEGMENT FLEXIBILITIES} \quad L_1 = 2 \text{ m} \quad L_2 = 3 \text{ m}$$

$$f_1 = \frac{L_1}{EA_1} = 0.02105 \text{ mm/N} \quad f_2 = \frac{L_2}{EA_2} = 0.041 \text{ mm/N} \quad \frac{f_1}{f_2} = 0.513$$

$$\text{TENSILE stress } (\sigma_1) \text{ is known in upper segment so } R_1 = \sigma_1 \times A_1 \quad \sigma_1 = 10.5 \text{ MPa} \quad R_1 = \sigma_1 A_1 = 9.07 \text{ kN}$$

$$\delta_{1a} = -Pf_2 \quad \delta_{1b} = R_1(f_1 + f_2) \quad \text{Compatibility: } \delta_{1a} + \delta_{1b} = 0$$

$$\text{Solve for } P: \quad P = R_1 \left(\frac{f_1 + f_2}{f_2} \right) = 13.73 \text{ kN}$$

$$\text{Finally, use statics to find } R_2: \quad R_2 = P - R_1 = 4.66 \text{ kN} \quad \sigma_2 = \frac{R_2}{A_2} = 7 \text{ MPa} < \text{compressive since } R_2 \text{ is positive (upward)}$$

$$P = 13.73 \text{ kN}$$

$$R_1 = 9.07 \text{ kN}$$

$$R_2 = 4.66 \text{ kN}$$

$$\sigma_2 = 7 \text{ MPa}$$

- (b) DISPLACEMENT AT CAP PLATE

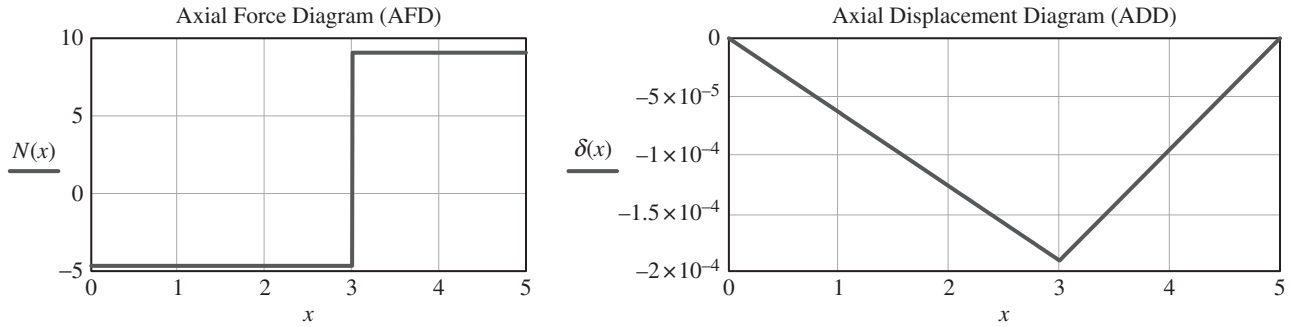
$$\delta_c = R_1 f_1 = 190.909 \text{ mm} < \text{downward} \quad \text{OR} \quad \delta_c = (R_2) f_2 = 190.909 \text{ mm} < \text{downward (neg. } x\text{-direction)}$$

$$\delta_{\text{cap}} = \delta_c = 0.191 \text{ m} \quad \boxed{\delta_{\text{cap}} = 190.9 \text{ mm}}$$

$$\text{AFD and ADD:} \quad R_1 = 9.071 \quad R_2 = 4.657 \quad L_1 = 2 \quad A_1 = 863.938 \quad A_2 = 665.232 \quad E = 110$$

$$L_2 = 3$$

NOTE: x is measured up from lower support.



(c) UNIFORM LOAD Q ON SEGMENT 2 SUCH THAT $R_2 = 0$

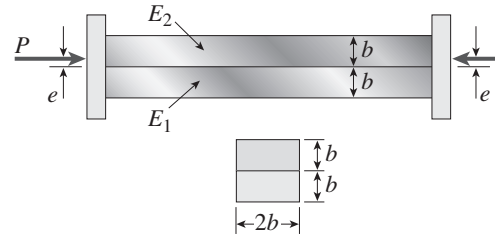
$$P = 13.728 \text{ kN} \quad R_1 = \sigma_1 A_1 = 9.071 \text{ kN} \quad L_2 = 3 \text{ m}$$

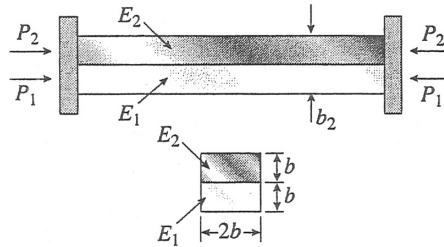
$$\text{Equilibrium: } R_1 + R_2 = P - qL_2 < \text{set } R_2 = 0, \text{ solve for req'd } q \quad q = \frac{P - R_1}{L_2} = 1.552 \text{ kN/m}$$

$$q = 1.552 \text{ kN/m}$$

Problem 2.4-11 A bimetallic bar (or composite bar) of square cross section with dimensions $2b \times 2b$ is constructed of two different metals having moduli of elasticity E_1 and E_2 (see figure). The two parts of the bar have the same cross-sectional dimensions. The bar is compressed by forces P acting through rigid end plates. The line of action of the loads has an eccentricity e of such magnitude that each part of the bar is stressed uniformly in compression.

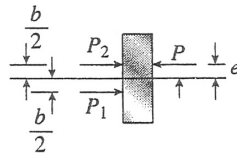
- Determine the axial forces P_1 and P_2 in the two parts of the bar.
- Determine the eccentricity e of the loads.
- Determine the ratio σ_1/σ_2 of the stresses in the two parts of the bar.



Solution 2.4-11 Bimetallic bar in compression

FREE-BODY DIAGRAM

(Plate at right-hand end)



EQUATIONS OF EQUILIBRIUM

$$\Sigma F = 0 \quad P_1 + P_2 = P \quad (\text{Eq. 1})$$

$$\Sigma M = 0 \quad P e + P_1 \left(\frac{b}{2} \right) - P_2 \left(\frac{b}{2} \right) = 0 \quad (\text{Eq. 2})$$

EQUATION OF COMPATIBILITY

$$\delta_2 = \delta_1$$

$$\frac{P_2 L}{E_2 A} = \frac{P_1 L}{E_1 A} \quad \text{or} \quad \frac{P_2}{E_2} = \frac{P_1}{E_1} \quad (\text{Eq. 3})$$

(a) AXIAL FORCES

Solve simultaneously Eqs. (1) and (3):

$$P_1 = \frac{P E_1}{E_1 + E_2} \quad P_2 = \frac{P E_2}{E_1 + E_2} \quad \leftarrow$$

(b) ECCENTRICITY OF LOAD P Substitute P_1 and P_2 into Eq. (2) and solve for e :

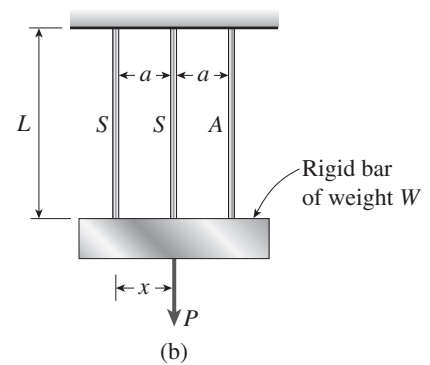
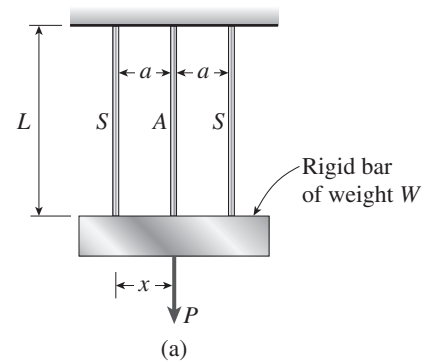
$$e = \frac{b(E_2 - E_1)}{2(E_2 + E_1)} \quad \leftarrow$$

(c) RATIO OF STRESSES

$$\sigma_1 = \frac{P_1}{A} \quad \sigma_2 = \frac{P_2}{A} \quad \frac{\sigma_1}{\sigma_2} = \frac{P_1}{P_2} = \frac{E_1}{E_2} \quad \leftarrow$$

Problem 2.4-12 A rigid bar of weight $W = 800$ N hangs from three equally spaced vertical wires (length $L = 150$ mm, spacing $a = 50$ mm): two of steel and one of aluminum. The wires also support a load P acting on the bar. The diameter of the steel wires is $d_s = 2$ mm, and the diameter of the aluminum wire is $d_a = 4$ mm. Assume $E_s = 210$ GPa and $E_a = 70$ GPa.

- What load P_{allow} can be supported at the midpoint of the bar ($x = a$) if the allowable stress in the steel wires is 220 MPa and in the aluminum wire is 80 MPa? (See figure part a.)
- What is P_{allow} if the load is positioned at $x = a/2$? (See figure part a.)
- Repeat (b) above if the second and third wires are switched as shown in figure part b.



Solution 2.4-12

Numerical data:

$$\begin{aligned} W &= 800 \text{ N} & L &= 150 \text{ mm} \\ a &= 50 \text{ mm} & d_s &= 2 \text{ mm} \\ d_a &= 4 \text{ mm} & E_s &= 210 \text{ GPa} \\ E_a &= 70 \text{ GPa} \\ \sigma_{sa} &= 220 \text{ MPa} & \sigma_{aa} &= 80 \text{ MPa} \end{aligned}$$

$$A_a = \frac{\pi}{4} d_a^2 \quad A_s = \frac{\pi}{4} d_s^2$$

$$A_a = 13 \text{ mm}^2 \quad A_s = 3 \text{ mm}^2$$

- (a) P_{allow} AT CENTER OF BAR

One-degree statically indeterminate - use reaction (R_a) at top of aluminum bar as the redundant

compatibility: $\delta_1 - \delta_2 = 0$ Statics: $2R_s + R_a = P + W$

$$\delta_1 = \frac{P + W}{2} \left(\frac{L}{E_s A_s} \right) \quad \begin{array}{l} < \text{downward displacement due to elongation of each steel wire under } P + W \\ \text{aluminum wire is cut at top} \end{array}$$

$$\delta_2 = R_A \left(\frac{L}{2E_S A_S} + \frac{L}{E_A A_A} \right) \quad < \text{upward displ. due to shortening of steel wires and elongation of aluminum wire under redundant } R_A$$

Enforce compatibility and then solve for R_A :

$$\delta_1 = \delta_2 \quad \text{so} \quad R_A = \frac{\frac{P+W}{2} \left(\frac{L}{E_S A_S} \right)}{\frac{L}{2E_S A_S} + \frac{L}{E_A A_A}} \quad R_A = (P+W) \frac{E_A A_A}{E_A A_A + 2E_S A_S} \quad \text{and} \quad \sigma_{Aa} = \frac{R_A}{A_A}$$

Now use statics to find R_S :

$$R_S = \frac{P+W-R_A}{2} \quad R_S = \frac{P+W - (P+W) \frac{E_A A_A}{E_A A_A + 2E_S A_S}}{2} \quad R_S = (P+W) \frac{E_S A_S}{E_A A_A + 2E_S A_S}$$

$$\text{and} \quad \sigma_{Sa} = \frac{R_S}{A_S}$$

Compute stresses and apply allowable stress values:

$$\sigma_{Aa} = (P+W) \frac{E_A}{E_A A_A + 2E_S A_S} \quad \sigma_{Sa} = (P+W) \frac{E_S}{E_A A_A + 2E_S A_S}$$

Solve for allowable load P :

$$P_{Aa} = \sigma_{Aa} \left(\frac{E_A A_A + 2E_S A_S}{E_A} \right) - W \quad P_{Sa} = \sigma_{Sa} \left(\frac{E_A A_A + 2E_S A_S}{E_S} \right) - W \quad (\text{lower value of } P \text{ controls})$$

$$P_{Aa} = 1713 \text{ N}$$

$$P_{Sa} = 1504 \text{ N} \quad \leftarrow P_{\text{allow}} \text{ is controlled by steel wires}$$

(b) P_{allow} IF LOAD P AT $x = a/2$

Again, cut aluminum wire at top, then compute elongations of left and right steel wires:

$$\delta_{1L} = \left(\frac{3P}{4} + \frac{W}{2} \right) \left(\frac{L}{E_S A_S} \right) \quad \delta_{1R} = \left(\frac{P}{4} + \frac{W}{2} \right) \left(\frac{L}{E_S A_S} \right)$$

$$\delta_1 = \frac{\delta_{1L} + \delta_{1R}}{2} \quad \delta_1 = \frac{P+W}{2} \left(\frac{L}{E_S A_S} \right) \quad \text{where } \delta_1 = \text{displacement at } x = a$$

Use δ_2 from part (a):

$$\delta_2 = R_A \left(\frac{L}{2E_S A_S} + \frac{L}{E_A A_A} \right)$$

$$\text{So equating } \delta_1 \text{ and } \delta_2, \text{ solve for } R_A: \quad R_A = (P+W) \frac{E_A A_A}{E_A A_A + 2E_S A_S}$$

^ same as in part (a)

$$R_{SL} = \frac{3P}{4} + \frac{W}{2} - \frac{R_A}{2} \quad < \text{stress in left steel wire exceeds that in right steel wire}$$

$$R_{SL} = \frac{3P}{4} + \frac{W}{2} - \frac{(P+W) \frac{E_A A_A}{E_A A_A + 2E_S A_S}}{2}$$

$$R_{SL} = \frac{PE_AA_A + 6PE_SA_S + 4WE_SA_S}{4E_AA_A + 8E_SA_S} \quad \sigma_{Sa} = \frac{PE_AA_A + 6PE_SA_S + 4WE_SA_S}{4E_AA_A + 8E_SA_S} \left(\frac{1}{A_S} \right)$$

Solve for P_{allow} based on allowable stresses in steel and aluminum:

$$P_{Sa} = \frac{\sigma_{Sa}(4A_SE_AA_A + 8E_SA_S^2) - (4WE_SA_S)}{E_AA_A + 6E_SA_S} \quad P_{Aa} = 1713 \text{ N} \quad < \text{same as in part(a)}$$

$$P_{Sa} = 820 \text{ N} \quad \leftarrow \text{steel controls}$$

(c) P_{allow} IF WIRES ARE SWITCHED AS SHOWN AND $x = a/2$

Select R_A as the redundant; statics on the two released structures:

(1) Cut aluminum wire—apply P and W , compute forces in left and right steel wires, then compute displacements at each steel wire:

$$R_{SL} = \frac{P}{2} \quad R_{SR} = \frac{P}{2} + W$$

$$\delta_{1L} = \frac{P}{2} \left(\frac{L}{E_SA_S} \right) \quad \delta_{1R} = \left(\frac{P}{2} + W \right) \left(\frac{L}{E_SA_S} \right)$$

By geometry, δ at aluminum wire location at far right is $\delta_1 = \left(\frac{P}{2} + 2W \right) \left(\frac{L}{E_SA_S} \right)$

(2) Next apply redundant R_A at right wire, compute wire force and displacement at aluminum wire:

$$R_{SL} = -R_A \quad R_{SR} = 2R_A \quad \delta_2 = R_A \left(\frac{5L}{E_SA_S} + \frac{L}{E_AA_A} \right)$$

(3) Compatibility equate δ_1 , δ_2 and solve for R_A , then P_{allow} for aluminum wire:

$$R_A = \frac{\left(\frac{P}{2} + 2W \right) \left(\frac{L}{E_SA_S} \right)}{\frac{5L}{E_SA_S} + \frac{L}{E_AA_A}} \quad R_A = \frac{E_AA_AP + 4E_AA_AW}{10E_AA_A + 2E_SA_S} \quad \sigma_{Aa} = \frac{R_A}{A_A}$$

$$\sigma_{Aa} = \frac{E_AP + 4E_AW}{10E_AA_A + 2E_SA_S}$$

$$P_{Aa} = \frac{\sigma_{Aa}(10E_AA_A + 2E_SA_S) - 4E_AW}{E_A} \quad P_{Aa} = 1713 \text{ N}$$

(4) Statics or superposition—find forces in steel wires, then P_{allow} for steel wires:

$$R_{SL} = \frac{P}{2} + R_A \quad R_{SR} = \frac{P}{2} + \frac{E_AA_AP + 4E_AA_AW}{10E_AA_A + 2E_SA_S}$$

$$R_{SL} = \frac{6E_AA_AP + PE_SA_S + 4E_AA_AW}{10E_AA_A + 2E_SA_S} \quad < \text{larger than } R_{SR}, \text{ so use in allowable stress calculations}$$

$$R_{SR} = \frac{P}{2} + W - 2R_A \quad R_{SR} = \frac{P}{2} + W - \frac{E_A A_A P + 4E_A A_A W}{5E_A A_A + E_S A_S}$$

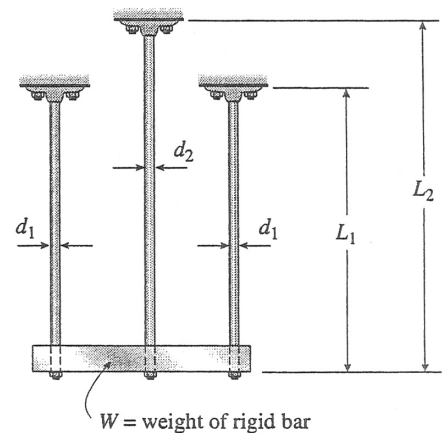
$$R_{SR} = \frac{3E_A A_A P + P E_S A_S + 2E_A A_A W + 2W E_S A_S}{10E_A A_A + 2E_S A_S}$$

$$\sigma_{Sa} = \frac{R_{SL}}{A_S} \quad P_{Sa} = \sigma_{Sa} A_S \left(\frac{10E_A A_A + 2E_S A_S}{6E_A A_A + E_S A_S} \right) - \frac{4E_A A_A W}{6E_A A_A + E_S A_S}$$

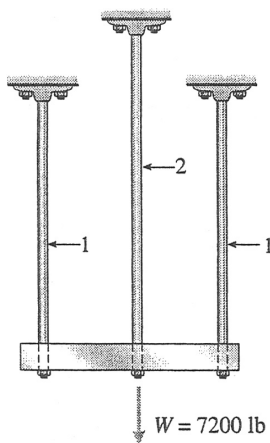
$$P_{Sa} = \frac{10\sigma_{Sa} A_S E_A A_A + 2\sigma_{Sa} A_S^2 E_S - 4E_A A_A W}{6E_A A_A + E_S A_S} \quad P_{Sa} = 703 \text{ N} \quad \leftarrow \text{^ steel controls}$$

Problem 2.4-13 A horizontal rigid bar of weight $W = 7200 \text{ lb}$ is supported by three slender circular rods that are equally spaced (see figure). The two outer rods are made of aluminum ($E_1 = 10 \times 10^6 \text{ psi}$) with diameter $d_1 = 0.4 \text{ in.}$ and length $L_1 = 40 \text{ in.}$ The inner rod is magnesium ($E_2 = 6.5 \times 10^6 \text{ psi}$) with diameter d_2 and length L_2 . The allowable stresses in the aluminum and magnesium are 24,000 psi and 13,000 psi, respectively.

If it is desired to have all three rods loaded to their maximum allowable values, what should be the diameter d_2 and length L_2 of the middle rod?



Solution 2.4-13 Bar supported by three rods



BAR 1 ALUMINUM

$$E_1 = 10 \times 10^6 \text{ psi}$$

$$d_1 = 0.4 \text{ in.}$$

$$L_1 = 40 \text{ in.}$$

$$\sigma_1 = 24,000 \text{ psi}$$

BAR 2 MAGNESIUM

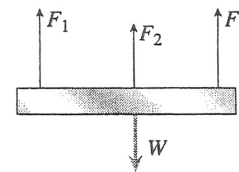
$$E_2 = 6.5 \times 10^6 \text{ psi}$$

$$d_2 = ? \quad L_2 = ?$$

$$\sigma_2 = 13,000 \text{ psi}$$

FREE-BODY DIAGRAM OF RIGID BAR

EQUATION OF EQUILIBRIUM



$$\Sigma F_{\text{vert}} = 0$$

$$2F_1 + F_2 - W = 0 \quad (\text{Eq. 1})$$

FULLY STRESSED RODS

$$F_1 = \sigma_1 A_1 \quad F_2 = \sigma_2 A_2$$

$$A_1 = \frac{\pi d_1^2}{4} \quad A_2 = \frac{\pi d_2^2}{4}$$

Substitute into Eq. (1):

$$2\sigma_1 \left(\frac{\pi d_1^2}{4} \right) + \sigma_2 \left(\frac{\pi d_2^2}{4} \right) = W$$

Diameter d_1 is known; solve for d_2 :

$$d_2^2 = \frac{4W}{\pi\sigma_2} - \frac{2\sigma_1 d_1^2}{\sigma_2} \quad \leftarrow \quad (\text{Eq. 2})$$

SUBSTITUTE NUMERICAL VALUES:

$$\begin{aligned} d_2^2 &= \frac{4(7200 \text{ lb})}{\pi(13,000 \text{ psi})} - \frac{2(24,000 \text{ psi})(0.4 \text{ in.})^2}{13,000 \text{ psi}} \\ &= 0.70518 \text{ in.}^2 - 0.59077 \text{ in.}^2 = 0.11441 \text{ in.}^2 \\ d_2 &= 0.338 \text{ in.} \quad \leftarrow \end{aligned}$$

EQUATION OF COMPATIBILITY

$$\delta_1 = \delta_2 \quad (\text{Eq. 3})$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_1 = \frac{F_1 L_1}{E_1 A_1} = \sigma_1 \left(\frac{L_1}{E_1} \right) \quad (\text{Eq. 4})$$

$$\delta_2 = \frac{F_2 L_2}{E_2 A_2} = \sigma_2 \left(\frac{L_2}{E_2} \right) \quad (\text{Eq. 5})$$

Substitute (4) and (5) into Eq. (3):

$$\sigma_1 \left(\frac{L_1}{E_1} \right) = \sigma_2 \left(\frac{L_2}{E_2} \right)$$

Length L_1 is known; solve for L_2 :

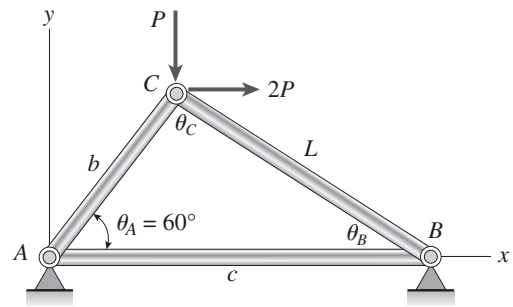
$$L_2 = L_1 \left(\frac{\sigma_1 E_2}{\sigma_2 E_1} \right) \quad \leftarrow \quad (\text{Eq. 6})$$

SUBSTITUTE NUMERICAL VALUES:

$$\begin{aligned} L_2 &= (40 \text{ in.}) \left(\frac{24,000 \text{ psi}}{13,000 \text{ psi}} \right) \left(\frac{6.5 \times 10^6 \text{ psi}}{10 \times 10^6 \text{ psi}} \right) \\ &= 48.0 \text{ in.} \end{aligned}$$

Problem 2.4-14 Three-bar truss ABC (see figure) is constructed of steel pipes having a cross-sectional area $A = 3500 \text{ mm}^2$ and a modulus of elasticity $E = 210 \text{ GPa}$. Member BC is of length $L = 2.5 \text{ m}$, and the angle between members AC and AB is known to be 60° . Member AC length is $b = 0.71L$. Loads $P = 185 \text{ kN}$ and $2P = 370 \text{ kN}$ act vertically and horizontally at joint C , as shown. Joints A and B are pinned supports. (Use the law of sines and law of cosines to find missing dimensions and angles in the figure.)

- Find the support reactions at joints A and B . Use horizontal reaction B_x as the redundant.
- What is the maximum permissible value of load variable P if the allowable normal stress in each truss member is 150 MPa ?



Solution 2.4-14

NUMERICAL DATA

$$\begin{aligned} L &= 2.5 \text{ m} & b &= 0.71 & L &= 1.775 \text{ m} & E &= 210 \text{ GPa} & A &= 3500 \text{ mm}^2 & P &= 185 \text{ kN} & \theta_A &= 60^\circ \\ \sigma_a &= 150 \text{ MPa} \end{aligned}$$

FIND MISSING DIMENSIONS AND ANGLES IN PLANE TRUSS FIGURE

$$x_c = b \cos(\theta_A) = 0.8875 \text{ m} \quad y_c = b \sin(\theta_A) = 1.5372 \text{ m}$$

$$\frac{b}{\sin(\theta_B)} = \frac{L}{\sin(\theta_A)} \quad \text{so} \quad \theta_B = \sin^{-1} \left(\frac{b \sin(\theta_A)}{L} \right) = 37.94306^\circ$$

$$\theta_C = 180^\circ - (\theta_A + \theta_B) = 82.05694^\circ$$

$$c = \frac{L}{\sin(\theta_A)} \sin(\theta_C) = 2.85906 \text{ m} \quad \text{or} \quad c = \sqrt{b^2 + L^2 - 2bL \cos(\theta_C)} = 2.85906 \text{ m}$$

- (a) SELECT B_x AS THE REDUNDANT; PERFORM SUPERPOSITION ANALYSIS TO FIND B_x THEN USE STATICS TO FIND REMAINING REACTIONS. FINALLY USE METHOD OF JOINTS TO FIND MEMBER FORCES (SEE EXAMPLE 1-1)

δ_{Bx1} = displacement in x -direction in released structure acted upon by loads P and $2P$ at joint C :

$$\delta_{Bx1} = 1.2789911 \text{ mm} < \text{this displacement equals force in } AB \text{ divided by flexibility of } AB$$

$$\delta_{Bx2} = \text{displacement in } x\text{-direction in released structure acted upon by redundant } B_x: \quad \delta_{Bx2} = B_x \frac{c}{EA}$$

$$\text{COMPATIBILITY EQUATION:} \quad \delta_{Bx1} + \delta_{Bx2} = 0 \quad \text{so} \quad B_x = \frac{-EA}{c} \delta_{Bx1} = -328.8 \text{ kN}$$

$$\text{STATICS:} \quad \Sigma F_x = 0 \quad A_x = -B_x - 2P = -41.2 \text{ kN}$$

$$\Sigma M_A = 0 \quad B_y = \frac{1}{c} [2P(b \sin(\theta_A)) + P(b \cos(\theta_A))] = 256.361 \text{ kN}$$

$$\Sigma F_y = 0 \quad A_y = P - B_y = -71.361 \text{ kN}$$

REACTIONS:

$$\boxed{A_x = -41.2 \text{ kN}} \quad \boxed{A_y = -71.4 \text{ kN}} \quad \boxed{B_x = -329 \text{ kN}} \quad \boxed{B_y = 256 \text{ kN}}$$

- (b) FIND MAXIMUM PERMISSIBLE VALUE OF LOAD VARIABLE P IF ALLOWABLE NORMAL STRESS IS 150 MPa

(1) Use reactions and Method of Joints to find member forces in each member for above loading.

$$\text{Results: } F_{AB} = 0 \quad F_{BC} = -416.929 \text{ kN} \quad F_{AC} = 82.40 \text{ kN}$$

(2) Compute member stresses:

$$\sigma_{AB} = 0 \quad \sigma_{BC} = \frac{-416.93 \text{ kN}}{A} = -119.123 \text{ MPa} \quad \sigma_{AC} = \frac{82.4 \text{ kN}}{A} = 23.543 \text{ MPa}$$

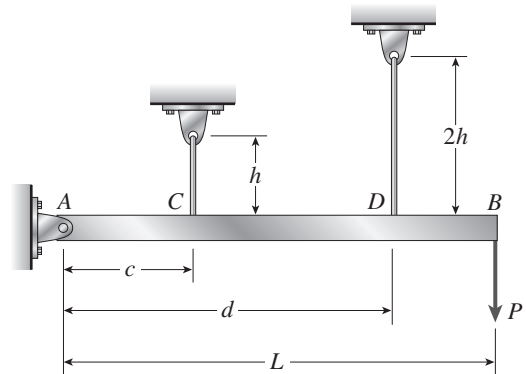
(3) Maximum stress occurs in member BC . For linear analysis, the stress is proportional to the load so

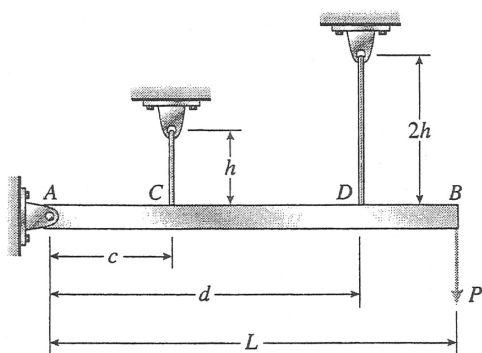
$$P_{\max} = \frac{\sigma_a}{\sigma_{BC}} P = 233 \text{ kN}$$

So when downward load $P = 233 \text{ kN}$ is applied at C and horizontal load $2P = 466 \text{ kN}$ is applied to the right at C , the stress in BC is 150 MPa

Problem 2.4-15 A rigid bar AB of length $L = 66 \text{ in.}$ is hinged to a support at A and supported by two vertical wires attached at points C and D (see figure). Both wires have the same cross-sectional area ($A = 0.0272 \text{ in.}^2$) and are made of the same material (modulus $E = 30 \times 10^6 \text{ psi}$). The wire at C has length $h = 18 \text{ in.}$ and the wire at D has length twice that amount. The horizontal distances are $c = 20 \text{ in.}$ and $d = 50 \text{ in.}$

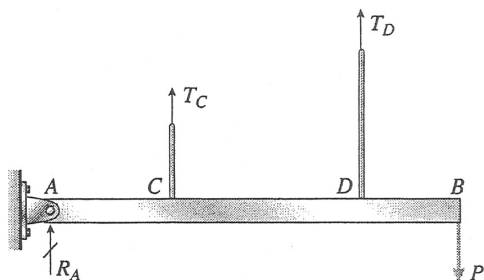
- (a) Determine the tensile stresses σ_C and σ_D in the wires due to the load $P = 340 \text{ lb}$ acting at end B of the bar.
 (b) Find the downward displacement δ_B at end B of the bar.



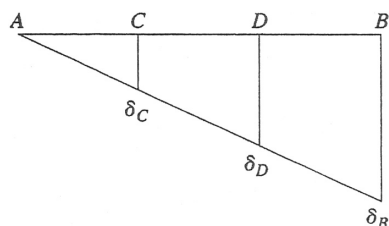
Solution 2.4-15 Bar supported by two wires

$$\begin{aligned} h &= 18 \text{ in.} \\ 2h &= 36 \text{ in.} \\ c &= 20 \text{ in.} \\ d &= 50 \text{ in.} \\ L &= 66 \text{ in.} \\ E &= 30 \times 10^6 \text{ psi} \\ A &= 0.0272 \text{ in.}^2 \\ P &= 340 \text{ lb} \end{aligned}$$

FREE-BODY DIAGRAM



DISPLACEMENT DIAGRAM



EQUATION OF EQUILIBRIUM

$$\Sigma M_A = 0 \quad \curvearrowright \quad T_C(c) + T_D(d) = PL \quad (\text{Eq. 1})$$

EQUATION OF COMPATIBILITY

$$\frac{\delta_C}{c} = \frac{\delta_D}{d} \quad (\text{Eq. 2})$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_C = \frac{T_C h}{EA} \quad \delta_D = \frac{T_D (2h)}{EA} \quad (\text{Eqs. 3, 4})$$

SOLUTION OF EQUATIONS

Substitute (3) and (4) into Eq. (2):

$$\frac{T_C h}{cEA} = \frac{T_D (2h)}{dEA} \quad \text{or} \quad \frac{T_C}{c} = \frac{2T_D}{d} \quad (\text{Eq. 5})$$

TENSILE FORCES IN THE WIRES

Solve simultaneously Eqs. (1) and (5):

$$T_C = \frac{2cPL}{2c^2 + d^2} \quad T_D = \frac{dPL}{2c^2 + d^2} \quad (\text{Eqs. 6, 7})$$

TENSILE STRESSES IN THE WIRES

$$\sigma_C = \frac{T_C}{A} = \frac{2cPL}{A(2c^2 + d^2)} \quad (\text{Eq. 8})$$

$$\sigma_D = \frac{T_D}{A} = \frac{dPL}{A(2c^2 + d^2)} \quad (\text{Eq. 9})$$

DISPLACEMENT AT END OF BAR

$$\delta_B = \delta_D \left(\frac{L}{d} \right) = \frac{2hT_D}{EA} \left(\frac{L}{d} \right) = \frac{2hPL^2}{EA(2c^2 + d^2)} \quad (\text{Eq. 10})$$

SUBSTITUTE NUMERICAL VALUES

$$2c^2 + d^2 = 2(20 \text{ in.})^2 + (50 \text{ in.})^2 = 3300 \text{ in.}^2$$

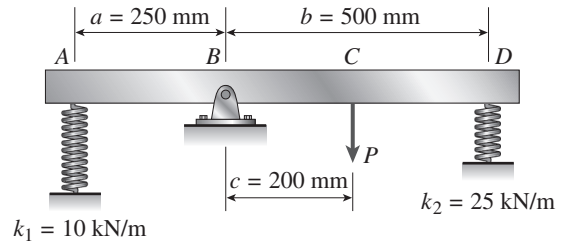
$$\begin{aligned} \text{(a) } \sigma_C &= \frac{2cPL}{A(2c^2 + d^2)} = \frac{2(20 \text{ in.})(340 \text{ lb})(66 \text{ in.})}{(0.0272 \text{ in.}^2)(3300 \text{ in.}^2)} \\ &= 10,000 \text{ psi} \quad \leftarrow \end{aligned}$$

$$\begin{aligned} \sigma_D &= \frac{dPL}{A(2c^2 + d^2)} = \frac{(50 \text{ in.})(340 \text{ lb})(66 \text{ in.})}{(0.0272 \text{ in.}^2)(3300 \text{ in.}^2)} \\ &= 12,500 \text{ psi} \quad \leftarrow \end{aligned}$$

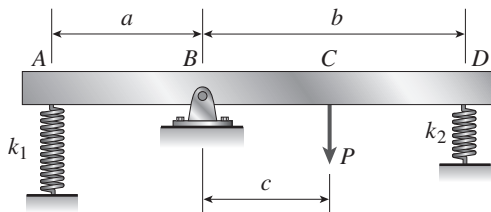
$$\begin{aligned} \text{(b) } \delta_B &= \frac{2hPL^2}{EA(2c^2 + d^2)} \\ &= \frac{2(18 \text{ in.})(340 \text{ lb})(66 \text{ in.})^2}{(30 \times 10^6 \text{ psi})(0.0272 \text{ in.}^2)(3300 \text{ in.}^2)} \\ &= 0.0198 \text{ in.} \quad \leftarrow \end{aligned}$$

Problem 2.4-16 A rigid bar $ABCD$ is pinned at point B and supported by springs at A and D (see figure). The springs at A and D have stiffnesses $k_1 = 10 \text{ kN/m}$ and $k_2 = 25 \text{ kN/m}$, respectively, and the dimensions a , b , and c are 250 mm, 500 mm, and 200 mm, respectively. A load P acts at point C .

If the angle of rotation of the bar due to the action of the load P is limited to 3° , what is the maximum permissible load P_{\max} ?



Solution 2.4-16 Rigid bar supported by springs



NUMERICAL DATA

$$a = 250 \text{ mm}$$

$$b = 500 \text{ mm}$$

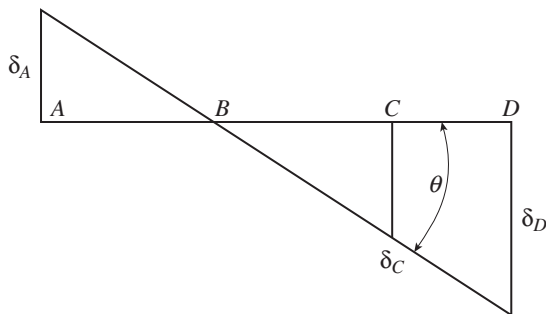
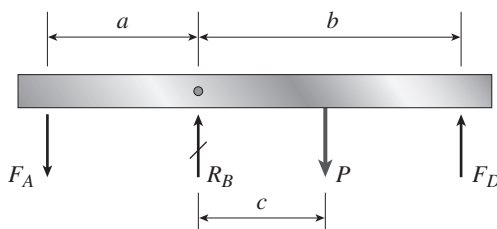
$$c = 200 \text{ mm}$$

$$k_1 = 10 \text{ kN/m}$$

$$k_2 = 25 \text{ kN/m}$$

$$\theta_{\max} = 3^\circ = \frac{\pi}{60} \text{ rad}$$

FREE-BODY DIAGRAM AND DISPLACEMENT DIAGRAM



EQUATION OF EQUILIBRIUM

$$\Sigma M_B = 0 + -F_A(a) - P(c) + F_D(b) = 0 \quad (\text{Eq. 1})$$

EQUATION OF COMPATIBILITY

$$\frac{\delta_A}{a} = \frac{\delta_D}{b} \quad (\text{Eq. 2})$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_A = \frac{F_A}{k_1} \quad \delta_D = \frac{F_D}{k_2} \quad (\text{Eqs. 3, 4})$$

SOLUTION OF EQUATIONS

Substitute (3) and (4) into Eq. (2):

$$\frac{F_A}{ak_1} = \frac{F_D}{bk_2} \quad (\text{Eq. 5})$$

SOLVE SIMULTANEOUSLY EQS. (1) AND (5):

$$F_A = \frac{ack_1P}{a^2k_1 + b^2k_2} \quad F_D = \frac{bck_2P}{a^2k_1 + b^2k_2}$$

ANGLE OF ROTATION

$$\delta_D = \frac{F_D}{k_2} = \frac{bcP}{a^2k_1 + b^2k_2} \quad \theta = \frac{\delta_D}{b} = \frac{cP}{a^2k_1 + b^2k_2}$$

MAXIMUM LOAD

$$P = \frac{\theta}{c} (a^2k_1 + b^2k_2)$$

$$P_{\max} = \frac{\theta_{\max}}{c} (a^2k_1 + b^2k_2) \quad \leftarrow$$

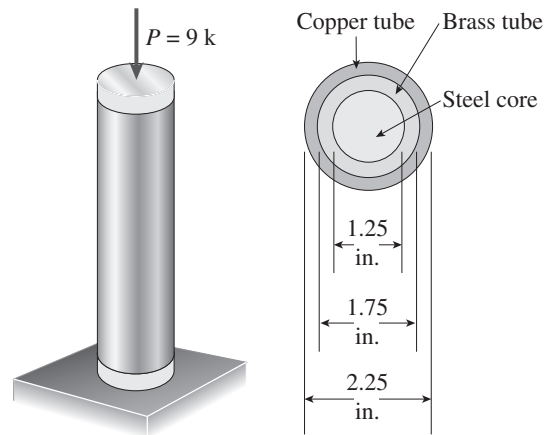
SUBSTITUTE NUMERICAL VALUES:

$$P_{\max} = \frac{\pi/60 \text{ rad}}{200 \text{ mm}} [(250 \text{ mm})^2(10 \text{ kN/m}) + (500 \text{ mm})^2(25 \text{ kN/m})]$$

$$= 1800 \text{ N} \quad \leftarrow$$

Problem 2.4-17 A trimetallic bar is uniformly compressed by an axial force $P = 9$ kips applied through a rigid end plate (see figure). The bar consists of a circular steel core surrounded by brass and copper tubes. The steel core has diameter 1.25 in., the brass tube has outer diameter 1.75 in., and the copper tube has outer diameter 2.25 in. The corresponding moduli of elasticity are $E_s = 30,000$ ksi, $E_b = 16,000$ ksi, and $E_c = 18,000$ ksi.

Calculate the compressive stresses σ_s , σ_b , and σ_c in the steel, brass, and copper, respectively, due to the force P .



Solution 2.4-17

Numerical properties (kips, inches):

$$d_c = 2.25 \text{ in.} \quad d_b = 1.75 \text{ in.} \quad d_s = 1.25 \text{ in.}$$

$$E_c = 18,000 \text{ ksi} \quad E_b = 16,000 \text{ ksi}$$

$$E_s = 30,000 \text{ ksi}$$

$$P = 9 \text{ k}$$

EQUATION OF EQUILIBRIUM

$$\Sigma F_{\text{vert}} = 0 \quad P_s + P_b + P_c = P \quad (\text{Eq. 1})$$

EQUATIONS OF COMPATIBILITY

$$\delta_s = \delta_b \quad \delta_c = \delta_s \quad (\text{Eqs. 2})$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_s = \frac{P_s L}{E_s A_s} \quad \delta_b = \frac{P_b L}{E_b A_b} \quad \delta_c = \frac{P_c L}{E_c A_c} \quad (\text{Eqs. 3, 4, 5})$$

SOLUTION OF EQUATIONS

Substitute (3), (4), and (5) into Eqs. (2):

$$P_b = P_s \frac{E_b A_b}{E_s A_s} \quad P_c = P_s \frac{E_c A_c}{E_s A_s} \quad (\text{Eqs. 6, 7})$$

SOLVE SIMULTANEOUSLY EQS. (1), (6), AND (7):

$$P_s = P \frac{E_s A_s}{E_s A_s + E_b A_b + E_c A_c} = 3.95 \text{ k}$$

$$P_b = P \frac{E_b A_b}{E_s A_s + E_b A_b + E_c A_c} = 2.02 \text{ k}$$

$$P_c = P \frac{E_c A_c}{E_s A_s + E_b A_b + E_c A_c} = 3.03 \text{ k}$$

$$P_s + P_b + P_c = 9 \quad \text{statics check}$$

COMPRESSIVE STRESSES

$$\text{Let } \Sigma EA = E_s A_s + E_b A_b + E_c A_c$$

$$\sigma_s = \frac{P_s}{A_s} = \frac{PE_s}{\Sigma EA} \quad \sigma_s = 3.22 \text{ ksi} \quad \leftarrow$$

$$\sigma_b = \frac{P_b}{A_b} = \frac{PE_b}{\Sigma EA} \quad \sigma_b = 1.716 \text{ ksi} \quad \leftarrow$$

$$\sigma_c = \frac{P_c}{A_c} = \frac{PE_c}{\Sigma EA} \quad \sigma_c = 1.93 \text{ ksi} \quad \leftarrow$$

Thermal Effects

Problem 2.5-1 The rails of a railroad track are welded together at their ends (to form continuous rails and thus eliminate the clacking sound of the wheels) when the temperature is 60°F.

What compressive stress σ is produced in the rails when they are heated by the sun to 120°F if the coefficient of thermal expansion $\alpha = 6.5 \times 10^{-6}/^\circ\text{F}$ and the modulus of elasticity $E = 30 \times 10^6$ psi?

Solution 2.5-1 Expansion of railroad rails

The rails are prevented from expanding because of their great length and lack of expansion joints.

Therefore, each rail is in the same condition as a bar with fixed ends (see Example 2-7).

The compressive stress in the rails may be calculated as follows.

$$\Delta T = 120^\circ\text{F} - 60^\circ\text{F} = 60^\circ\text{F}$$

$$\sigma = E\alpha(\Delta T)$$

$$= (30 \times 10^6 \text{ psi})(6.5 \times 10^{-6}/^\circ\text{F})(60^\circ\text{F})$$

$$\sigma = 11,700 \text{ psi} \quad \leftarrow$$

Problem 2.5-2 An aluminum pipe has a length of 60 m at a temperature of 10°C. An adjacent steel pipe at the same temperature is 5 mm longer than the aluminum pipe.

At what temperature (degrees Celsius) will the aluminum pipe be 15 mm longer than the steel pipe? (Assume that the coefficients of thermal expansion of aluminum and steel are $\alpha_a = 23 \times 10^{-6}/^\circ\text{C}$ and $\alpha_s = 12 \times 10^{-6}/^\circ\text{C}$, respectively.)

Solution 2.5-2 Aluminum and steel pipes

INITIAL CONDITIONS

$$L_a = 60 \text{ m} \quad T_0 = 10^\circ\text{C}$$

$$L_s = 60.005 \text{ m} \quad T_0 = 10^\circ\text{C}$$

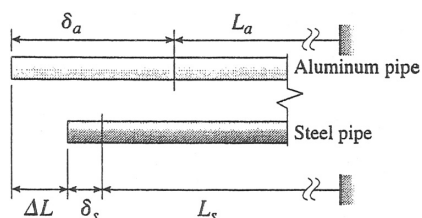
$$\alpha_a = 23 \times 10^{-6}/^\circ\text{C} \quad \alpha_s = 12 \times 10^{-6}/^\circ\text{C}$$

FINAL CONDITIONS

Aluminum pipe is longer than the steel pipe by the amount $\Delta L = 15$ mm.

ΔT = increase in temperature

$$\delta_a = \alpha_a(\Delta T)L_a \quad \delta_s = \alpha_s(\Delta T)L_s$$



From the figure above:

$$\delta_a + L_a = \Delta L + \delta_s + L_s$$

$$\text{or, } \alpha_a(\Delta T)L_a + L_a = \Delta L + \alpha_s(\Delta T)L_s + L_s$$

Solve for ΔT :

$$\Delta T = \frac{\Delta L + (L_s - L_a)}{\alpha_a L_a - \alpha_s L_s} \quad \leftarrow$$

Substitute numerical values:

$$\alpha_a L_a - \alpha_s L_s = 659.9 \times 10^{-6} \text{ m}/^\circ\text{C}$$

$$\Delta T = \frac{15 \text{ mm} + 5 \text{ mm}}{659.9 \times 10^{-6} \text{ m}/^\circ\text{C}} = 30.31^\circ\text{C}$$

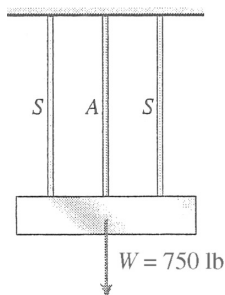
$$T = T_0 + \Delta T = 10^\circ\text{C} + 30.31^\circ\text{C}$$

$$= 40.3^\circ\text{C} \quad \leftarrow$$

Problem 2.5-3 A rigid bar of weight $W = 750$ lb hangs from three equally spaced wires, two of steel and one of aluminum (see figure). The diameter of the wires is $\frac{1}{8}$ in. Before they were loaded, all three wires had the same length.

What temperature increase ΔT in all three wires will result in the entire load being carried by the steel wires? (Assume $E_s = 30 \times 10^6$ psi, $\alpha_s = 6.5 \times 10^{-6}/^\circ\text{F}$, and $\alpha_a = 12 \times 10^{-6}/^\circ\text{F}$.)

Solution 2.5-3 Bar supported by three wires



S = steel A = aluminum

$W = 750$ lb

$$d = \frac{1}{8} \text{ in.}$$

$$A_s = \frac{\pi d^2}{4} = 0.012272 \text{ in.}^2$$

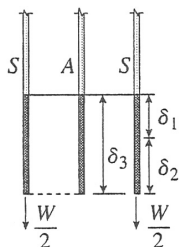
$$E_s = 30 \times 10^6 \text{ psi}$$

$$E_s A_s = 368,155 \text{ lb}$$

$$\alpha_s = 6.5 \times 10^{-6}/^\circ\text{F}$$

$$\alpha_a = 12 \times 10^{-6}/^\circ\text{F}$$

L = Initial length of wires



δ_1 = increase in length of a steel wire due to temperature increase ΔT

$$= \alpha_s (\Delta T) L$$

δ_2 = increase in length of a steel wire due to load $W/2$

$$= \frac{WL}{2E_s A_s}$$

δ_3 = increase in length of aluminum wire due to temperature increase ΔT

$$= \alpha_a (\Delta T) L$$

For no load in the aluminum wire:

$$\delta_1 + \delta_2 = \delta_3$$

$$\alpha_s (\Delta T) L + \frac{WL}{2E_s A_s} = \alpha_a (\Delta T) L$$

or

$$\Delta T = \frac{W}{2E_s A_s (\alpha_a - \alpha_s)} \quad \leftarrow$$

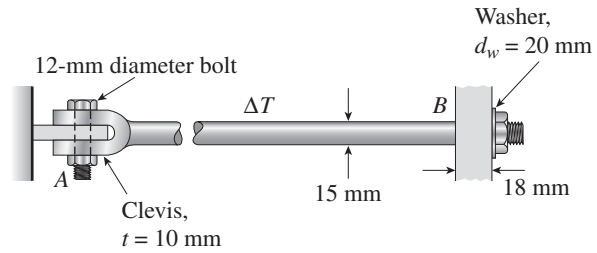
Substitute numerical values:

$$\begin{aligned} \Delta T &= \frac{750 \text{ lb}}{(2)(368,155 \text{ lb})(5.5 \times 10^{-6}/^\circ\text{F})} \\ &= 185^\circ\text{F} \quad \leftarrow \end{aligned}$$

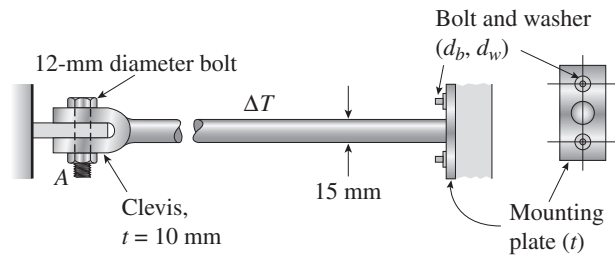
NOTE: If the temperature increase is larger than ΔT , the aluminum wire would be in compression, which is not possible. Therefore, the steel wires continue to carry all of the load. If the temperature increase is less than ΔT , the aluminum wire will be in tension and carry part of the load.

Problem 2.5-4 A steel rod of 15-mm diameter is held snugly (but without any initial stresses) between rigid walls by the arrangement shown in figure part a. (For the steel rod, use $\alpha = 12 \times 10^{-6}/^\circ\text{C}$ and $E = 200 \text{ GPa}$.)

- Calculate the temperature drop ΔT (degrees Celsius) at which the average shear stress in the 12-mm diameter bolt becomes 45 MPa. Also, what is the normal stress in the rod?
- What are the average bearing stresses in the bolt and clevis at A and between the washer ($d_w = 20 \text{ mm}$) and wall ($t = 18 \text{ mm}$) at B?
- If the connection to the wall at B is changed to an end plate with two bolts (see figure part b), what is the required diameter d_b of each bolt if the temperature drop is $\Delta T = 38^\circ\text{C}$ and the allowable bolt stress is 90 MPa?



(a)



(b)

Solution 2.5-4

NUMERICAL PROPERTIES

$$d_r = 15 \text{ mm} \quad d_b = 12 \text{ mm} \quad d_w = 20 \text{ mm} \quad t_c = 10 \text{ mm} \quad t_{\text{wall}} = 18 \text{ mm}$$

$$\tau_b = 45 \text{ MPa} \quad \alpha = 12 (10^{-6}) \quad E = 200 \text{ GPa}$$

(a) TEMPERATURE DROP RESULTING IN BOLT SHEAR STRESS $\epsilon = \alpha \Delta T \quad \sigma = E \alpha \Delta T$

$$\text{Rod force} = P = (E \alpha \Delta T) \frac{\pi}{4} d_r^2 \quad \text{and bolt in double shear with shear stress} \quad \tau = \frac{P}{A_s} \quad \tau = \frac{P}{2 \frac{\pi}{4} d_b^2}$$

$$\tau_b = \frac{2}{\pi d_b^2} \left[(E \alpha \Delta T) \frac{\pi}{4} d_r^2 \right] \quad \tau_b = \frac{E \alpha \Delta T}{2} \left(\frac{d_r}{d_b} \right)^2$$

$$\tau_b = 45 \text{ MPa}$$

$$\Delta T = \frac{2 \tau_b}{E (1000) \alpha} \left(\frac{d_b}{d_r} \right)^2 \quad \Delta T = 24^\circ\text{C} \quad P = (E \alpha \Delta T) \frac{\pi}{4} d_r^2 \quad P = 10 \text{ kN}$$

$$\sigma_{\text{rod}} = \frac{P 1000}{\frac{\pi}{4} d_r^2} \quad \boxed{\sigma_{\text{rod}} = 57.6 \text{ MPa}}$$

(b) BEARING STRESSES

$$\text{BOLT AND CLEVIS} \quad \sigma_{bc} = \frac{\frac{P}{2}}{d_b t_c} \quad \boxed{\sigma_{bc} = 42.4 \text{ MPa}}$$

$$\text{WASHER AT WALL} \quad \sigma_{bw} = \frac{P}{\frac{\pi}{4}(d_w^2 - d_r^2)} \quad \boxed{\sigma_{bw} = 74.1 \text{ MPa}}$$

- (c) If the connection to the wall at B is changed to an end plate with two bolts (see Fig. b), what is the required diameter d_b of each bolt if temperature drop $\Delta T = 38^\circ\text{C}$ and the allowable bolt stress is 90 MPa?

Find force in rod due to temperature drop.

$$\Delta T = 38^\circ\text{C} \quad P = (E\alpha\Delta T) \frac{\pi}{4} d_r^2$$

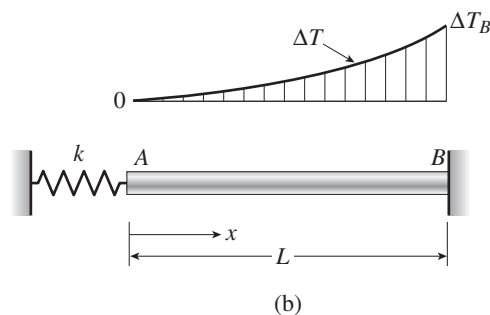
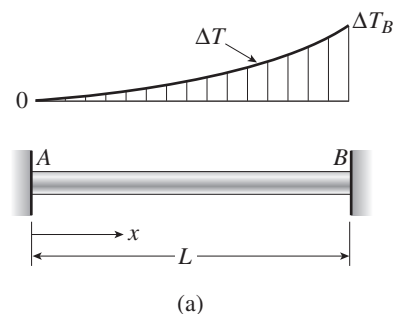
$$P = 200 \text{ GPa} \frac{\pi}{4} (15 \text{ mm})^2 [12(10^{-6})](38) = 16116 \text{ N} \quad P = 16.12 \text{ kN}$$

Each bolt carries one half of the force P :

$$d_b = \sqrt{\frac{\frac{16.12 \text{ kN}}{2}}{\frac{\pi}{4}(90 \text{ MPa})}} = 10.68 \text{ mm} \quad \boxed{d_b = 10.68 \text{ mm}}$$

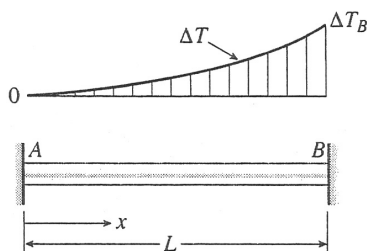
Problem 2.5-5 A bar AB of length L is held between rigid supports and heated nonuniformly in such a manner that the temperature increase ΔT at distance x from end A is given by the expression $\Delta T = \Delta T_B x^3/L^3$, where ΔT_B is the increase in temperature at end B of the bar (see figure part a).

- Derive a formula for the compressive stress σ_c in the bar. (Assume that the material has modulus of elasticity E and coefficient of thermal expansion α).
- Now modify the formula in (a) if the rigid support at A is replaced by an elastic support at A having a spring constant k (see figure part b). Assume that only bar AB is subject to the temperature increase.



Solution 2.5-5

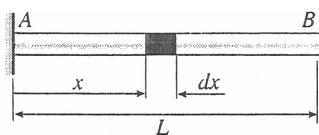
- (a) ONE DEGREE STATICALLY INDETERMINATE—USE SUPERPOSITION SELECT REACTION R_B AS THE REDUNDANT; FOLLOW PROCEDURE
Bar with nonuniform temperature change.



At distance x :

$$\Delta T = \Delta T_B \left(\frac{x^3}{L^3} \right)$$

REMOVE THE SUPPORT AT THE END B OF THE BAR:



Consider an element dx at a distance x from end A .

$d\delta$ = Elongation of element dx

$$d\delta = \alpha(\Delta T)dx = \alpha(\Delta T_B) \left(\frac{x^3}{L^3} \right) dx$$

$d\delta$ = elongation of bar

$$\delta = \int_0^L d\delta = \int_0^L \alpha(\Delta T_B) \left(\frac{x^3}{L^3} \right) dx = \frac{1}{4} \alpha(\Delta T_B) L$$

COMPRESSIVE FORCE P REQUIRED TO SHORTEN THE BAR BY THE AMOUNT δ

$$P = \frac{EA\delta}{L} = \frac{1}{4} EA\alpha(\Delta T_B)$$

COMPRESSIVE STRESS IN THE BAR

$$\sigma_c = \frac{P}{A} = \frac{E\alpha(\Delta T_B)}{4} \leftarrow$$

- (b) ONE DEGREE STATICALLY INDETERMINATE—USE SUPERPOSITION.

Select reaction R_B as the redundant then compute bar elongations due to ΔT and due to R_B

$$\delta_{B1} = \alpha\Delta T_B \frac{L}{4} \quad \text{due to temperature from above}$$

$$\delta_{B2} = R_B \left(\frac{1}{k} + \frac{L}{EA} \right)$$

Compatibility: solve for R_B : $\delta_{B1} + \delta_{B2} = 0$

$$R_B = \frac{-\left(\alpha\Delta T_B \frac{L}{4} \right)}{\left(\frac{1}{k} + \frac{L}{EA} \right)}$$

$$R_B = -\alpha\Delta T_B \left[\frac{EA}{4 \left(\frac{EA}{kL} + 1 \right)} \right]$$

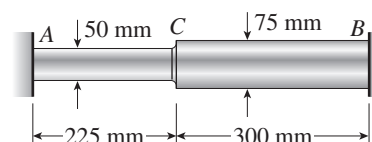
So compressive stress in bar is

$$\sigma_c = \frac{R_B}{A} \quad \sigma_c = \frac{E\alpha(\Delta T_B)}{4 \left(\frac{EA}{kL} + 1 \right)} \leftarrow$$

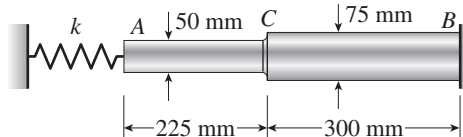
NOTE: σ_c in part (b) is the same as in part (a) if spring constant k goes to infinity.

Problem 2.5-6 A plastic bar ACB having two different solid circular cross sections is held between rigid supports as shown in the figure. The diameters in the left- and right-hand parts are 50 mm and 75 mm, respectively. The corresponding lengths are 225 mm and 300 mm. Also, the modulus of elasticity E is 6.0 GPa, and the coefficient of thermal expansion α is $100 \times 10^{-6}/^\circ\text{C}$. The bar is subjected to a uniform temperature increase of 30°C .

- (a) Calculate the following quantities: (1) the compressive force N in the bar; (2) the maximum compressive stress σ_c ; and (3) the displacement δ_C of point C .
- (b) Repeat (a) if the rigid support at A is replaced by an elastic support having spring constant $k = 50 \text{ MN/m}$ (see figure part b; assume that only the bar ACB is subject to the temperature increase).



(a)



(b)

Solution

NUMERICAL DATA

$$d_1 = 50 \text{ mm} \quad d_2 = 75 \text{ mm}$$

$$L_1 = 225 \text{ mm} \quad L_2 = 300 \text{ mm}$$

$$E = 6.0 \text{ GPa} \quad \alpha = 100 \times 10^{-6}/^\circ\text{C}$$

$$\Delta T = 30^\circ\text{C} \quad k = 50 \text{ MN/m}$$

- (a) COMPRESSIVE FORCE N , MAXIMUM COMPRESSIVE STRESS AND DISPLACEMENT OF PT. C

$$A_1 = \frac{\pi}{4} d_1^2 \quad A_2 = \frac{\pi}{4} d_2^2$$

One-degree statically indeterminate—use R_B as redundant

$$\delta_{B1} = \alpha \Delta T (L_1 + L_2)$$

$$\delta_{B2} = R_B \left(\frac{L_1}{EA_1} + \frac{L_2}{EA_2} \right)$$

Compatibility: $\delta_{B1} = \delta_{B2}$, solve for R_B

$$R_B = \frac{\alpha \Delta T (L_1 + L_2)}{\frac{L_1}{EA_1} + \frac{L_2}{EA_2}} \quad N = R_B$$

$$N = 51.8 \text{ kN} \quad \leftarrow$$

Maximum compressive stress in AC since it has the smaller area ($A_1 < A_2$):

$$\sigma_{c\max} = \frac{N}{A_1} \quad \sigma_{c\max} = 26.4 \text{ MPa}$$

Displacement δ_C of point C = superposition of displacements in two released structures at C :

$$\delta_C = \alpha \Delta T (L_1) - R_B \frac{L_1}{EA_1}$$

$$\delta_C = -0.314 \text{ mm} \quad \leftarrow (-) \text{ sign means joint } C \text{ moves left}$$

- (b) COMPRESSIVE FORCE N , MAXIMUM COMPRESSIVE STRESS AND DISPLACEMENT OF PART C FOR ELASTIC SUPPORT CASE

Use R_B as redundant as in part (a):

$$\delta_{B1} = \alpha \Delta T (L_1 + L_2)$$

$$\delta_{B2} = R_B \left(\frac{L_1}{EA_1} + \frac{L_2}{EA_2} + \frac{1}{k} \right)$$

Now add effect of elastic support; equate δ_{B1} and δ_{B2} then solve for R_B :

$$R_B = \frac{\alpha \Delta T (L_1 + L_2)}{\frac{L_1}{EA_1} + \frac{L_2}{EA_2} + \frac{1}{k}} \quad N = R_B$$

$$N = 31.2 \text{ kN} \quad \leftarrow$$

$$\sigma_{c\max} = \frac{N}{A_1} \quad \sigma_{c\max} = 15.91 \text{ MPa} \quad \leftarrow$$

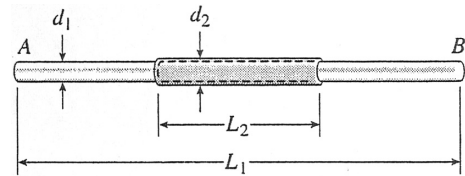
Superposition:

$$\delta_C = \alpha \Delta T (L_1) - R_B \left(\frac{L_1}{EA_1} + \frac{1}{k} \right)$$

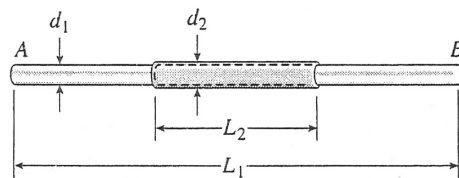
$$\delta_C = -0.546 \text{ mm} \quad \leftarrow (-) \text{ sign means joint } C \text{ moves left}$$

Problem 2.5-7 A circular steel rod AB (diameter $d_1 = 1.0$ in., length $L_1 = 3.0$ ft) has a bronze sleeve (outer diameter $d_2 = 1.25$ in., length $L_2 = 1.0$ ft) shrunk onto it so that the two parts are securely bonded (see figure).

Calculate the total elongation δ of the steel bar due to a temperature rise $\Delta T = 500^\circ\text{F}$. (Material properties are as follows: for steel, $E_s = 30 \times 10^6$ psi and $\alpha_s = 6.5 \times 10^{-6}/^\circ\text{F}$; for bronze, $E_b = 15 \times 10^6$ psi, and $\alpha_b = 11 \times 10^{-6}/^\circ\text{F}$.)



Solution 2.5-7 Steel rod with bronze sleeve



$$L_1 = 36 \text{ in.} \quad L_2 = 12 \text{ in.}$$

ELONGATION OF THE TWO OUTER PARTS OF THE BAR

$$\begin{aligned} \delta_1 &= \alpha_s (\Delta T) (L_1 - L_2) \\ &= (6.5 \times 10^{-6}/^\circ\text{F})(500^\circ\text{F})(36 \text{ in.} - 12 \text{ in.}) \\ &= 0.07800 \text{ in.} \end{aligned}$$

ELONGATION OF THE MIDDLE PART OF THE BAR

The steel rod and bronze sleeve lengthen the same amount, so they are in the same condition as the bolt and sleeve of Example 2-8. Thus, we can calculate the elongation from Eq. (2-23):

$$\delta_2 = \frac{(\alpha_s E_s A_s + \alpha_b E_b A_b)(\Delta T)L_2}{E_s A_s + E_b A_b}$$

SUBSTITUTE NUMERICAL VALUES

$$\alpha_s = 6.5 \times 10^{-6}/^\circ\text{F} \quad \alpha_b = 11 \times 10^{-6}/^\circ\text{F}$$

$$E_s = 30 \times 10^6 \text{ psi} \quad E_b = 15 \times 10^6 \text{ psi}$$

$$d_1 = 1.0 \text{ in.}$$

$$A_s = \frac{\pi}{4} d_1^2 = 0.78540 \text{ in.}^2$$

$$d_2 = 1.25 \text{ in.}$$

$$A_b = \frac{\pi}{4} (d_2^2 - d_1^2) = 0.44179 \text{ in.}^2$$

$$\Delta T = 500^\circ\text{F} \quad L_2 = 12.0 \text{ in.}$$

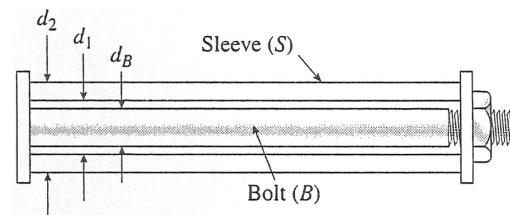
$$\delta_2 = 0.04493 \text{ in.}$$

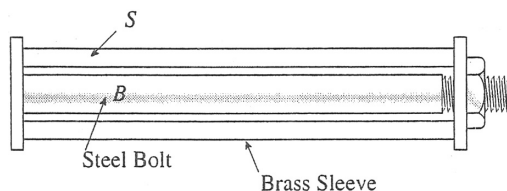
TOTAL ELONGATION

$$\delta = \delta_1 + \delta_2 = 0.123 \text{ in.} \quad \leftarrow$$

Problem 2.5-8 A brass sleeve S is fitted over a steel bolt B (see figure), and the nut is tightened until it is just snug. The bolt has a diameter $d_B = 25$ mm, and the sleeve has inside and outside diameters $d_1 = 26$ mm and $d_2 = 36$ mm, respectively.

Calculate the temperature rise ΔT that is required to produce a compressive stress of 25 MPa in the sleeve. (Use material properties as follows: for the sleeve, $\alpha_s = 21 \times 10^{-6}/^\circ\text{C}$ and $E_s = 100$ GPa; for the bolt, $\alpha_b = 10 \times 10^{-6}/^\circ\text{C}$ and $E_b = 200$ GPa.) (Suggestion: Use the results of Example 2-8.)



Solution 2.5-8 Brass sleeve fitted over a Steel bolt

Subscript S means “sleeve”.

Subscript B means “bolt”.

Use the results of Example 2-8.

σ_S = compressive force in sleeve

EQUATION (2-22a):

$$\sigma_S = \frac{(\alpha_S - \alpha_B)(\Delta T)E_S E_B A_B}{E_S A_S + E_B A_B} \text{ (Compression)}$$

SOLVE FOR ΔT :

$$\Delta T = \frac{\sigma_S(E_S A_S + E_B A_B)}{(\alpha_S - \alpha_B)E_S E_B A_B}$$

or

$$\Delta T = \frac{\sigma_S}{E_S(\alpha_S - \alpha_B)} \left(1 + \frac{E_S A_S}{E_B A_B} \right) \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

$$\sigma_S = 25 \text{ MPa}$$

$$d_2 = 36 \text{ mm} \quad d_1 = 26 \text{ mm} \quad d_B = 25 \text{ mm}$$

$$E_S = 100 \text{ GPa} \quad E_B = 200 \text{ GPa}$$

$$\alpha_S = 21 \times 10^{-6}/^\circ\text{C} \quad \alpha_B = 10 \times 10^{-6}/^\circ\text{C}$$

$$A_S = \frac{\pi}{4} (d_2^2 - d_1^2) = \frac{\pi}{4} (620 \text{ mm}^2)$$

$$A_B = \frac{\pi}{4} (d_B)^2 = \frac{\pi}{4} (625 \text{ mm}^2) \quad 1 + \frac{E_S A_S}{E_B A_B} = 1.496$$

$$\Delta T = \frac{25 \text{ MPa} (1.496)}{(100 \text{ GPa})(11 \times 10^{-6}/^\circ\text{C})}$$

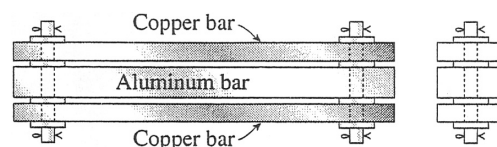
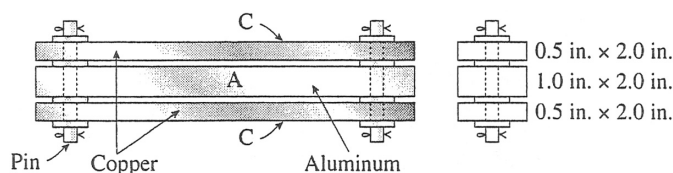
$$\Delta T = 34^\circ\text{C} \quad \leftarrow$$

(Increase in temperature)

Problem 2.5-9 Rectangular bars of copper and aluminum are held by pins at their ends, as shown in the figure. Thin spacers provide a separation between the bars. The copper bars have cross-sectional dimensions 0.5 in. \times 2.0 in., and the aluminum bar has dimensions 1.0 in. \times 2.0 in.

Determine the shear stress in the 7/16 in. diameter pins if the temperature is raised by 100°F. (For copper, $E_c = 18,000$ ksi and $\alpha_c = 9.5 \times 10^{-6}/^\circ\text{F}$; for aluminum, $E_a = 10,000$ ksi, and $\alpha_a = 13 \times 10^{-6}/^\circ\text{F}$.)

Suggestion: Use the results of Example 2-8.

**Solution 2.5-9 Rectangular bars held by pins**

$$\text{Diameter of pin: } d_P = \frac{7}{16} \text{ in.} = 0.4375 \text{ in.}$$

$$\text{Area of pin: } A_P = \frac{\pi}{4} d_P^2 = 0.15033 \text{ in.}^2$$

$$\text{Area of two copper bars: } A_c = 2.0 \text{ in.}^2$$

$$\text{Area of aluminum bar: } A_a = 2.0 \text{ in.}^2$$

$$\Delta T = 100^\circ\text{F}$$

Copper: $E_c = 18,000 \text{ ksi}$ $\alpha_c = 9.5 \times 10^{-6}/^\circ\text{F}$

Aluminum: $E_a = 10,000 \text{ ksi}$

$\alpha_a = 13 \times 10^{-6}/^\circ\text{F}$

Use the results of Example 2-8.

Find the forces P_a and P_c in the aluminum bar and copper bar, respectively, from Eq. (2-21).

Replace the subscript “S” in that equation by “a” (for aluminum) and replace the subscript “B” by “c” (for copper), because α for aluminum is larger than α for copper.

$$P_a = P_c = \frac{(\alpha_a - \alpha_c)(\Delta T)E_a A_a E_c A_c}{E_a A_a + E_c A_c}$$

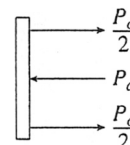
Note that P_a is the compressive force in the aluminum bar and P_c is the combined tensile force in the two copper bars.

$$P_a = P_c = \frac{(\alpha_a - \alpha_c)(\Delta T)E_c A_c}{1 + \frac{E_c A_c}{E_a A_a}}$$

SUBSTITUTE NUMERICAL VALUES:

$$P_a = P_c = \frac{(3.5 \times 10^{-6}/^\circ\text{F})(100^\circ\text{F})(18,000 \text{ ksi})(2 \text{ in.}^2)}{1 + \left(\frac{18}{10}\right)\left(\frac{2.0}{2.0}\right)} = 4,500 \text{ lb}$$

FREE-BODY DIAGRAM OF PIN AT THE LEFT END



V = shear force in pin

$$= P_c/2$$

$$= 2,250 \text{ lb}$$

τ = average shear stress on cross section of pin

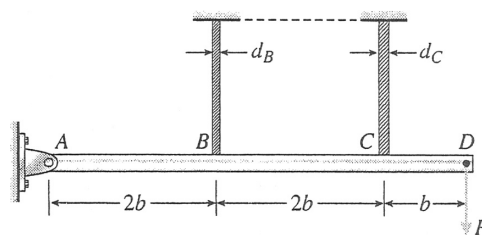
$$\tau = \frac{V}{A_p} = \frac{2,250 \text{ lb}}{0.15033 \text{ in.}^2}$$

$$\tau = 15.0 \text{ ksi} \quad \leftarrow$$

Problem 2.5-10 A rigid bar $ABCD$ is pinned at end A and supported by two cables at points B and C (see figure). The cable at B has nominal diameter $d_B = 12 \text{ mm}$ and the cable at C has nominal diameter $d_C = 20 \text{ mm}$. A load P acts at end D of the bar.

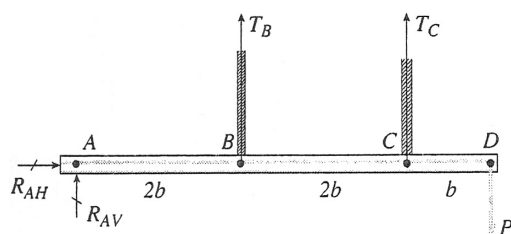
What is the allowable load P if the temperature rises by 60°C and each cable is required to have a factor of safety of at least 5 against its ultimate load?

(Note: The cables have effective modulus of elasticity $E = 140 \text{ GPa}$ and coefficient of thermal expansion $\alpha = 12 \times 10^{-6}/^\circ\text{C}$. Other properties of the cables can be found in Table 2-1, Section 2.2.)



Solution 2.5-10 Rigid bar supported by two cables

FREE-BODY DIAGRAM OF BAR $ABCD$



T_B = force in cable B T_C = force in cable C

$d_B = 12 \text{ mm}$ $d_C = 20 \text{ mm}$

From Table 2-1:

$$A_B = 76.7 \text{ mm}^2 \quad E = 140 \text{ GPa}$$

$$\Delta T = 60^\circ\text{C} \quad A_C = 173 \text{ mm}^2$$

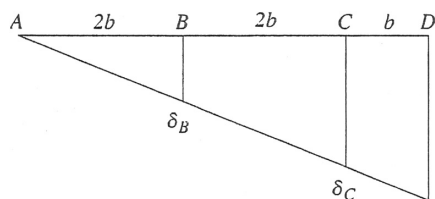
$$\alpha = 12 \times 10^{-6}/^\circ\text{C}$$

EQUATION OF EQUILIBRIUM

$$\Sigma M_A = 0 \quad \curvearrowright \quad T_B(2b) + T_C(4b) - P(5b) = 0$$

$$\text{or } 2T_B + 4T_C = 5P \quad (\text{Eq. 1})$$

DISPLACEMENT DIAGRAM



COMPATIBILITY:

$$\delta_C = 2\delta_B \quad (\text{Eq. 2})$$

FORCE-DISPLACEMENT AND TEMPERATURE-DISPLACEMENT RELATIONS

$$\delta_B = \frac{T_B L}{EA_B} + \alpha(\Delta T)L \quad (\text{Eq. 3})$$

$$\delta_C = \frac{T_C L}{EA_C} + \alpha(\Delta T)L \quad (\text{Eq. 4})$$

SUBSTITUTE EQS. (3) AND (4) INTO EQ. (2):

$$\frac{T_C L}{EA_C} + \alpha(\Delta T)L = \frac{2T_B L}{EA_B} + 2\alpha(\Delta T)L$$

or

$$2T_B A_C - T_C A_B = -E\alpha(\Delta T)A_B A_C \quad (\text{Eq. 5})$$

SUBSTITUTE NUMERICAL VALUES INTO EQ. (5):

$$T_B(346) - T_C(76.7) = -1,338,000 \quad (\text{Eq. 6})$$

in which T_B and T_C have units of newtons.

SOLVE SIMULTANEOUSLY EQS. (1) AND (6):

$$T_B = 0.2494 P - 3,480 \quad (\text{Eq. 7})$$

$$T_C = 1.1253 P + 1,740 \quad (\text{Eq. 8})$$

in which P has units of newtons.SOLVE EQS. (7) AND (8) FOR THE LOAD P :

$$P_B = 4.0096 T_B + 13,953 \quad (\text{Eq. 9})$$

$$P_C = 0.8887 T_C - 1,546 \quad (\text{Eq. 10})$$

ALLOWABLE LOADS

From Table 2-1:

$$(T_B)_{\text{ULT}} = 102,000 \text{ N} \quad (T_C)_{\text{ULT}} = 231,000 \text{ N}$$

Factor of safety = 5

$$(T_B)_{\text{allow}} = 20,400 \text{ N} \quad (T_C)_{\text{allow}} = 46,200 \text{ N}$$

$$\text{From Eq. (9): } P_B = (4.0096)(20,400 \text{ N}) + 13,953 \text{ N} = 95,700 \text{ N}$$

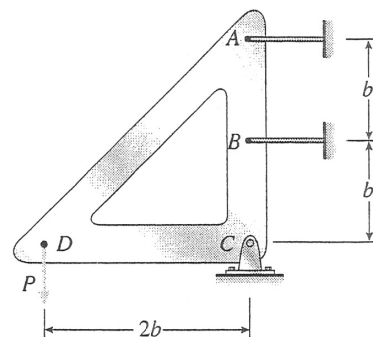
$$\text{From Eq. (10): } P_C = (0.8887)(46,200 \text{ N}) - 1,546 \text{ N} = 39,500 \text{ N}$$

Cable C governs.

$$P_{\text{allow}} = 39.5 \text{ kN} \quad \leftarrow$$

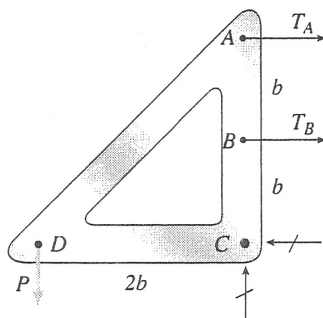
Problem 2.5-11 A rigid triangular frame is pivoted at C and held by two identical horizontal wires at points A and B (see figure). Each wire has axial rigidity $EA = 120 \text{ k}$ and coefficient of thermal expansion $\alpha = 12.5 \times 10^{-6}/^\circ\text{F}$.

- If a vertical load $P = 500 \text{ lb}$ acts at point D , what are the tensile forces T_A and T_B in the wires at A and B , respectively?
- If, while the load P is acting, both wires have their temperatures raised by 180°F , what are the forces T_A and T_B ?
- What further increase in temperature will cause the wire at B to become slack?



Solution 2.5-11 Triangular frame held by two wires

FREE-BODY DIAGRAM OF FRAME

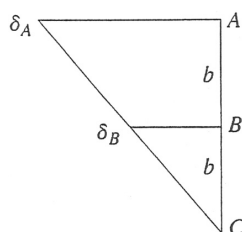


EQUATION OF EQUILIBRIUM

$$\sum M_C = 0 \quad \curvearrowright$$

$$P(2b) - T_A(2b) - T_B(b) = 0 \quad \text{or} \quad 2T_A + T_B = 2P \quad (\text{Eq. 1})$$

DISPLACEMENT DIAGRAM



EQUATION OF COMPATIBILITY

$$\delta_A = 2\delta_B \quad (\text{Eq. 2})$$

(a) LOAD P ONLY

Force-displacement relations:

$$\delta_A = \frac{T_A L}{EA} \quad \delta_B = \frac{T_B L}{EA} \quad (\text{Eq. 3, 4})$$

 $(L = \text{length of wires at A and B.})$

Substitute (3) and (4) into Eq. (2):

$$\frac{T_A L}{EA} = \frac{2T_B L}{EA} \quad (\text{Eq. 5})$$

$$\text{or} \quad T_A = 2T_B$$

Solve simultaneously Eqs. (1) and (5):

$$T_A = \frac{4P}{5} \quad T_B = \frac{2P}{5} \quad (\text{Eqs. 6, 7})$$

Numerical values:

$$P = 500 \text{ lb}$$

$$\therefore T_A = 400 \text{ lb} \quad T_B = 200 \text{ lb} \quad \leftarrow$$

(b) LOAD P AND TEMPERATURE INCREASE ΔT

Force-displacement and temperature-displacement relations:

$$\delta_A = \frac{T_A L}{EA} + \alpha(\Delta T)L \quad (\text{Eq. 8})$$

$$\delta_B = \frac{T_B L}{EA} + \alpha(\Delta T)L \quad (\text{Eq. 9})$$

Substitute (8) and (9) into Eq. (2):

$$\frac{T_A L}{EA} + \alpha(\Delta T)L = \frac{2T_B L}{EA} + 2\alpha(\Delta T)L$$

$$\text{or} \quad T_A - 2T_B = EA\alpha(\Delta T) \quad (\text{Eq. 10})$$

Solve simultaneously Eqs. (1) and (10):

$$T_A = \frac{1}{5}[4P + EA\alpha(\Delta T)] \quad (\text{Eq. 11})$$

$$T_B = \frac{2}{5}[P - EA\alpha(\Delta T)] \quad (\text{Eq. 12})$$

Substitute numerical values:

$$P = 500 \text{ lb} \quad EA = 120,000 \text{ lb}$$

$$\Delta T = 180^\circ\text{F}$$

$$\alpha = 12.5 \times 10^{-6}/^\circ\text{F}$$

$$T_A = \frac{1}{5}(2000 \text{ lb} + 270 \text{ lb}) = 454 \text{ lb} \quad \leftarrow$$

$$T_B = \frac{2}{5}(500 \text{ lb} - 270 \text{ lb}) = 92 \text{ lb} \quad \leftarrow$$

(c) WIRE B BECOMES SLACKSet $T_B = 0$ in Eq. (12):

$$P = EA\alpha(\Delta T)$$

or

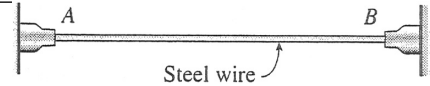
$$\Delta T = \frac{P}{EA\alpha} = \frac{500 \text{ lb}}{(120,000 \text{ lb})(12.5 \times 10^{-6}/^\circ\text{F})} = 333.3^\circ\text{F}$$

Further increase in temperature:

$$\Delta T = 333.3^\circ\text{F} - 180^\circ\text{F} = 153^\circ\text{F} \quad \leftarrow$$

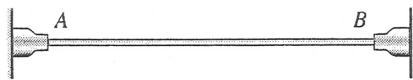
Misfits and Prestrains

Problem 2.5-12 A steel wire AB is stretched between rigid supports (see figure). The initial prestress in the wire is 42 MPa when the temperature is 20°C .



- What is the stress σ in the wire when the temperature drops to 0°C ?
- At what temperature T will the stress in the wire become zero?
(Assume $\alpha = 14 \times 10^{-6}/^\circ\text{C}$ and $E = 200 \text{ GPa}$.)

Solution 2.5-12 Steel wire with initial prestress



Initial prestress: $\sigma_1 = 42 \text{ MPa}$

Initial temperature: $T_1 = 20^\circ\text{C}$

$E = 200 \text{ GPa}$

$\alpha = 14 \times 10^{-6}/^\circ\text{C}$

- STRESS σ WHEN TEMPERATURE DROPS TO 0°C

$$T_2 = 0^\circ\text{C} \quad \Delta T = 20^\circ\text{C}$$

NOTE: Positive ΔT means a *decrease* in temperature and an *increase* in the stress in the wire.

Negative ΔT means an *increase* in temperature and a *decrease* in the stress.

Stress σ equals the initial stress σ_1 plus the additional stress σ_2 due to the temperature drop.

$$\sigma_2 = E\alpha(\Delta T)$$

$$\begin{aligned} \sigma &= \sigma_1 + \sigma_2 = \sigma_1 + E\alpha(\Delta T) \\ &= 42 \text{ MPa} + (200 \text{ GPa})(14 \times 10^{-6}/^\circ\text{C})(20^\circ\text{C}) \\ &= 42 \text{ MPa} + 56 \text{ MPa} = 98 \text{ MPa} \quad \leftarrow \end{aligned}$$

- TEMPERATURE WHEN STRESS EQUALS ZERO

$$\sigma = \sigma_1 + \sigma_2 = 0 \quad \sigma_1 + E\alpha(\Delta T) = 0$$

$$\Delta T = -\frac{\sigma_1}{E\alpha}$$

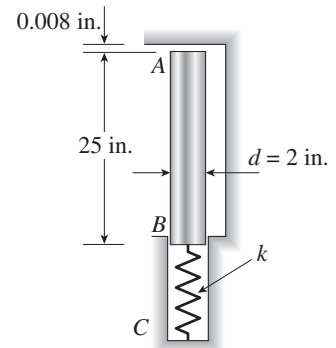
(Negative means increase in temp.)

$$\Delta T = -\frac{42 \text{ MPa}}{(200 \text{ GPa})(14 \times 10^{-6}/^\circ\text{C})} = -15^\circ\text{C}$$

$$T = 20^\circ\text{C} + 15^\circ\text{C} = 35^\circ\text{C} \quad \leftarrow$$

Problem 2.5-13 A copper bar AB of length 25 in. and diameter 2 in. is placed in position at room temperature with a gap of 0.008 in. between end A and a rigid restraint (see figure). The bar is supported at end B by an elastic spring with spring constant $k = 1.2 \times 10^6 \text{ lb/in.}$

- Calculate the axial compressive stress σ_c in the bar if the temperature rises 50°F . (For copper, use $\alpha = 9.6 \times 10^{-6}/^\circ\text{F}$ and $E = 16 \times 10^6 \text{ psi}$.)
- What is the force in the spring? (Neglect gravity effects.)
- Repeat (a) if $k \rightarrow \infty$.



Solution 2.5-13

Numerical data:

$$L = 25 \text{ in.} \quad d = 2 \text{ in.} \quad \delta = 0.008 \text{ in.}$$

$$k = 1.2 \times (10^6) \text{ lb/in.} \quad E = 16 \times (10^6) \text{ psi}$$

$$\alpha = 9.6 \times (10^{-6})/^{\circ}\text{F} \quad \Delta T = 50^{\circ}\text{F}$$

$$A = \frac{\pi}{4}d^2 \quad A = 3.14159 \text{ in.}^2$$

(a) ONE-DEGREE STATICALLY INDETERMINATE IF GAP CLOSSES

$$\Delta = \alpha \Delta T L \quad \Delta = 0.012 \text{ in.} \quad < \text{exceeds gap}$$

Select R_A as redundant and do superposition analysis:

$$\delta_{A1} = \Delta \quad \delta_{A2} = R_A \left(\frac{L}{EA} + \frac{1}{k} \right)$$

$$\text{Compatibility: } \delta_{A1} + \delta_{A2} = \delta \quad \delta_{A2} = \delta - \delta_{A1}$$

$$R_A = \frac{\delta - \Delta}{\frac{L}{EA} + \frac{1}{k}} \quad R_A = -3006 \text{ lb}$$

Compressive stress in bar:

$$\sigma = \frac{R_A}{A} \quad \sigma = -957 \text{ psi}$$

(b) FORCE IN SPRING $F_k = R_C$

STATICS

$$R_A + R_C = 0$$

$$R_C = -R_A$$

$$R_C = 3006 \text{ lb} \quad \leftarrow$$

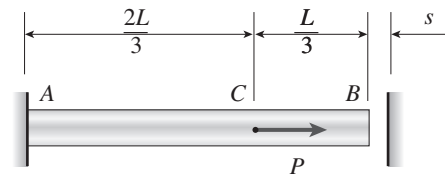
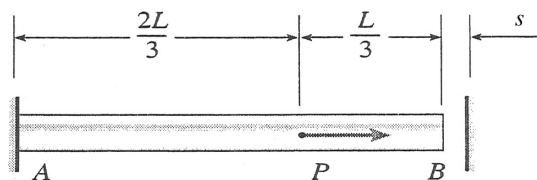
(c) FIND COMPRESSIVE STRESS IN BAR IF k GOES TO INFINITY FROM EXPRESSION FOR R_A ABOVE, $1/k$ GOES TO ZERO

$$R_A = \frac{\delta - \Delta}{\frac{L}{EA}} \quad R_A = -8042 \text{ lb} \quad \sigma = \frac{R_A}{A}$$

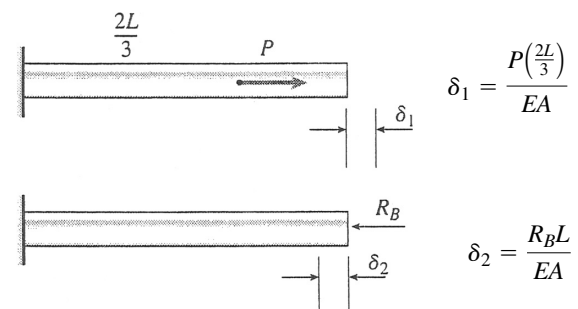
$$\sigma = -2560 \text{ psi} \quad \leftarrow$$

Problem 2.5-14 A bar AB having length L and axial rigidity EA is fixed at end A (see figure). At the other end a small gap of dimension s exists between the end of the bar and a rigid surface. A load P acts on the bar at point C , which is two-thirds of the length from the fixed end.

If the support reactions produced by the load P are to be equal in magnitude, what should be the size s of the gap?

**Solution 2.5-14 Bar with a gap (load P)** L = length of bar s = size of gap EA = axial rigidityReactions must be equal; find s .

FORCE-DISPLACEMENT RELATIONS



COMPATIBILITY EQUATION

$$\delta_1 - \delta_2 = s \quad \text{or}$$

$$\frac{2PL}{3EA} - \frac{R_B L}{EA} = s \quad (\text{Eq. 1})$$

EQUILIBRIUM EQUATION

R_A = reaction at end A (to the left)

R_B = reaction at end B (to the left)

$$P = R_A + R_B$$

Reactions must be equal.

$$\therefore R_A = R_B \quad P = 2R_B \quad R_B = \frac{P}{2}$$

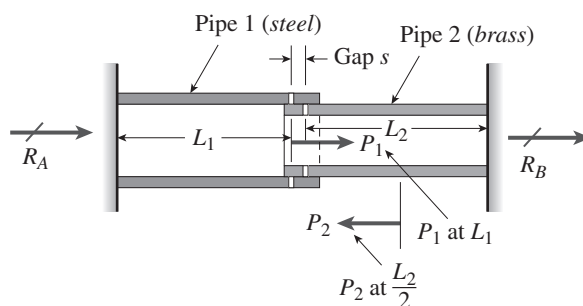
Substitute for R_B in Eq. (1):

$$\frac{2PL}{3EA} - \frac{PL}{2EA} = s \quad \text{or} \quad s = \frac{PL}{6EA} \quad \leftarrow$$

NOTE: The gap closes when the load reaches the value $P/4$. When the load reaches the value P , equal to $6EA s/L$, the reactions are equal ($R_A = R_B = P/2$). When the load is between $P/4$ and P , R_A is greater than R_B . If the load exceeds P , R_B is greater than R_A .

Problem 2.5-15 Pipe 2 has been inserted snugly into Pipe 1, but the holes for a connecting pin do not line up: there is a gap s . The user decides to apply *either* force P_1 to Pipe 1 *or* force P_2 to Pipe 2, whichever is smaller. Determine the following using the numerical properties in the box.

- If only P_1 is applied, find P_1 (kips) required to close gap s ; if a pin is then inserted and P_1 removed, what are reaction forces R_A and R_B for this load case?
- If only P_2 is applied, find P_2 (kips) required to close gap s ; if a pin is inserted and P_2 removed, what are reaction forces R_A and R_B for this load case?
- What is the maximum *shear* stress in the pipes, for the loads in parts (a) and (b)?
- If a temperature increase ΔT is to be applied to the entire structure to close gap s (*instead of applying forces* P_1 and P_2), find the ΔT required to close the gap. If a pin is inserted after the gap has closed, what are reaction forces R_A and R_B for this case?
- Finally, if the structure (with pin inserted) then cools to the *original* ambient temperature, what are reaction forces R_A and R_B ?



Numerical properties

$E_1 = 30,000$ ksi, $E_2 = 14,000$ ksi
 $\alpha_1 = 6.5 \times 10^{-6}/^\circ\text{F}$, $\alpha_2 = 11 \times 10^{-6}/^\circ\text{F}$
 Gap $s = 0.05$ in.
 $L_1 = 56$ in., $d_1 = 6$ in., $t_1 = 0.5$ in., $A_1 = 8.64$ in.²
 $L_2 = 36$ in., $d_2 = 5$ in., $t_2 = 0.25$ in., $A_2 = 3.73$ in.²

Solution 2.5-15

- (a) FIND REACTIONS AT A AND B FOR APPLIED FORCE P_1
 First compute P_1 , required to close gap:

$$P_1 = \frac{E_1 A_1}{L_1} s \quad P_1 = 231.4 \text{ k} \quad \leftarrow$$

Statically indeterminate analysis with R_B as the redundant:

$$\delta_{B1} = -s \quad \delta_{B2} = R_B \left(\frac{L_1}{E_1 A_1} + \frac{L_2}{E_2 A_2} \right)$$

$$\text{Compatibility: } \delta_{B1} + \delta_{B2} = 0$$

$$R_B = \frac{s}{\left(\frac{L_1}{E_1 A_1} + \frac{L_2}{E_2 A_2} \right)} \quad R_B = 55.2 \text{ k} \quad \leftarrow$$

$$R_A = -R_B \quad \leftarrow$$

- (b) FIND REACTIONS AT A AND B FOR APPLIED FORCE P_2

$$P_2 = \frac{E_2 A_2}{L_2} s \quad P_2 = 145.1 \text{ k} \quad \leftarrow$$

Analysis after removing P_2 is same as in part (a), so reaction forces are the same

- (c) MAXIMUM SHEAR STRESS IN PIPE 1 OR 2 WHEN EITHER P_1 OR P_2

$$\text{IS APPLIED } \tau_{\max a} = \frac{P_1}{2A_1} \quad \tau_{\max a} = 13.39 \text{ ksi} \quad \leftarrow$$

$$\tau_{\max b} = \frac{P_2}{2A_2} \quad \tau_{\max b} = 19.44 \text{ ksi} \quad \leftarrow$$

- (d) REQUIRED ΔT AND REACTIONS AT A AND B

$$\Delta T_{\text{reqd}} = \frac{s}{\alpha_1 L_1 + \alpha_2 L_2} \quad \Delta T_{\text{reqd}} = 65.8^\circ\text{F} \quad \leftarrow$$

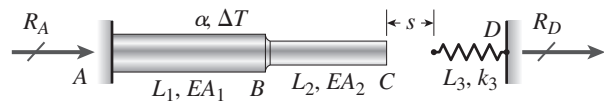
If pin is inserted but temperature remains at ΔT above ambient temperature, reactions are zero.

- (e) IF TEMPERATURE RETURNS TO ORIGINAL AMBIENT TEMPERATURE, FIND REACTIONS AT A AND B
statically indeterminate analysis with R_B as the redundant Compatibility: $\delta_{B1} + \delta_{B2} = 0$
Analysis is the same as in parts (a) and (b) above since gap s is the same, so reactions are the same.

Problem 2.5-16 A nonprismatic bar ABC made up of segments AB (length L_1 , cross-sectional area A_1) and BC (length L_2 , cross-sectional area A_2) is fixed at end A and free at end C (see figure). The modulus of elasticity of the bar is E .

A small gap of dimension s exists between the end of the bar and an elastic spring of length L_3 and spring constant k_3 . If bar ABC only (not the spring) is subjected to temperature increase ΔT determine the following.

- Write an expression for reaction forces R_A and R_D if the elongation of ABC exceeds gap length s .
- Find expressions for the displacements of points B and C if the elongation of ABC exceeds gap length s .



Solution 2.5-16

With gap s closed due to ΔT , structure is one-degree statically-indeterminate; select internal force (Q) at juncture of bar and spring as the redundant. Use superposition of two released structures in the solution.

δ_{rel1} = relative displacement between end of bar at C and end of spring due to ΔT

$$\delta_{\text{rel1}} = \alpha \Delta T (L_1 + L_2)$$

δ_{rel1} is greater than gap length s

δ_{rel2} = relative displacement between ends of bar and spring due to pair of forces Q , one on end of bar at C and the other on end of spring

$$\delta_{\text{rel2}} = Q \left(\frac{L_1}{EA_1} + \frac{L_2}{EA_2} \right) + \frac{Q}{k_3}$$

$$\delta_{\text{rel2}} = Q \left(\frac{L_1}{EA_1} + \frac{L_2}{EA_2} + \frac{1}{k_3} \right)$$

$$\text{Compatibility: } \delta_{\text{rel1}} + \delta_{\text{rel2}} = s \quad \delta_{\text{rel2}} = s - \delta_{\text{rel1}}$$

$$\delta_{\text{rel2}} = s - \alpha \Delta T (L_1 + L_2)$$

$$Q = \frac{s - \alpha \Delta T (L_1 + L_2)}{\frac{L_1}{EA_1} + \frac{L_2}{EA_2} + \frac{1}{k_3}}$$

$$Q = \frac{EA_1 A_2 k_3}{L_1 A_2 k_3 + L_2 A_1 k_3 + EA_1 A_2} [s - \alpha \Delta T (L_1 + L_2)]$$

- (a) REACTIONS AT A AND D

$$\text{Statics: } R_A = -Q \quad R_D = Q$$

$$R_A = \frac{-s + \alpha \Delta T (L_1 + L_2)}{\frac{L_1}{EA_1} + \frac{L_2}{EA_2} + \frac{1}{k_3}} \quad \leftarrow$$

$$R_D = -R_A \quad \leftarrow$$

(b) DISPLACEMENTS AT B AND C

Use superposition of displacements in the two released structures:

$$\delta_B = \alpha \Delta T (L_1) - R_A \left(\frac{L_1}{EA_1} \right) \quad \leftarrow$$

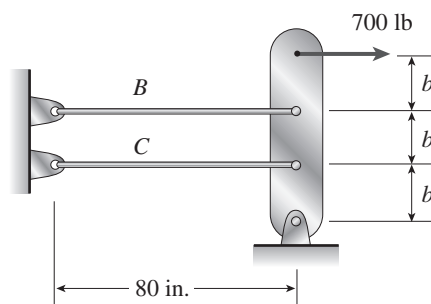
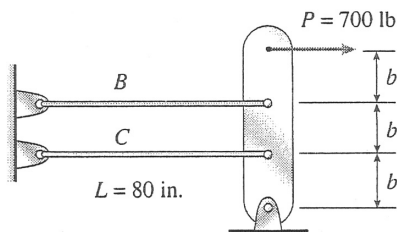
$$\delta_B = \alpha \Delta T (L_1) - \frac{[-s + \alpha \Delta T (L_1 + L_2)] \left(\frac{L_1}{EA_1} \right)}{\frac{L_1}{EA_1} + \frac{L_2}{EA_2} + \frac{1}{k_3}}$$

$$\delta_C = \alpha \Delta T (L_1 + L_2) - R_A \left(\frac{L_1}{EA_1} + \frac{L_2}{EA_2} \right) \quad \leftarrow$$

$$\delta_C = \alpha \Delta T (L_1 + L_2) - \frac{[-s + \alpha \Delta T (L_1 + L_2)] \left(\frac{L_1}{EA_1} + \frac{L_2}{EA_2} \right)}{\frac{L_1}{EA_1} + \frac{L_2}{EA_2} + \frac{1}{k_3}}$$

Problem 2.5-17 Wires B and C are attached to a support at the left-hand end and to a pin-supported rigid bar at the right-hand end (see figure). Each wire has cross-sectional area $A = 0.03 \text{ in.}^2$ and modulus of elasticity $E = 30 \times 10^6 \text{ psi}$. When the bar is in a vertical position, the length of each wire is $L = 80 \text{ in.}$ However, before being attached to the bar, the length of wire B was 79.98 in. and of wire C was 79.95 in.

Find the tensile forces T_B and T_C in the wires under the action of a force $P = 700 \text{ lb}$ acting at the upper end of the bar.

**Solution 2.5-17 Wires B and C attached to a bar**

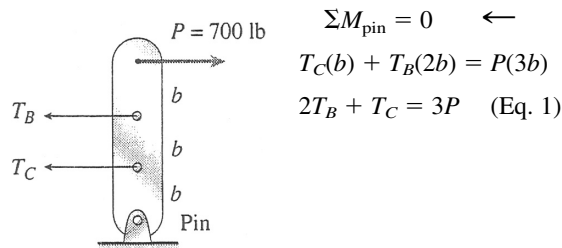
$$P = 700 \text{ lb}$$

$$A = 0.03 \text{ in.}^2$$

$$E = 30 \times 10^6 \text{ psi}$$

$$L_B = 79.98 \text{ in.}$$

$$L_C = 79.95 \text{ in.}$$

EQUILIBRIUM EQUATION

$$\Sigma M_{\text{pin}} = 0 \quad \leftarrow$$

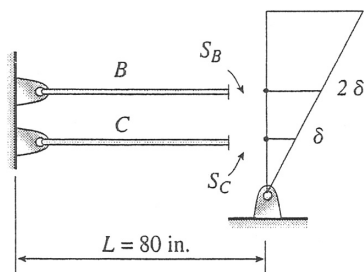
$$T_C(b) + T_B(2b) = P(3b)$$

$$2T_B + T_C = 3P \quad (\text{Eq. 1})$$

DISPLACEMENT DIAGRAM

$$S_B = 80 \text{ in.} - L_B = 0.02 \text{ in.}$$

$$S_C = 80 \text{ in.} - L_C = 0.05 \text{ in.}$$



Elongation of wires:

$$\delta_B = S_B + 2\delta \quad (\text{Eq. 2})$$

$$\delta_C = S_C + \delta \quad (\text{Eq. 3})$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_B = \frac{T_B L}{EA} \quad \delta_C = \frac{T_C L}{EA} \quad (\text{Eqs. 4, 5})$$

SOLUTION OF EQUATIONS

Combine Eqs. (2) and (4):

$$\frac{T_B L}{EA} = S_B + 2\delta \quad (\text{Eq. 6})$$

Combine Eqs. (3) and (5):

$$\frac{T_C L}{EA} = S_C + \delta \quad (\text{Eq. 7})$$

Eliminate δ between Eqs. (6) and (7):

$$T_B - 2T_C = \frac{EAS_B}{L} - \frac{2EAS_C}{L} \quad (\text{Eq. 8})$$

Solve simultaneously Eqs. (1) and (8):

$$T_B = \frac{6P}{5} + \frac{EAS_B}{5L} - \frac{2EAS_C}{5L} \quad \leftarrow$$

$$T_C = \frac{3P}{5} - \frac{2EAS_B}{5L} + \frac{4EAS_C}{5L} \quad \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

$$\frac{EA}{5L} = 2250 \text{ lb/in.}$$

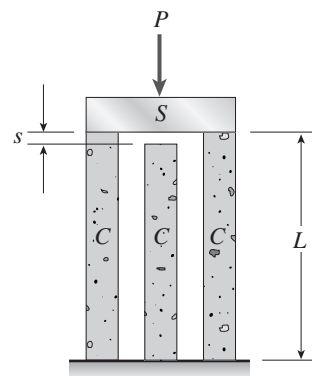
$$T_B = 840 \text{ lb} + 45 \text{ lb} - 225 \text{ lb} = 660 \text{ lb} \quad \leftarrow$$

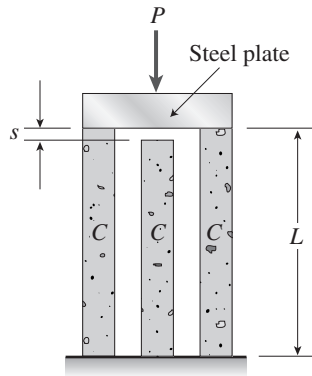
$$T_C = 420 \text{ lb} - 90 \text{ lb} + 450 \text{ lb} = 780 \text{ lb} \quad \leftarrow$$

(Both forces are positive, which means tension, as required for wires.)

Problem 2.5-18 A rigid steel plate is supported by three posts of high-strength concrete each having an effective cross-sectional area $A = 40,000 \text{ mm}^2$ and length $L = 2 \text{ m}$ (see figure). Before the load P is applied, the middle post is shorter than the others by an amount $s = 1.0 \text{ mm}$.

Determine the maximum allowable load P_{allow} if the allowable compressive stress in the concrete is $\sigma_{\text{allow}} = 20 \text{ MPa}$. (Use $E = 30 \text{ GPa}$ for concrete.)



Solution 2.5-18 Plate supported by three posts

s = size of gap = 1.0 mm

L = length of posts = 2.0 m

A = 40,000 mm²

σ_{allow} = 20 MPa

E = 30 GPa

C = concrete post

DOES THE GAP CLOSE?

Stress in the two outer posts when the gap is just closed:

$$\sigma = E\varepsilon = E\left(\frac{s}{L}\right) = (30 \text{ GPa})\left(\frac{1.0 \text{ mm}}{2.0 \text{ m}}\right)$$

$$= 15 \text{ MPa}$$

Since this stress is less than the allowable stress, the allowable force P will close the gap.

EQUILIBRIUM EQUATION

$$2P_1 + P_2 = P \quad (\text{Eq. 1})$$

COMPATIBILITY EQUATION

δ_1 = shortening of outer posts

δ_2 = shortening of inner post

$$\delta_1 = \delta_2 + s \quad (\text{Eq. 2})$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_1 = \frac{P_1 L}{EA} \quad \delta_2 = \frac{P_2 L}{EA} \quad (\text{Eqs. 3, 4})$$

SOLUTION OF EQUATIONS

Substitute (3) and (4) into Eq. (2):

$$\frac{P_1 L}{EA} = \frac{P_2 L}{EA} + s \quad \text{or} \quad P_1 - P_2 = \frac{EAs}{L} \quad (\text{Eq. 5})$$

Solve simultaneously Eqs. (1) and (5):

$$P = 3P_1 - \frac{EAs}{L}$$

By inspection, we know that P_1 is larger than P_2 .

Therefore, P_1 will control and will be equal to $\sigma_{\text{allow}} A$.

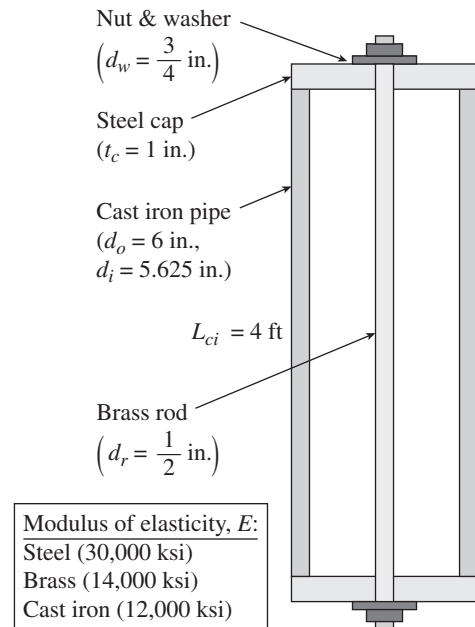
$$P_{\text{allow}} = 3\sigma_{\text{allow}} A - \frac{EAs}{L}$$

$$= 2400 \text{ kN} - 600 \text{ kN} = 1800 \text{ kN}$$

$$= 1.8 \text{ MN} \quad \leftarrow$$

Problem 2.5-19 A capped cast-iron pipe is compressed by a brass rod, as shown. The nut is turned until it is just snug, then add an additional quarter turn to pre-compress the CI pipe. The pitch of the threads of the bolt is $p = 52$ mils (a mil is one-thousandth of an inch). Use the numerical properties provided.

- What stresses σ_p and σ_r will be produced in the cast-iron pipe and brass rod, respectively, by the additional quarter turn of the nut?
- Find the bearing stress σ_b beneath the washer and the shear stress τ_c in the steel cap.



Solution 2.5-19

The figure shows a section through the pipe, cap and rod

NUMERICAL PROPERTIES

$$L_{ci} = 48 \text{ in.} \quad E_s = 30000 \text{ ksi} \quad E_b = 14,000 \text{ ksi}$$

$$E_c = 12,000 \text{ ksi} \quad t_c = 1 \text{ in.} \quad p = 52 \times (10^{-3}) \text{ in.} \quad n = \frac{1}{4}$$

$$d_w = \frac{3}{4} \text{ in.} \quad d_r = \frac{1}{2} \text{ in.} \quad d_o = 6 \text{ in.} \quad d_i = 5.625 \text{ in.}$$

(a) FORCES AND STRESSES IN PIPE AND ROD

One degree statically indeterminate—cut rod at cap and use force in rod (Q) as the redundant:

δ_{rel1} = relative displacement between cut ends of rod due to 1/4 turn of nut

$$\delta_{rel1} = -np \quad \text{Ends of rod move apart, not together, so this is (-).}$$

δ_{rel2} = relative displacement between cut ends of rod due pair of forces Q

$$\delta_{rel2} = Q \left(\frac{L + 2t_c}{E_b A_{rod}} + \frac{L_{ci}}{E_c A_{pipe}} \right)$$

$$A_{rod} = \frac{\pi}{4} d_r^2 \quad A_{pipe} = \frac{\pi}{4} (d_o^2 - d_i^2)$$

$$A_{rod} = 0.196 \text{ in.}^2 \quad A_{pipe} = 3.424 \text{ in.}^2$$

$$\text{Compatibility equation: } \delta_{rel1} + \delta_{rel2} = 0$$

$$Q = \frac{np}{\frac{L_{ci} + 2t_c}{E_b A_{rod}} + \frac{L_{ci}}{E_c A_{pipe}}}$$

$$Q = 0.672 \text{ k} \quad F_{rod} = Q$$

$$\text{Statics: } F_{pipe} = -Q$$

$$\text{Stresses: } \sigma_c = \frac{F_{pipe}}{A_{pipe}} \quad \sigma_c = -0.196 \text{ ksi} \quad \leftarrow$$

$$\sigma_b = \frac{F_{rod}}{A_{rod}} \quad \sigma_b = 3.42 \text{ ksi} \quad \leftarrow$$

(b) BEARING AND SHEAR STRESSES IN STEEL CAP

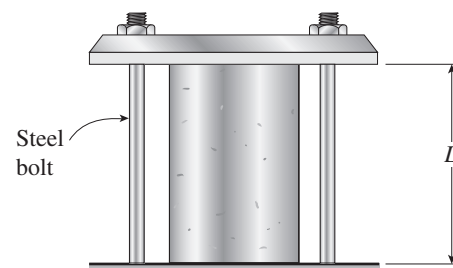
$$\sigma_b = \frac{F_{rod}}{\frac{\pi}{4} (d_w^2 - d_r^2)} \quad \sigma_b = 2.74 \text{ ksi} \quad \leftarrow$$

$$\tau_c = \frac{F_{rod}}{\pi d_w t_c} \quad \tau_c = 0.285 \text{ ksi} \quad \leftarrow$$

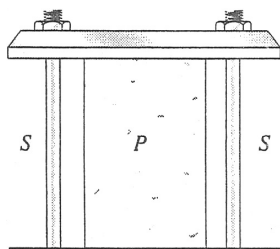
Problem 2.5-20 A plastic cylinder is held snugly between a rigid plate and a foundation by two steel bolts (see figure).

Determine the compressive stress σ_p in the plastic when the nuts on the steel bolts are tightened by one complete turn.

Data for the assembly are as follows: length $L = 200$ mm, pitch of the bolt threads $p = 1.0$ mm, modulus of elasticity for steel $E_s = 200$ GPa, modulus of elasticity for the plastic $E_p = 7.5$ GPa, cross-sectional area of one bolt $A_s = 36.0$ mm², and cross-sectional area of the plastic cylinder $A_p = 960$ mm².



Solution 2.5-20 Plastic cylinder and two steel bolts



$$L = 200 \text{ mm}$$

$$p = 1.0 \text{ mm}$$

$$E_s = 200 \text{ GPa}$$

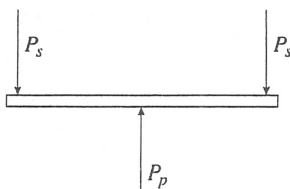
$$A_s = 36.0 \text{ mm}^2 \text{ (for one bolt)}$$

$$E_p = 7.5 \text{ GPa}$$

$$A_p = 960 \text{ mm}^2$$

$$n = 1 \text{ (See Eq. 2-24)}$$

EQUILIBRIUM EQUATION



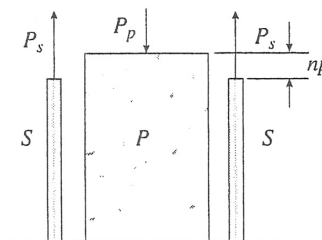
P_s = tensile force in one steel bolt

P_p = compressive force in plastic cylinder

$$P_p = 2P_s$$

(Eq. 1)

COMPATIBILITY EQUATION



δ_s = elongation of steel bolt

δ_p = shortening of plastic cylinder

$$\delta_s + \delta_p = np$$

(Eq. 2)

FORCE-DISPLACEMENT RELATIONS

$$\delta_s = \frac{P_s L}{E_s A_s} \quad \delta_p = \frac{P_p L}{E_p A_p} \quad (\text{Eq. 3, Eq. 4})$$

SOLUTION OF EQUATIONS

Substitute (3) and (4) into Eq. (2):

$$\frac{P_s L}{E_s A_s} + \frac{P_p L}{E_p A_p} = np \quad (\text{Eq. 5})$$

Solve simultaneously Eqs. (1) and (5):

$$P_p = \frac{2npE_s A_s E_p A_p}{L(E_p A_p + 2E_s A_s)}$$

STRESS IN THE PLASTIC CYLINDER

$$\sigma_p = \frac{P_p}{A_p} = \frac{2np E_s A_s E_p}{L(E_p A_p + 2E_s A_s)}$$

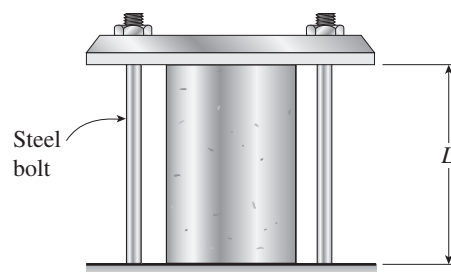
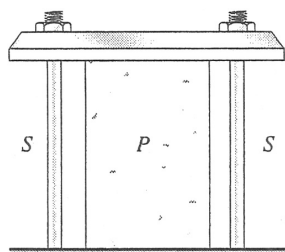
SUBSTITUTE NUMERICAL VALUES:

$$N = E_s A_s E_p = 54.0 \times 10^{15} \text{ N}^2/\text{m}^2$$

$$D = E_p A_p + 2E_s A_s = 21.6 \times 10^6 \text{ N}$$

$$\sigma_p = \frac{2np \left(\frac{N}{D} \right)}{L \left(\frac{N}{D} \right)} = \frac{2(1)(1.0 \text{ mm}) \left(\frac{N}{D} \right)}{200 \text{ mm} \left(\frac{N}{D} \right)} = 25.0 \text{ MPa} \quad \leftarrow$$

Problem 2.5-21 Solve the preceding problem if the data for the assembly are as follows: length $L = 10$ in., pitch of the bolt threads $p = 0.058$ in., modulus of elasticity for steel $E_s = 30 \times 10^6$ psi, modulus of elasticity for the plastic $E_p = 500$ ksi, cross-sectional area of one bolt $A_s = 0.06$ in.², and cross-sectional area of the plastic cylinder $A_p = 1.5$ in.²

**Solution 2.5-21 Plastic cylinder and two steel bolts**

$$\begin{aligned} L &= 10 \text{ in.} \\ p &= 0.058 \text{ in.} \\ E_s &= 30 \times 10^6 \text{ psi} \end{aligned}$$

$$A_s = 0.06 \text{ in.}^2 \text{ (for one bolt)}$$

$$E_p = 500 \text{ ksi}$$

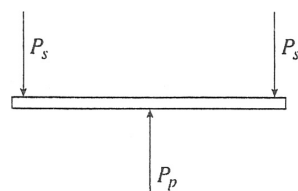
$$A_p = 1.5 \text{ in.}^2$$

$$n = 1 \text{ (see Eq. 2-24)}$$

EQUILIBRIUM EQUATION

 P_s = tensile force in one steel bolt P_p = compressive force in plastic cylinder

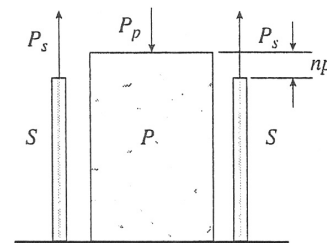
$$P_p = 2P_s \quad (\text{Eq. 1})$$



COMPATIBILITY EQUATION

 δ_s = elongation of steel bolt δ_p = shortening of plastic cylinder

$$\delta_s + \delta_p = np \quad (\text{Eq. 2})$$



FORCE-DISPLACEMENT RELATIONS

$$\delta_s = \frac{P_s L}{E_s A_s} \quad \delta_p = \frac{P_p L}{E_p A_p} \quad (\text{Eq. 3, Eq. 4})$$

SOLUTION OF EQUATIONS

Substitute (3) and (4) into Eq. (2):

$$\frac{P_s L}{E_s A_s} + \frac{P_p L}{E_p A_p} = np \quad (\text{Eq. 5})$$

Solve simultaneously Eqs. (1) and (5):

$$P_p = \frac{2np E_s A_s E_p A_p}{L(E_p A_p + 2E_s A_s)}$$

STRESS IN THE PLASTIC CYLINDER

$$\sigma_p = \frac{P_p}{A_p} = \frac{2np E_s A_s E_p}{L(E_p A_p + 2E_s A_s)} \quad \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

$$N = E_s A_s E_p = 900 \times 10^9 \text{ lb}^2/\text{in.}^2$$

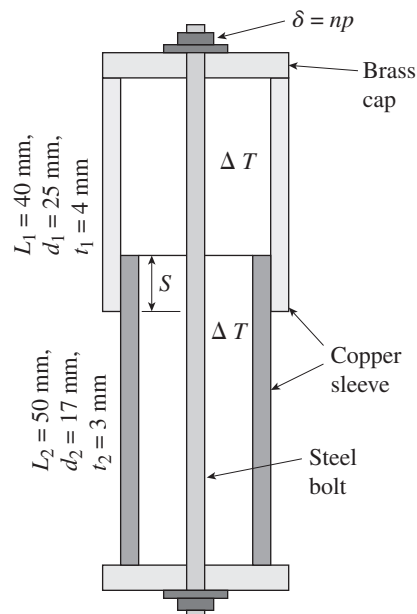
$$D = E_p A_p + 2E_s A_s = 4350 \times 10^3 \text{ lb}$$

$$\sigma_p = \frac{2np \left(\frac{N}{D} \right)}{L \left(\frac{N}{D} \right)} = \frac{2(1)(0.058 \text{ in.}) \left(\frac{N}{D} \right)}{10 \text{ in.}} \left(\frac{N}{D} \right)$$

$$= 2400 \text{ psi} \quad \leftarrow$$

Problem 2.5-22 Consider the sleeve made from two copper tubes joined by tin-lead solder over distance s . The sleeve has brass caps at both ends, which are held in place by a steel bolt and washer with the nut turned just snug at the outset. Then, two “loadings” are applied: $n = 1/2$ turn applied to the nut; at the same time the internal temperature is raised by $\Delta T = 30^\circ\text{C}$.

- Find the forces in the sleeve and bolt, P_s and P_B , due to both the prestress in the bolt and the temperature increase. For copper, use $E_c = 120 \text{ GPa}$ and $\alpha_c = 17 \times 10^{-6}/^\circ\text{C}$; for steel, use $E_s = 200 \text{ GPa}$ and $\alpha_s = 12 \times 10^{-6}/^\circ\text{C}$. The pitch of the bolt threads is $p = 1.0 \text{ mm}$. Assume $s = 26 \text{ mm}$ and bolt diameter $d_b = 5 \text{ mm}$.
- Find the required length of the solder joint, s , if shear stress in the sweated joint cannot exceed the allowable shear stress $\tau_{aj} = 18.5 \text{ MPa}$.
- What is the final elongation of the entire assemblage due to both temperature change ΔT and the initial prestress in the bolt?



Solution 2.5-22

The figure shows a section through the sleeve, cap, and bolt.

NUMERICAL PROPERTIES

$$n = \frac{1}{2} \quad p = 1.0 \text{ mm} \quad \Delta T = 30^\circ\text{C}$$

$$E_c = 120 \text{ GPa} \quad \alpha_c = 17 \times (10^{-6})/^\circ\text{C}$$

$$E_s = 200 \text{ GPa} \quad \alpha_s = 12 \times (10^{-6})/^\circ\text{C}$$

$$\tau_{aj} = 18.5 \text{ MPa} \quad s = 26 \text{ mm} \quad d_b = 5 \text{ mm}$$

$$L_1 = 40 \text{ mm} \quad t_1 = 4 \text{ mm} \quad L_2 = 50 \text{ mm} \quad t_2 = 3 \text{ mm}$$

$$d_1 = 25 \text{ mm} \quad d_1 - 2t_1 = 17 \text{ mm} \quad d_2 = 17 \text{ mm}$$

$$A_b = \frac{\pi}{4} d_b^2 \quad A_1 = \frac{\pi}{4} [d_1^2 - (d_1 - 2t_1)^2]$$

$$A_b = 19.635 \text{ mm}^2 \quad A_1 = 263.894 \text{ mm}^2$$

$$A_2 = \frac{\pi}{4} [d_2^2 - (d_2 - 2t_2)^2] \quad A_2 = 131.947 \text{ mm}^2$$

(a) FORCES IN SLEEVE AND BOLT

One-degree statically indeterminate—cut bolt and use force in bolt (P_B) as redundant (see sketches):

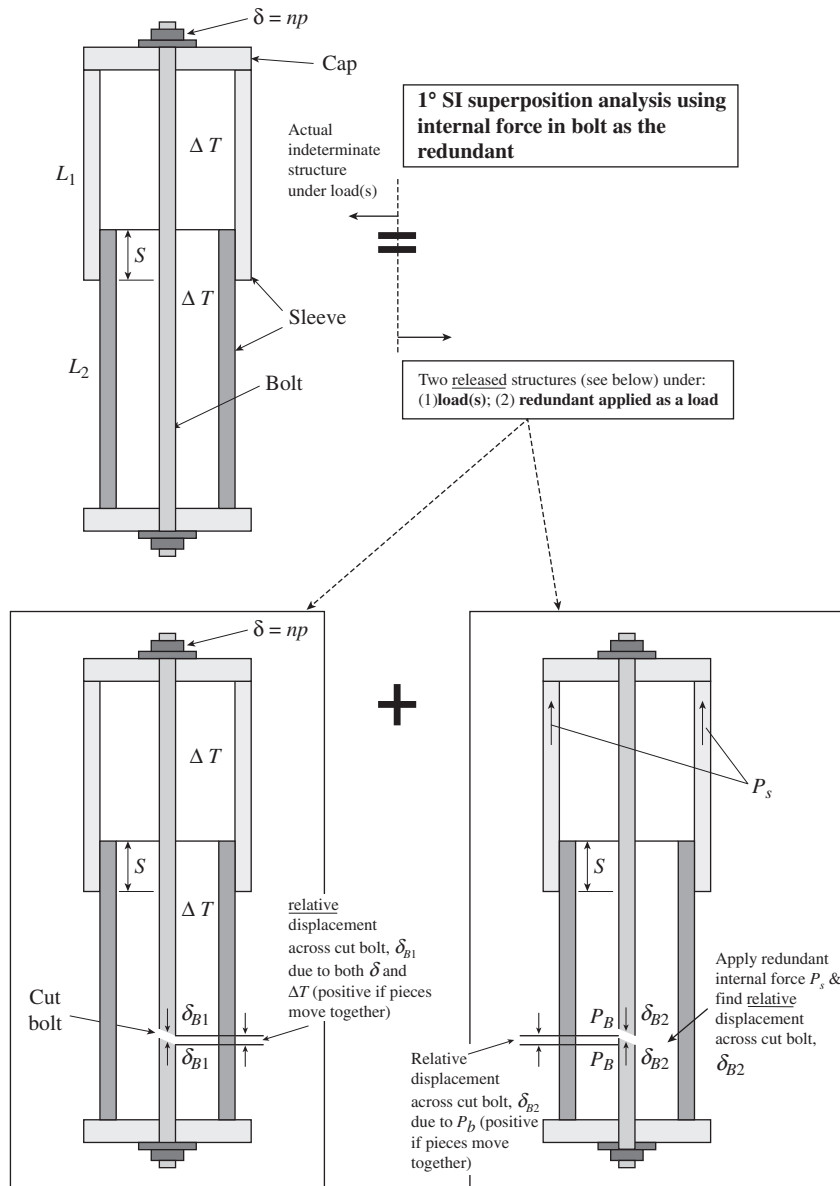
$$\delta_{B1} = -np + \alpha_s \Delta T (L_1 + L_2 - s)$$

$$\delta_{B2} = P_B \left[\frac{L_1 + L_2 - s}{E_s A_b} + \frac{L_1 - s}{E_c A_1} + \frac{L_2 - s}{E_c A_2} + \frac{s}{E_c (A_1 + A_2)} \right]$$

Compatibility: $\delta_{B1} + \delta_{B2} = 0$

$$P_B = \frac{-[-np + \alpha_s \Delta T (L_1 + L_2 - s)]}{\left[\frac{L_1 + L_2 - s}{E_s A_b} + \frac{L_1 - s}{E_c A_1} + \frac{L_2 - s}{E_c A_2} + \frac{s}{E_c (A_1 + A_2)} \right]} \quad P_B = 25.4 \text{ kN} \quad \leftarrow \quad P_s = -P_B \quad \leftarrow$$

Sketches illustrating superposition procedure for statically-indeterminate analysis



(b) REQUIRED LENGTH OF SOLDER JOINT \approx

$$\tau = \frac{P}{A_s} \quad A_s = \pi d_2 s$$

$$s_{\text{reqd}} = \frac{P_B}{\pi d_2 \tau_{aj}} \quad s_{\text{reqd}} = 25.7 \text{ mm}$$

(c) FINAL ELONGATION

δ_f = net of elongation of bolt (δ_b) and shortening of sleeve (δ_s)

$$\delta_b = P_B \left(\frac{L_1 + L_2 - s}{E_s A_b} \right) \quad \delta_b = 0.413 \text{ mm}$$

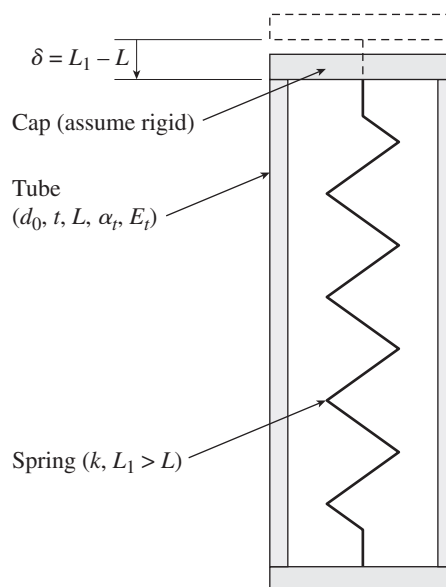
$$\delta_s = P_s \left[\frac{L_1 - s}{E_c A_1} + \frac{L_2 - s}{E_c A_2} + \frac{s}{E_c (A_1 + A_2)} \right]$$

$$\delta_s = -0.064 \text{ mm}$$

$$\delta_f = \delta_b + \delta_s \quad \delta_f = 0.35 \text{ mm} \quad \leftarrow$$

Problem 2.5-23 A polyethylene tube (length L) has a cap which when installed compresses a spring (with undeformed length $L_1 > L$) by amount $\delta = (L_1 - L)$. Ignore deformations of the cap and base. Use the force at the base of the spring as the redundant. Use numerical properties in the boxes given.

- What is the resulting force in the spring, F_k ?
- What is the resulting force in the tube, F_t ?
- What is the final length of the tube, L_f ?
- What temperature change ΔT inside the tube will result in zero force in the spring?



Modulus of elasticity
Polyethylene tube ($E_t = 100 \text{ ksi}$)
Coefficients of thermal expansion
$\alpha_t = 80 \times 10^{-6}/^\circ\text{F}$, $\alpha_k = 6.5 \times 10^{-6}/^\circ\text{F}$

Properties and dimensions
$d_0 = 6 \text{ in.}$ $t = \frac{1}{8} \text{ in.}$
$L_1 = 12.125 \text{ in.} > L = 12 \text{ in.}$ $k = 1.5 \text{ k/in.}$

Solution 2.5-23

The figure shows a section through the tube, cap, and spring.

Properties and dimensions:

$$d_o = 6 \text{ in.} \quad t = \frac{1}{8} \text{ in.} \quad E_t = 100 \text{ ksi}$$

$$A_t = \frac{\pi}{4} [d_o^2 - (d_o - 2t)^2] \quad A_t = 2.307 \text{ in.}^2$$

$$L_1 = 12.125 \text{ in.} > L = 12 \text{ in.} \quad k = 1.5 \text{ k/in.}$$

Spring is 1/8 in. longer than tube

$$\delta = L_1 - L \quad \delta = 0.125 \text{ in.}$$

$$\alpha_k = 6.5(10^{-6})/^{\circ}\text{F} < \alpha_t = 80 \times (10^{-6})/^{\circ}\text{F}$$

$$\Delta T = 0 \quad < \text{note that } Q \text{ result below is for zero temperature (until part(d))}$$

(a) FORCE IN SPRING $F_k = \text{REDUNDANT } Q$

$$\text{Flexibilities:} \quad f = \frac{1}{k} \quad f_t = \frac{L}{E_t A_t}$$

$\delta_2 = \text{relative displacement across cut spring due to redundant} = Q(f + f_t)$

$\delta_1 = \text{relative displacement across cut spring due to precompression and } \Delta T = \delta + \alpha_k \Delta T L_1 - \alpha_t \Delta T L$

$$\text{Compatibility: } \delta_1 + \delta_2 = 0$$

Solve for redundant Q :

$$Q = \frac{-\delta + \Delta T(-\alpha_k L_1 + \alpha_t L)}{f + f_t} = F_k$$

$$F_k = -0.174 \text{ k} \quad \leftarrow \text{compressive force in spring } (F_k) \text{ and also tensile force in tube}$$

$$(b) F_t = \text{force in tube} = -Q \quad \leftarrow$$

NOTE: If tube is rigid, $F_k = -k\delta = -0.1875 \text{ k}$

(c) FINAL LENGTH OF TUBE

$$L_f = L + \delta_{c1} + \delta_{c2} \quad < \text{i.e., add displacements for the two released structures to initial tube length } L$$

$$L_f = L - Qf_t + \alpha_t(\Delta T)L \quad L_f = 12.01 \text{ in.} \quad \leftarrow$$

(d) SET $Q = 0$ TO FIND ΔT REQUIRED TO REDUCE SPRING FORCE TO ZERO

$$\Delta T_{\text{reqd}} = \frac{\delta}{(-\alpha_k L_1 + \alpha_t L)}$$

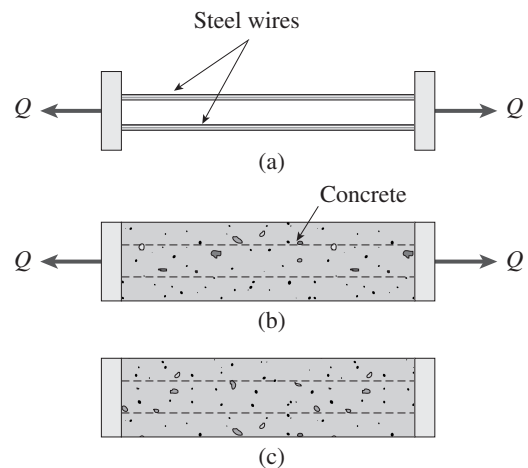
$$\Delta T_{\text{reqd}} = 141.9^{\circ}\text{F}$$

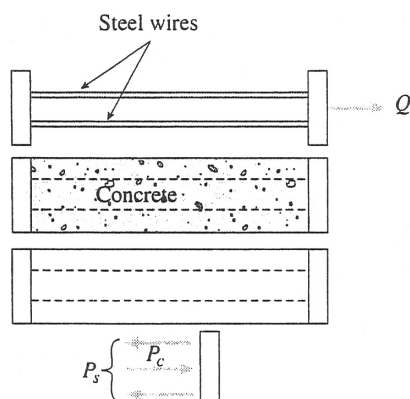
Since $\alpha_t > \alpha_k$, a temp. increase is req'd to expand tube so that spring force goes to zero.

Problem 2.5-24 Prestressed concrete beams are sometimes manufactured in the following manner. High-strength steel wires are stretched by a jacking mechanism that applies a force Q , as represented schematically in part (a) of the figure. Concrete is then poured around the wires to form a beam, as shown in part (b).

After the concrete sets properly, the jacks are released and the force Q is removed [see part (c) of the figure]. Thus, the beam is left in a prestressed condition, with the wires in tension and the concrete in compression.

Let us assume that the prestressing force Q produces in the steel wires an initial stress $\sigma_0 = 620 \text{ MPa}$. If the moduli of elasticity of the steel and concrete are in the ratio 12:1 and the cross-sectional areas are in the ratio 1:50, what are the final stresses σ_s and σ_c in the two materials?



Solution 2.5-24 Prestressed concrete beam

EQUILIBRIUM EQUATION

$$P_s = P_c$$

COMPATIBILITY EQUATION AND
FORCE-DISPLACEMENT RELATIONS δ_1 = initial elongation of steel wires

$$= \frac{QL}{E_s A_s} = \frac{\sigma_0 L}{E_s}$$

 δ_2 = final elongation of steel wires

$$= \frac{P_s L}{E_s A_s}$$

 δ_3 = shortening of concrete

$$= \frac{P_c L}{E_c A_c}$$

 $\delta_1 - \delta_2 = \delta_3$ or

$$\frac{\sigma_0 L}{E_s} - \frac{P_s L}{E_s A_s} = \frac{P_c L}{E_c A_c} \quad (\text{Eq. 2, Eq. 3})$$

Solve simultaneously Eqs. (1) and (3):

$$P_s = P_c = \frac{\sigma_0 A_s}{1 + \frac{E_s A_s}{E_c A_c}}$$

 L = length σ_0 = initial stress in wires

$$= \frac{Q}{A_s} = 620 \text{ MPa}$$

 A_s = total area of steel wires A_c = area of concrete

$$= 50 A_s$$

 $E_s = 12 E_c$ P_s = final tensile force in steel wires P_c = final compressive force in concrete

STRESSES

$$\sigma_s = \frac{P_s}{A_s} = \frac{\sigma_0}{1 + \frac{E_s A_s}{E_c A_c}} \quad \leftarrow$$

$$\sigma_c = \frac{P_c}{A_c} = \frac{\sigma_0}{\frac{A_c}{A_s} + \frac{E_s}{E_c}} \quad \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

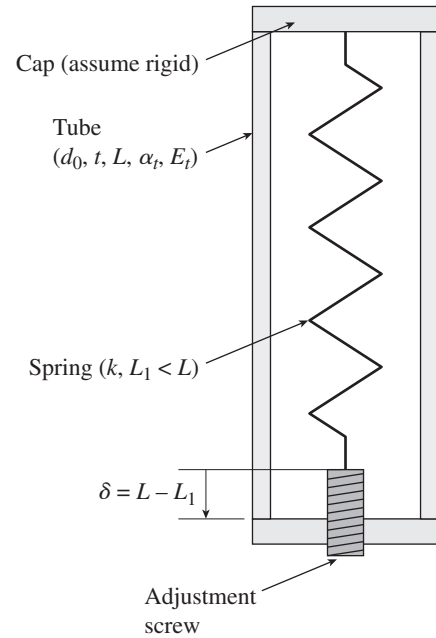
$$\sigma_0 = 620 \text{ MPa} \quad \frac{E_s}{E_c} = 12 \quad \frac{A_s}{A_c} = \frac{1}{50}$$

$$\sigma_s = \frac{620 \text{ MPa}}{1 + \frac{12}{50}} = 500 \text{ MPa (Tension)} \quad \leftarrow$$

$$\sigma_c = \frac{620 \text{ MPa}}{50 + 12} = 10 \text{ MPa (Compression)} \quad \leftarrow$$

Problem 2.5-25 A polyethylene tube (length L) has a cap which is held in place by a spring (with undeformed length $L_1 < L$). After installing the cap, the spring is post-tensioned by turning an adjustment screw by amount δ . Ignore deformations of the cap and base. Use the force at the base of the spring as the redundant. Use numerical properties in the boxes below.

- What is the resulting force in the spring, F_k ?
- What is the resulting force in the tube, F_t ?
- What is the final length of the tube, L_f ?
- What temperature change ΔT inside the tube will result in zero force in the spring?



Modulus of elasticity Polyethylene tube ($E_t = 100$ ksi)
Coefficients of thermal expansion $\alpha_t = 80 \times 10^{-6}/^\circ\text{F}$, $\alpha_k = 6.5 \times 10^{-6}/^\circ\text{F}$

Properties and dimensions
$d_o = 6$ in. $t = \frac{1}{8}$ in.
$L = 12$ in. $L_1 = 11.875$ in. $k = 1.5$ k/in.

Solution 2.5-25

The figure shows a section through the tube, cap, and spring.

Properties and dimensions:

$$d_o = 6 \text{ in.} \quad t = \frac{1}{8} \text{ in.} \quad E_t = 100 \text{ ksi}$$

$$L = 12 \text{ in.} > L_1 = 11.875 \text{ in.} \quad k = 1.5 \text{ k/in.}$$

$$\alpha_k = 6.5(10^{-6}) < \alpha_t = 80 \times (10^{-6})$$

$$A_t = \frac{\pi}{4} [d_o^2 - (d_o - 2t)^2]$$

$$A_t = 2.307 \text{ in.}^2$$

Pretension and temperature:

Spring is $1/8$ in. shorter than tube.

$$\delta = L - L_1 \quad \delta = 0.125 \text{ in.} \quad \Delta T = 0$$

Note that Q result below is for zero temperature (until part (d)).

$$\text{Flexibilities:} \quad f = \frac{1}{k} \quad f_t = \frac{L}{E_t A_t}$$

(a) FORCE IN SPRING (F_k) = REDUNDANT (Q)

Follow solution procedure outlined in Prob. 2.5-23 solution:

$$Q = \frac{\delta + \Delta T(-\alpha_k L_1 + \alpha_t L)}{f + f_t} = F_k$$

$F_k = 0.174 \text{ k}$ ← also the compressive force in the tube

(b) FORCE IN TUBE $F_t = -Q = -0.174 \text{ k}$ ←

(c) FINAL LENGTH OF TUBE AND SPRING $L_f = L + \delta_{c1} + \delta_{c2}$

$$L_f = L - Qf_t + \alpha_t(\Delta T)L \quad L_f = 11.99 \text{ in.} \quad \leftarrow$$

(d) SET $Q = 0$ TO FIND ΔT REQUIRED TO REDUCE SPRING FORCE TO ZERO

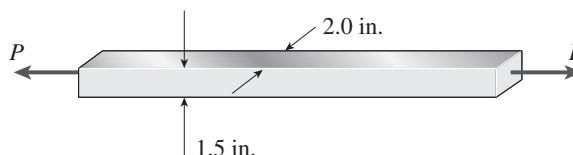
$$\Delta T_{\text{reqd}} = \frac{-\delta}{(-\alpha_k L_1 + \alpha_t L)}$$

$$\Delta T_{\text{reqd}} = -141.6^\circ\text{F}$$

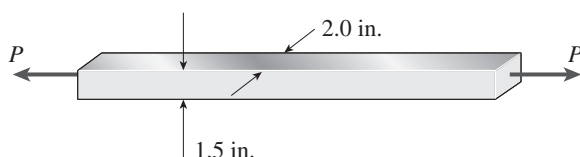
Since $\alpha_t > \alpha_k$, a temperature drop is required to shrink tube so that spring force goes to zero.

Stresses on Inclined Sections

Problem 2.6-1 A steel bar of rectangular cross section (1.5 in. \times 2.0 in.) carries a tensile load P (see figure). The allowable stresses in tension and shear are 14,500 psi and 7,100 psi, respectively. Determine the maximum permissible load P_{\max} .



Solution 2.6-1



NUMERICAL DATA

$$A = 3 \text{ in.}^2 \quad \sigma_a = 14500 \text{ psi}$$

$$\tau_a = 7100 \text{ psi}$$

MAXIMUM LOAD—TENSION

$$P_{\max 1} = \sigma_a A \quad P_{\max 1} = 43500 \text{ lbs}$$

MAXIMUM LOAD—SHEAR

$$P_{\max 2} = 2\tau_a A \quad P_{\max 2} = 42,600 \text{ lbs}$$

Because τ_{allow} is less than one-half of σ_{allow} , the shear stress governs.

Problem 2.6-2 A circular steel rod of diameter d is subjected to a tensile force $P = 3.5 \text{ kN}$ (see figure). The allowable stresses in tension and shear are 118 MPa and 48 MPa, respectively. What is the minimum permissible diameter d_{\min} of the rod?



Solution 2.6-2



NUMERICAL DATA $P = 3.5 \text{ kN}$ $\sigma_a = 118 \text{ MPa}$
 $\tau_a = 48 \text{ MPa}$

Find P_{\max} then rod diameter.

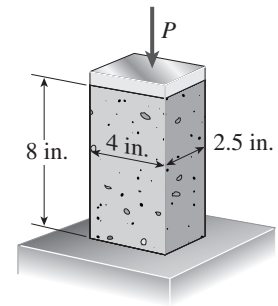
since τ_a is less than $1/2$ of σ_a , shear governs.

$$P_{\max} = 2\tau_a \left(\frac{\pi}{4} d_{\min}^2 \right)$$

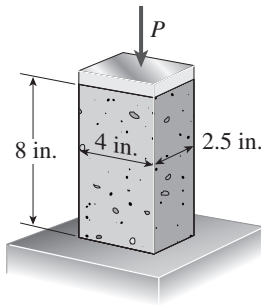
$$d_{\min} = \sqrt{\frac{2}{\pi \tau_a} P}$$

$$d_{\min} = 6.81 \text{ mm} \quad \leftarrow$$

Problem 2.6-3 A standard brick (dimensions 8 in. \times 4 in. \times 2.5 in.) is compressed lengthwise by a force P , as shown in the figure. If the ultimate shear stress for brick is 1200 psi and the ultimate compressive stress is 3600 psi, what force P_{\max} is required to break the brick?



Solution 2.6-3 Standard brick in compression



$$A = 2.5 \text{ in.} \times 4.0 \text{ in.} = 10.0 \text{ in.}^2$$

Maximum normal stress:

$$\sigma_x = \frac{P}{A}$$

Maximum shear stress:

$$\tau_{\max} = \frac{\sigma_x}{2} = \frac{P}{2A}$$

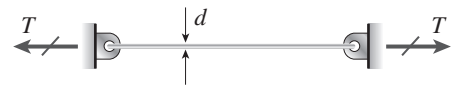
$$\sigma_{\text{ult}} = 3600 \text{ psi} \quad \tau_{\text{ult}} = 1200 \text{ psi}$$

Because τ_{ult} is less than one-half of σ_{ult} , the shear stress governs.

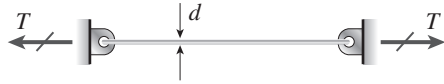
$$\tau_{\max} = \frac{P}{2A} \quad \text{or} \quad P_{\max} = 2A\tau_{\text{ult}}$$

$$P_{\max} = 2(10.0 \text{ in.}^2)(1200 \text{ psi}) = 24,000 \text{ lb} \quad \leftarrow$$

Problem 2.6-4 A brass wire of diameter $d = 2.42 \text{ mm}$ is stretched tightly between rigid supports so that the tensile force is $T = 98 \text{ N}$ (see figure). The coefficient of thermal expansion for the wire is $19.5 \times 10^{-6}/^\circ\text{C}$ and the modulus of elasticity is $E = 110 \text{ GPa}$.



- What is the maximum permissible temperature drop ΔT if the allowable shear stress in the wire is 60 MPa?
- At what temperature changes does the wire go slack?

Solution 2.6-4 Brass wire in tension

NUMERICAL DATA

$$d = 2.42 \text{ mm} \quad T = 98 \text{ N} \\ \alpha = 19.5 (10^{-6})/^{\circ}\text{C} \quad E = 110 \text{ GPa}$$

(a) ΔT_{\max} (DROP IN TEMPERATURE)

$$\sigma = \frac{T}{A} - (E\alpha \Delta T) \quad \tau_{\max} = \frac{\sigma}{2}$$

$$\tau_a = \frac{T}{2A} - \frac{E\alpha \Delta T}{2}$$

$$\tau_a = 60 \text{ MPa} \quad A = \frac{\pi}{4} d^2$$

$$\Delta T_{\max} = \frac{\frac{T}{A} - 2\tau_a}{E\alpha}$$

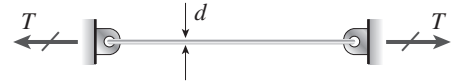
$$\Delta T_{\max} = -46^{\circ}\text{C (drop)}$$

(b) ΔT AT WHICH WIRE GOES SLACKIncrease ΔT until $\sigma = 0$:

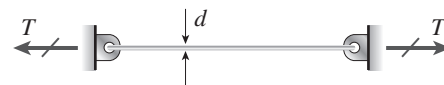
$$\Delta T = \frac{T}{E\alpha A}$$

$$\Delta T = 9.93^{\circ}\text{C (increase)}$$

Problem 2.6-5 A brass wire of diameter $d = 1/16$ in. is stretched between rigid supports with an initial tension T of 37 lb (see figure). Assume that the coefficient of thermal expansion is $10.6 \times 10^{-6}/^{\circ}\text{F}$ and the modulus of elasticity is 15×10^6 psi.)



- If the temperature is lowered by 60°F , what is the maximum shear stress τ_{\max} in the wire?
- If the allowable shear stress is 10,000 psi, what is the maximum permissible temperature drop?
- At what temperature change ΔT does the wire go slack?

Solution 2.6-5

NUMERICAL DATA

$$d = \frac{1}{16} \text{ in.} \quad T = 37 \text{ lb} \quad \alpha = 10.6 \times (10^{-6})/^{\circ}\text{F}$$

$$E = 15 \times (10^6) \text{ psi} \quad \Delta T = -60^{\circ}\text{F}$$

$$A = \frac{\pi}{4} d^2$$

(a) τ_{\max} (DUE TO DROP IN TEMPERATURE)

$$\tau_{\max} = \frac{\sigma_x}{2} \quad \tau_{\max} = \frac{\frac{T}{A} - (E\alpha \Delta T)}{2}$$

$$\tau_{\max} = 10,800 \text{ psi} \quad \leftarrow$$

(b) ΔT_{\max} FOR ALLOWABLE SHEAR STRESS

$$\tau_a = 10000 \text{ psi}$$

$$\Delta T_{\max} = \frac{\frac{T}{A} - 2\tau_a}{E\alpha}$$

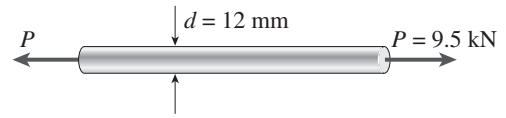
$$\Delta T_{\max} = -49.9^{\circ}\text{F} \quad \leftarrow$$

(c) ΔT AT WHICH WIRE GOES SLACKIncrease ΔT until $\sigma = 0$:

$$\Delta T = \frac{T}{E\alpha A}$$

$$\Delta T = 75.9^{\circ}\text{F (increase)} \quad \leftarrow$$

Problem 2.6-6 A steel bar with diameter $d = 12$ mm is subjected to a tensile load $P = 9.5$ kN (see figure).



- What is the maximum normal stress σ_{\max} in the bar?
- What is the maximum shear stress τ_{\max} ?
- Draw a stress element oriented at 45° to the axis of the bar and show all stresses acting on the faces of this element.
- Repeat part (c) for a stress element oriented at 22.5° to the axis of the bar.

Solution 2.6-6

(a) $d = 12$ mm $P = 9.5$ kN $A = \frac{\pi}{4}d^2 = 1.131 \times 10^{-4} \text{ m}^2$

$$\sigma_x = \frac{P}{A} = 84 \text{ MPa}$$

(b) $\tau_{\max} = \frac{\sigma_x}{2} = 42 \text{ MPa}$ On plane stress element rotated 45°

(c) ROTATED STRESS ELEMENT (45°) HAS NORMAL TENSILE STRESS $\sigma_x/2$ ON ALL FACES, $-T_{\max}$ (CW) ON $+x$ -FACE, AND $+T_{\max}$ (CCW) ON $+y$ -FACE

$$\tau_{xy1y1} = \tau_{\max} \quad \sigma_{x1} = \frac{\sigma_x}{2} \quad \sigma_{y1} = \sigma_{x1}$$

On rotated x -face: $\sigma_{x1} = 42 \text{ MPa}$ $\tau_{x1y1} = 42 \text{ MPa}$

On rotated y -face: $\sigma_{y1} = 42 \text{ MPa}$

(d) $\theta = 22.5^\circ < \text{CCW ROTATION OF ELEMENT}$

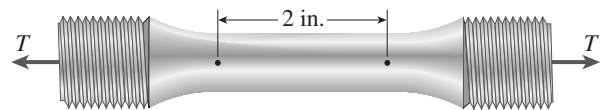
$$\sigma_\theta = \sigma_x \cos^2(\theta) = 71.7 \text{ MPa} < \text{on rotated } x \text{ face} \quad \sigma_y = \sigma_x \cos^2\left(\theta + \frac{\pi}{2}\right) = 12.3 \text{ MPa} < \text{on rotated } y \text{ face}$$

Eq. 2-31b $\tau_\theta = \frac{-\sigma_x}{2} \sin(2\theta) = -29.7 \text{ MPa} < \text{CW on rotated } x\text{-face}$

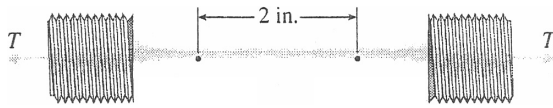
On rotated x -face: $\sigma_{x1} = 71.7 \text{ MPa}$ $\tau_{x1y1} = -29.7 \text{ MPa}$

On rotated y -face: $\sigma_{y1} = 12.3 \text{ MPa}$

Problem 2.6-7 During a tension test of a mild-steel specimen (see figure), the extensometer shows an elongation of 0.00120 in. with a gage length of 2 in. Assume that the steel is stressed below the proportional limit and that the modulus of elasticity $E = 30 \times 10^6$ psi.



- What is the maximum normal stress σ_{\max} in the specimen?
- What is the maximum shear stress τ_{\max} ?
- Draw a stress element oriented at an angle of 45° to the axis of the bar and show all stresses acting on the faces of this element.

Solution 2.6-7 Tension test

Elongation: $\delta = 0.00120$ in.

(2 in. gage length)

$$\text{Strain: } \varepsilon = \frac{\delta}{L} = \frac{0.00120 \text{ in.}}{2 \text{ in.}} = 0.00060$$

$$\begin{aligned} \text{Hooke's law: } \sigma_x &= E\varepsilon = (30 \times 10^6 \text{ psi})(0.00060) \\ &= 18,000 \text{ psi} \end{aligned}$$

(a) MAXIMUM NORMAL STRESS

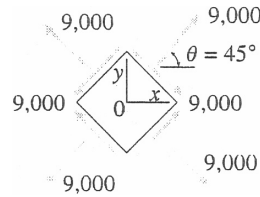
σ_x is the maximum normal stress.

$$\sigma_{\max} = 18,000 \text{ psi} \quad \leftarrow$$

(b) MAXIMUM SHEAR STRESS

The maximum shear stress is on a 45° plane and equals $\sigma_x/2$.

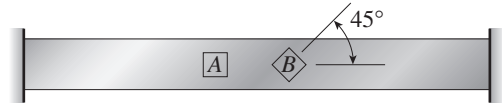
$$\tau_{\max} = \frac{\sigma_x}{2} = 9,000 \text{ psi} \quad \leftarrow$$

(c) STRESS ELEMENT AT $\theta = 45^\circ$ 

NOTE: All stresses have units of psi.

Problem 2.6-8 A copper bar with a rectangular cross section is held without stress between rigid supports (see figure). Subsequently, the temperature of the bar is raised 50°C .

- Determine the stresses on all faces of the elements *A* and *B*, and show these stresses on sketches of the elements. (Assume $\alpha = 17.5 \times 10^{-6}/^\circ\text{C}$ and $E = 120 \text{ GPa}$.)
- If the shear stress at *B* is known to be 48 MPa at some inclination θ , find angle θ and show the stresses on a sketch of a properly oriented element.

**Solution 2.6-8**

$$(a) \quad \alpha = 17.5(10^{-6}) \quad \Delta T = 50 \quad E = 120 \text{ GPa}$$

$$\sigma_x = -E\alpha\Delta T = -105 \text{ MPa} \quad \tau_{\max} = \frac{\sigma_x}{2} = -52.5 \text{ MPa} \quad < \text{ at } \theta = 45^\circ$$

(compression)

Element A: $\sigma_x = 105 \text{ MPa}$ (compression);
Element B: $\tau_{\max} = 52.5 \text{ MPa}$

$$(b) \quad \tau_\theta = 48 \text{ MPa}$$

$$\text{Eq. 2-31b} \quad \tau_\theta = \frac{-\sigma_x}{2} \sin(2\theta)$$

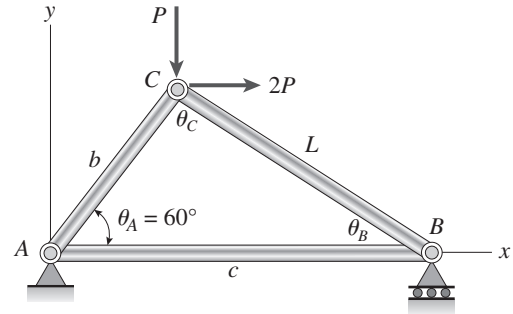
$$\text{so} \quad \theta = \frac{1}{2} \arcsin\left(\frac{2\tau_\theta}{-\sigma_x}\right) = 33.1^\circ < \text{CCW rotation of element} \quad \boxed{\theta = 33.1^\circ}$$

$$\sigma_\theta = \sigma_x \cos^2(\theta) = -73.8 \text{ MPa} < \text{on rotated } x \text{ face}$$

$$\sigma_y = \sigma_x \cos^2\left(\theta + \frac{\pi}{2}\right) = -31.2 \text{ MPa} < \text{on rotated } y \text{ face}$$

Problem 2.6-9 The plane truss below is assembled from steel C10 × 20 shapes (see Table 3(a) in Appendix F). Assume that $L = 10$ ft and $b = 0.71 L$.

- If load variable $P = 49$ k, what is the maximum shear stress τ_{\max} in each truss member?
- What is the maximum permissible value of load variable P if the allowable normal stress is 14 ksi and the allowable shear stress is 7.5 ksi?



Solution 2.6-9

NUMERICAL DATA

$$L = 10 \text{ ft} \quad b = 0.71 L \quad P = 49 \text{ k} \quad \sigma_a = 14 \text{ ksi} \quad \tau_a = 7.5 \text{ ksi} \quad A = 5.87 \text{ in.}^2$$

(a) FOR LINEAR ANALYSIS, MEMBER FORCES ARE PROPORTIONAL TO LOADING

$$\text{FROM EXAMPLE 1-1: } F_{AC} = \frac{P}{35} 15.59 = 21.826 \text{ k} \quad F_{AB} = \frac{P}{35} 62.2 = 87.08 \text{ k}$$

(solution for $P = 35$ k)

$$F_{BC} = \frac{P}{35} (-78.9) \quad F_{BC} = -110.46 \text{ k}$$

$$\text{Normal stresses in each member: } \sigma_{AC} = \frac{F_{AC}}{A} = 3.718 \text{ ksi} \quad \sigma_{AB} = \frac{F_{AB}}{A} = 14.835 \text{ ksi}$$

$$\text{From Eq. 2-33: } \sigma_{BC} = \frac{F_{BC}}{A} = -18.818 \text{ ksi}$$

$$\tau_{\max AC} = \frac{\sigma_{AC}}{2} = 1.859 \text{ ksi}$$

$$\tau_{\max AB} = \frac{\sigma_{AB}}{2} = 7.42 \text{ ksi}$$

$$\tau_{\max BC} = \frac{\sigma_{BC}}{2} = -9.41 \text{ ksi}$$

(b) $\sigma_a < 2 \times \tau_a$ so normal stress will control; lowest value governs here

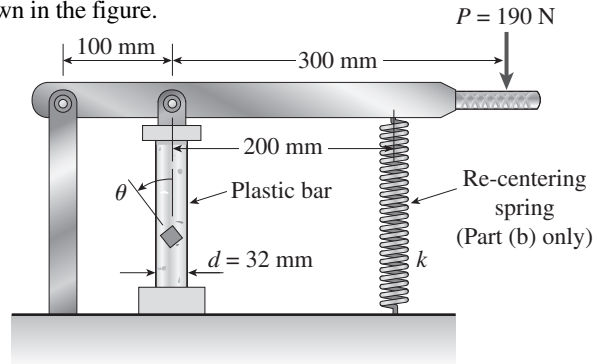
$$\text{MEMBER AC: } P_{\max \sigma} = \frac{P}{F_{AC}} (\sigma_a A) = 184.496 \text{ k} \quad P_{\max \tau} = \frac{P}{F_{AC}} (2 \tau_a A) = 197.675 \text{ k}$$

$$\text{MEMBER AB: } P_{\max \sigma} = \frac{P}{F_{AB}} (\sigma_a A) = 46.243 \text{ k} \quad P_{\max \tau} = \frac{P}{F_{AB}} (2 \tau_a A) = 49.546 \text{ k}$$

$$\text{MEMBER BC: } P_{\max \sigma} = \left| \frac{P}{F_{BC}} \right| (\sigma_a A) = 36.5 \text{ k} \quad P_{\max \tau} = \left| \frac{P}{F_{BC}} \right| (2 \tau_a A) = 39.059 \text{ k}$$

Problem 2.6-10 A plastic bar of diameter $d = 32$ mm is compressed in a testing device by a force $P = 190$ N applied as shown in the figure.

- (a) Determine the normal and shear stresses acting on all faces of stress elements oriented at (1) an angle $\theta = 0^\circ$, (2) an angle $\theta = 22.5^\circ$, and (3) an angle $\theta = 45^\circ$. In each case, show the stresses on a sketch of a properly oriented element. What are σ_{\max} and τ_{\max} ?
- (b) Find σ_{\max} and τ_{\max} in the plastic bar if a re-centering spring of stiffness k is inserted into the testing device, as shown in the figure. The spring stiffness is $1/6$ of the axial stiffness of the plastic bar.



Solution

NUMERICAL DATA

$$\begin{aligned} d &= 32 \text{ mm} & A &= \frac{\pi}{4} d^2 \\ P &= 190 \text{ N} & A &= 804.25 \text{ mm}^2 \\ a &= 100 \text{ mm} \\ b &= 300 \text{ mm} \end{aligned}$$

- (a) **STATICS—FIND COMPRESSIVE FORCE F AND STRESSES IN PLASTIC BAR**

$$F = \frac{P(a + b)}{a} \quad F = 760 \text{ N}$$

$$\sigma_x = \frac{F}{A} \quad \sigma_x = 0.945 \text{ MPa} \quad \text{or} \quad \sigma_x = 945 \text{ kPa}$$

From (1), (2), and (3) below:

$$\sigma_{\max} = \sigma_x \quad \sigma_{\max} = -945 \text{ kPa}$$

$$\tau_{\max} = 472 \text{ kPa} \quad \frac{\sigma_x}{2} = -472 \text{ kPa}$$

$$(1) \theta = 0^\circ \quad \sigma_x = -945 \text{ kPa} \quad \leftarrow$$

$$(2) \theta = 22.5^\circ$$

On $+x$ -face:

$$\begin{aligned} \sigma_\theta &= \sigma_x \cos^2(\theta) \\ \sigma_\theta &= -807 \text{ kPa} \quad \leftarrow \end{aligned}$$

$$\begin{aligned} \tau_\theta &= -\sigma_x \sin(\theta) \cos(\theta) \\ \tau_\theta &= 334 \text{ kPa} \quad \leftarrow \end{aligned}$$

$$\text{On } +y\text{-face:} \quad \theta = \theta + \frac{\pi}{2}$$

$$\begin{aligned} \sigma_\theta &= \sigma_x \cos^2(\theta) \\ \sigma_\theta &= -138.39 \text{ kPa} \end{aligned}$$

$$\begin{aligned} \tau_\theta &= -\sigma_x \sin(\theta) \cos(\theta) \\ \tau_\theta &= -334.1 \text{ kPa} \end{aligned}$$

$$(3) \theta = 45^\circ$$

On $+x$ -face:

$$\begin{aligned} \sigma_\theta &= \sigma_x \cos^2(\theta) \\ \sigma_\theta &= -472 \text{ kPa} \quad \leftarrow \end{aligned}$$

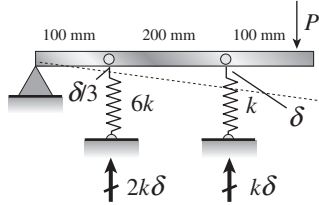
$$\begin{aligned} \tau_\theta &= -\sigma_x \sin(\theta) \cos(\theta) \\ \tau_\theta &= 472 \text{ kPa} \quad \leftarrow \end{aligned}$$

$$\text{On } +y\text{-face:} \quad \theta = \theta + \frac{\pi}{2}$$

$$\sigma_\theta = \sigma_x \cos^2(\theta) \quad \sigma_\theta = -472.49 \text{ kPa}$$

$$\tau_\theta = -\sigma_x \sin(\theta) \cos(\theta) \quad \tau_\theta = -472.49 \text{ kPa}$$

- (b) ADD SPRING—FIND MAXIMUM NORMAL AND SHEAR STRESSES IN PLASTIC BAR



$$\sum M_{\text{pin}} = 0$$

$$P(400) = [2k\delta(100) + k\delta(300)]$$

$$\delta = \frac{4}{5} \frac{P}{k}$$

$$\text{Force in plastic bar: } F = (2k) \left(\frac{4}{5} \frac{P}{k} \right)$$

$$F = \frac{8}{5} P \quad F = 304 \text{ N}$$

Normal and shear stresses in plastic bar:

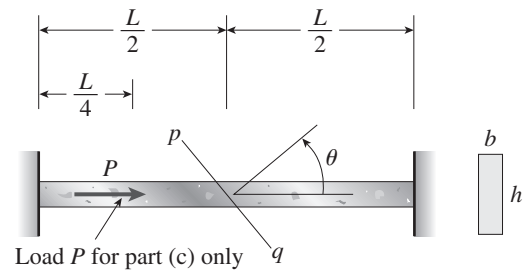
$$\sigma_x = \frac{F}{A} \quad \sigma_x = 0.38$$

$$\sigma_{\text{max}} = -378 \text{ kPa} \quad \leftarrow$$

$$\tau_{\text{max}} = \frac{\sigma_x}{2} \quad \tau_{\text{max}} = -189 \text{ kPa} \quad \leftarrow$$

Problem 2.6-11 A plastic bar of rectangular cross section ($b = 1.5$ in. and $h = 3$ in.) fits snugly between rigid supports at room temperature (68°F) but with no initial stress (see figure). When the temperature of the bar is raised to 160°F , the compressive stress on an inclined plane pq at midspan becomes 1700 psi.

- What is the shear stress on plane pq ? (Assume $\alpha = 60 \times 10^{-6}/^\circ\text{F}$ and $E = 450 \times 10^3$ psi.)
- Draw a stress element oriented to plane pq and show the stresses acting on all faces of this element.
- If the allowable normal stress is 3400 psi and the allowable shear stress is 1650 psi, what is the maximum load P (in $+x$ direction) which can be added at the quarter point (in addition to thermal effects above) without exceeding allowable stress values in the bar?



Solution 2.6-11

NUMERICAL DATA

$$b = 1.5 \text{ in.} \quad h = 3 \text{ in.} \quad A = bh \quad \Delta T = (160 - 68)^\circ\text{F}$$

$$\Delta T = 92^\circ\text{F}$$

$$A = 4.5 \text{ in.}^2 \quad \sigma_{pq} = -1700 \text{ psi}$$

$$\alpha = 60 \times (10^{-6})/^\circ\text{F}$$

$$E = 450 \times (10^3) \text{ psi}$$

- (a) SHEAR STRESS ON PLANE pq

Statically indeterminate analysis gives, for reaction at right support:

$$R = -EA\alpha\Delta T \quad R = -11178 \text{ lb}$$

$$\sigma_x = \frac{R}{A} \quad \sigma_x = -2484 \text{ psi}$$

$$\text{Using } \sigma_\theta = \sigma_x \cos(\theta)^2: \quad \cos(\theta)^2 = \frac{\sigma_{pq}}{\sigma_x}$$

$$\theta = \arccos\left(\sqrt{\frac{\sigma_{pq}}{\sigma_x}}\right) \quad \theta = 34.2^\circ$$

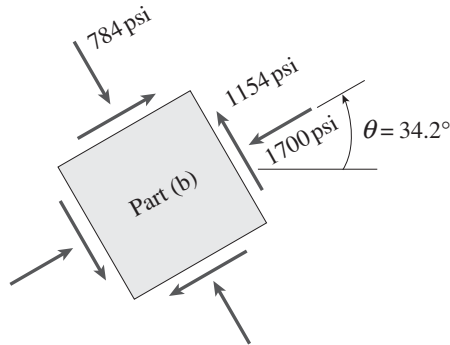
Now with θ , can find shear stress on plane pq :

$$\tau_{pq} = -\sigma_x \sin(\theta) \cos(\theta) \quad \tau_{pq} = 1154 \text{ psi} \quad \leftarrow$$

$$\sigma_{pq} = \sigma_x \cos(\theta)^2 \quad \sigma_{pq} = -1700 \text{ psi}$$

Stresses at $\theta + \pi/2$ (y-face):

$$\sigma_y = \sigma_x \cos\left(\theta + \frac{\pi}{2}\right)^2 \quad \sigma_y = -784 \text{ psi}$$

(b) STRESS ELEMENT FOR PLANE PQ

 (c) MAXIMUM LOAD AT QUARTER POINT $\sigma_a = 3400$ psi

$$\tau_a = 1650 \text{ psi} \quad 2\tau_a = 3300 < \text{less than } \sigma_a, \text{ so shear controls}$$

Statically indeterminate analysis for P at $L/4$ gives for reactions:

$$R_{R2} = \frac{-P}{4} \quad R_{L2} = \frac{3}{4}P$$

(tension for 0 to $L/4$ and compression for rest of bar)

From part (a) (for temperature increase ΔT):

$$R_{R1} = -EA\alpha\Delta T \quad R_{L1} = -EA\alpha\Delta T$$

Stresses in bar (0 to $L/4$):

$$\sigma_x = -E\alpha\Delta T + \frac{3P}{4A} \quad \tau_{\max} = \frac{\sigma_x}{2}$$

Set $\tau_{\max} = \tau_a$ and solve for $P_{\max 1}$:

$$\tau_a = \frac{-E\alpha\Delta T}{2} + \frac{3P}{8A}$$

$$P_{\max 1} = \frac{4A}{3}(2\tau_a + E\alpha\Delta T)$$

$$P_{\max 1} = 34,704 \text{ lb}$$

$$\tau_{\max} = \frac{-E\alpha\Delta T}{2} + \frac{3P_{\max 1}}{8A}$$

$$\tau_{\max} = 1650 \text{ psi} < \text{check}$$

$$\sigma_x = -E\alpha\Delta T + \frac{3P_{\max 1}}{4A}$$

$$\sigma_x = 3300 \text{ psi} < \text{less than } \sigma_a$$

Stresses in bar ($L/4$ to L):

$$\sigma_x = -E\alpha\Delta T - \frac{P}{4A} \quad \tau_{\max} = \frac{\sigma_x}{2}$$

Set $\tau_{\max} = \tau_a$ and solve for $P_{\max 2}$:

$$P_{\max 2} = -4A(-2\tau_a + E\alpha\Delta T)$$

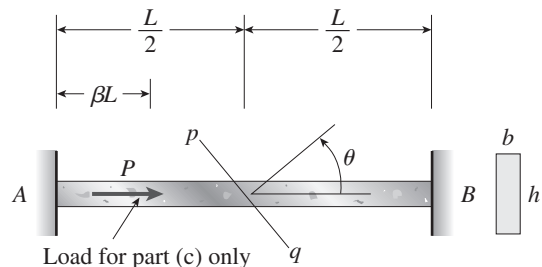
$$P_{\max 2} = 14,688 \text{ lb} \leftarrow \text{shear in segment } (L/4 \text{ to } L) \text{ controls}$$

$$\tau_{\max} = \frac{-E\alpha\Delta T}{2} - \frac{P_{\max 2}}{8A} \quad \tau_{\max} = -1650 \text{ psi}$$

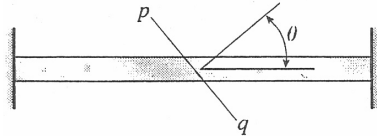
$$\sigma_x = -E\alpha\Delta T - \frac{P_{\max 2}}{4A} \quad \sigma_x = -3300 \text{ psi}$$

Problem 2.6-12 A copper bar of rectangular cross section ($b = 18$ mm and $h = 40$ mm) is held snugly (but without any initial stress) between rigid supports (see figure). The allowable stresses on the inclined plane pq at midspan, for which $\theta = 55^\circ$, are specified as 60 MPa in compression and 30 MPa in shear.

- What is the maximum permissible temperature rise ΔT if the allowable stresses on plane pq are not to be exceeded? (Assume $\alpha = 17 \times 10^{-6}/^\circ\text{C}$ and $E = 120$ GPa.)
- If the temperature increases by the maximum permissible amount, what are the stresses on plane pq ?
- If the temperature rise $\Delta T = 28^\circ\text{C}$, how far to the right of end A (distance βL , expressed as a fraction of length L) can load $P = 15$ kN be applied without exceeding allowable stress values in the bar? Assume that $\sigma_a = 75$ MPa and $\tau_a = 35$ MPa.



Solution 2.6-12



NUMERICAL DATA

$$\theta = 55 \left(\frac{\pi}{180} \right) \text{ rad}$$

$$b = 18 \text{ mm} \quad h = 40 \text{ mm}$$

$$A = bh \quad A = 720 \text{ mm}^2$$

$$\sigma_{pqa} = 60 \text{ MPa} \quad \tau_{pqa} = 30 \text{ MPa}$$

$$\alpha = 17 \times (10^{-6})/^{\circ}\text{C} \quad E = 120 \text{ GPa}$$

$$\Delta T = 20^{\circ}\text{C} \quad P = 15 \text{ kN}$$

- (a) FIND ΔT_{\max} BASED ON ALLOWABLE NORMAL AND SHEAR STRESS VALUES ON PLANE pq

$$\sigma_x = -E\alpha\Delta T_{\max} \quad \Delta T_{\max} = \frac{-\sigma_x}{E\alpha}$$

$$\sigma_{pq} = \sigma_x \cos(\theta)^2 \quad \tau_{pq} = -\sigma_x \sin(\theta) \cos(\theta)$$

Set each equal to corresponding allowable and solve for σ_x :

$$\sigma_{x1} = \frac{\sigma_{pqa}}{\cos(\theta)^2} \quad \sigma_{x1} = 182.38 \text{ MPa}$$

$$\sigma_{x2} = \frac{\tau_{pqa}}{-\sin(\theta) \cos(\theta)} \quad \sigma_{x2} = -63.85 \text{ MPa}$$

Lesser value controls, so allowable shear stress governs.

$$\Delta T_{\max} = \frac{-\sigma_{x2}}{E\alpha} \quad \Delta T_{\max} = 31.3^{\circ}\text{C} \quad \leftarrow$$

- (b) STRESSES ON PLANE pQ FOR MAXIMUM TEMPERATURE

$$\sigma_x = -E\alpha\Delta T_{\max} \quad \sigma_x = -63.85 \text{ MPa}$$

$$\sigma_{pq} = \sigma_x \cos(\theta)^2 \quad \sigma_{pq} = -21.0 \text{ MPa} \quad \leftarrow$$

$$\tau_{pq} = -\sigma_x \sin(\theta) \cos(\theta) \quad \tau_{pq} = 30 \text{ MPa} \quad \leftarrow$$

- (c) ADD LOAD P IN $+x$ -DIRECTION TO TEMPERATURE CHANGE AND FIND LOCATION OF LOAD

$$\Delta T = 28^{\circ}\text{C}$$

$P = 15 \text{ kN}$ from one-degree statically indeterminate analysis, reactions R_A and R_B due to load P :

$$R_A = -(1 - \beta)P \quad R_B = \beta P$$

Now add normal stresses due to P to thermal stresses due to ΔT (tension in segment 0 to βL , compression in segment βL to L).

Stresses in bar (0 to βL):

$$\sigma_x = -E\alpha\Delta T + \frac{R_A}{A} \quad \tau_{\max} = \frac{\sigma_x}{2}$$

Shear controls so set $\tau_{\max} = \tau_a$ and solve for β :

$$2\tau_a = -E\alpha\Delta T + \frac{(1 - \beta)P}{A}$$

$$\beta = 1 - \frac{A}{P} [2\tau_a + E\alpha\Delta T]$$

$$\beta = -5.1$$

Impossible so evaluate segment (βL to L):

Stresses in bar (βL to L):

$$\sigma_x = -E\alpha\Delta T - \frac{R_B}{A} \quad \tau_{\max} = \frac{\sigma_x}{2}$$

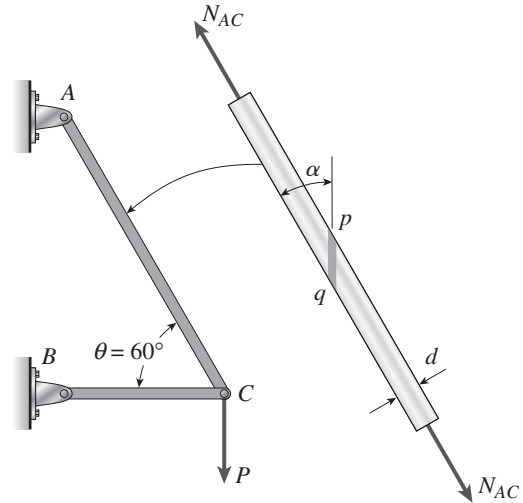
set $\tau_{\max} = \tau_a$ and solve for $P_{\max 2}$

$$2\tau_a = -E\alpha\Delta T - \frac{\beta P}{A}$$

$$\beta = \frac{-A}{P} [-2\tau_a + E\alpha\Delta T]$$

$$\beta = 0.62 \quad \leftarrow$$

Problem 2.6-13 A circular brass bar of diameter d is member AC in truss ABC which has load $P = 5000$ lb applied at joint C . Bar AC is composed of two segments brazed together on a plane pq making an angle $\alpha = 36^\circ$ with the axis of the bar (see figure). The allowable stresses in the brass are 13,500 psi in tension and 6500 psi in shear. On the brazed joint, the allowable stresses are 6000 psi in tension and 3000 psi in shear. What is the tensile force N_{AC} in bar AC ? What is the minimum required diameter d_{\min} of bar AC ?

**Solution 2.6-13**

NUMERICAL DATA

$$P = 5 \text{ k} \quad \alpha = 36^\circ \quad \sigma_a = 13.5 \text{ ksi}$$

$$\tau_a = 6.5 \text{ ksi}$$

$$\theta = \frac{\pi}{2} - \alpha \quad \theta = 54^\circ$$

$$\sigma_{ja} = 6.0 \text{ ksi}$$

$$\tau_{ja} = 3.0 \text{ ksi}$$

Tensile force N_{AC} using Method of Joints at C :

$$N_{AC} = \frac{P}{\sin(60^\circ)} \quad (\text{tension})$$

$$N_{AC} = 5.77 \text{ k} \quad \leftarrow$$

Minimum required diameter of bar AC :

(1) Check tension and shear in bars; $\tau_a < \sigma_a/2$ so shear

$$\text{controls } \tau_{\max} = \frac{\sigma_x}{2}:$$

$$2\tau_a = \frac{N_{AC}}{A} \quad \sigma_x = 2\tau_a = 13 \text{ ksi}$$

$$A_{\text{reqd}} = \frac{N_{AC}}{2\tau_a} \quad A_{\text{reqd}} = 0.44 \text{ in.}^2$$

$$d_{\min} = \sqrt{\frac{4}{\pi} A_{\text{reqd}}} \quad d_{\min} = 0.75 \text{ in.}$$

(2) Check tension and shear on brazed joint:

$$\sigma_x = \frac{N_{AC}}{A} \quad \sigma_x = \frac{N_{AC}}{\frac{\pi}{4} d^2} \quad d_{\text{reqd}} = \sqrt{\frac{4}{\pi} \frac{N_{AC}}{\sigma_x}}$$

Tension on brazed joint:

$$\sigma_\theta = \sigma_x \cos(\theta)^2$$

Set equal to σ_{ja} and solve for σ_x , then d_{reqd} :

$$\sigma_x = \frac{\sigma_{ja}}{\cos(\theta)^2} \quad \sigma_x = 17.37 \text{ ksi}$$

$$d_{\text{reqd}} = \sqrt{\frac{4}{\pi} \frac{N_{AC}}{\sigma_x}} \quad d_{\text{reqd}} = 0.65 \text{ in.}$$

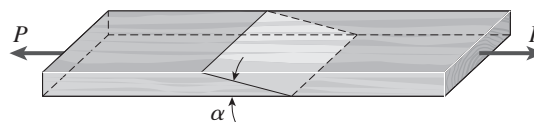
Shear on brazed joint:

$$\tau_\theta = -\sigma_x \sin(\theta) \cos(\theta)$$

$$\sigma_x = \left| \frac{\tau_{ja}}{-(\sin(\theta) \cos(\theta))} \right| \quad \sigma_x = -6.31 \text{ ksi}$$

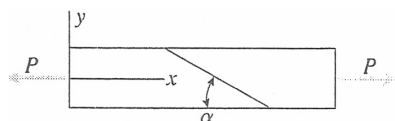
$$d_{\text{reqd}} = \sqrt{\frac{4}{\pi} \frac{N_{AC}}{\sigma_x}} \quad d_{\text{reqd}} = 1.08 \text{ in.} \quad \leftarrow$$

Problem 2.6-14 Two boards are joined by gluing along a scarf joint, as shown in the figure. For purposes of cutting and gluing, the angle α between the plane of the joint and the faces of the boards must be between 10° and 40° . Under a tensile load P , the normal stress in the boards is 4.9 MPa.



- What are the normal and shear stresses acting on the glued joint if $\alpha = 20^\circ$?
- If the allowable shear stress on the joint is 2.25 MPa, what is the largest permissible value of the angle α ?
- For what angle α will the shear stress on the glued joint be numerically equal to twice the normal stress on the joint?

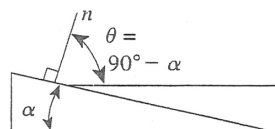
Solution 2.6-14 Two boards joined by a scarf joint



$$10^\circ \leq \alpha \leq 40^\circ$$

Due to load P : $\sigma_x = 4.9$ MPa

- (a) STRESSES ON JOINT WHEN $\alpha = 20^\circ$



$$\theta = 90^\circ - \alpha = 70^\circ$$

$$\begin{aligned}\sigma_\theta &= \sigma_x \cos^2 \theta = (4.9 \text{ MPa})(\cos 70^\circ)^2 \\ &= 0.57 \text{ MPa} \quad \leftarrow\end{aligned}$$

$$\begin{aligned}\tau_\theta &= -\sigma_x \sin \theta \cos \theta \\ &= (-4.9 \text{ MPa})(\sin 70^\circ)(\cos 70^\circ) \\ &= -1.58 \text{ MPa} \quad \leftarrow\end{aligned}$$

- (b) LARGEST ANGLE α IF $\tau_{\text{allow}} = 2.25$ MPa

$$\tau_{\text{allow}} = -\sigma_x \sin \theta \cos \theta$$

The shear stress on the joint has a negative sign. Its numerical value cannot exceed $\tau_{\text{allow}} = 2.25$ MPa. Therefore,

$$\begin{aligned}-2.25 \text{ MPa} &= -(4.9 \text{ MPa})(\sin \theta)(\cos \theta) \text{ or } \sin \theta \cos \theta \\ &= 0.4592\end{aligned}$$

$$\text{From trigonometry: } \sin \theta \cos \theta = \frac{1}{2} \sin 2\theta$$

$$\text{Therefore: } \sin 2\theta = 2(0.4592) = 0.9184$$

$$\text{Solving: } 2\theta = 66.69^\circ \quad \text{or} \quad 113.31^\circ$$

$$\theta = 33.34^\circ \quad \text{or} \quad 56.66^\circ$$

$$\alpha = 90^\circ - \theta \quad \therefore \alpha = 56.66^\circ \quad \text{or} \quad 33.34^\circ$$

Since α must be between 10° and 40° , we select

$$\alpha = 33.3^\circ \quad \leftarrow$$

NOTE: If α is between 10° and 33.3° ,

$$|\tau_\theta| < 2.25 \text{ MPa.}$$

If α is between 33.3° and 40° ,

$$|\tau_\theta| > 2.25 \text{ MPa.}$$

- (c) WHAT IS α IF $\tau_\theta = 2\sigma_\theta$?

Numerical values only:

$$|\tau_\theta| = \sigma_x \sin \theta \cos \theta \quad |\sigma_\theta| = \sigma_x \cos^2 \theta$$

$$\left| \frac{\tau_\theta}{\sigma_\theta} \right| = 2$$

$$\sigma_x \sin \theta \cos \theta = 2\sigma_x \cos^2 \theta$$

$$\sin \theta = 2 \cos \theta \quad \text{or} \quad \tan \theta = 2$$

$$\theta = 63.43^\circ \quad \alpha = 90^\circ - \theta$$

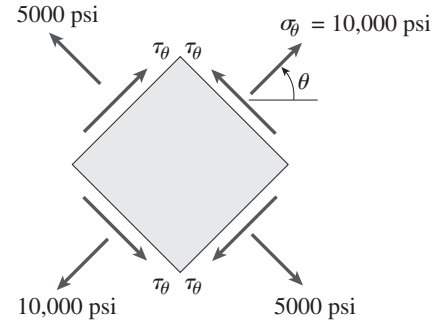
$$\alpha = 26.6^\circ \quad \leftarrow$$

NOTE: For $\alpha = 26.6^\circ$ and $\theta = 63.4^\circ$, we find $\sigma_\theta = 0.98$ MPa and $\tau_\theta = -1.96$ MPa.

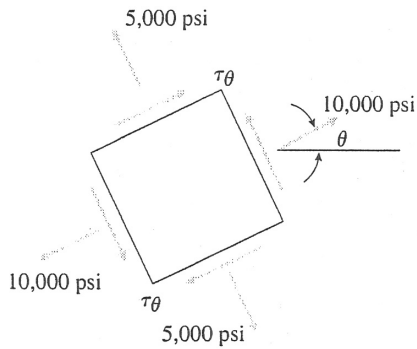
$$\text{Thus, } \left| \frac{\tau_\theta}{\sigma_\theta} \right| = 2 \text{ as required.}$$

Problem 2.6-15 Acting on the sides of a stress element cut from a bar in uniaxial stress are tensile stresses of 10,000 psi and 5,000 psi, as shown in the figure.

- Determine the angle θ and the shear stress τ_θ and show all stresses on a sketch of the element.
- Determine the maximum normal stress σ_{\max} and the maximum shear stress τ_{\max} in the material.



Solution 2.6-15 Bar in uniaxial stress



- (a) ANGLE θ AND SHEAR STRESS τ_θ

$$\sigma_\theta = \sigma_x \cos^2 \theta$$

$$\sigma_\theta = 10,000 \text{ psi}$$

$$\sigma_x = \frac{\sigma_\theta}{\cos^2 \theta} = \frac{10,000 \text{ psi}}{\cos^2 \theta}$$

PLANE AT ANGLE $\theta + 90^\circ$

$$\begin{aligned} \sigma_{\theta+90^\circ} &= \sigma_x [\cos(\theta + 90^\circ)]^2 = \sigma_x [-\sin \theta]^2 \\ &= \sigma_x \sin^2 \theta \end{aligned}$$

$$\sigma_{\theta+90^\circ} = 5,000 \text{ psi}$$

$$\sigma_x = \frac{\sigma_{\theta+90^\circ}}{\sin^2 \theta} = \frac{5,000 \text{ psi}}{\sin^2 \theta} \quad (2)$$

Equate (1) and (2):

$$\frac{10,000 \text{ psi}}{\cos^2 \theta} = \frac{5,000 \text{ psi}}{\sin^2 \theta}$$

$$\tan^2 \theta = \frac{1}{2} \quad \tan \theta = \frac{1}{\sqrt{2}} \quad \theta = 35.26^\circ \quad \leftarrow$$

From Eq. (1) or (2):

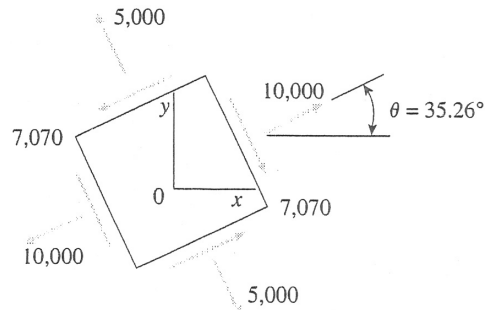
$$\sigma_x = 15,000 \text{ psi}$$

$$\tau_\theta = -\sigma_x \sin \theta \cos \theta$$

$$= (-15,000 \text{ psi})(\sin 35.26^\circ)(\cos 35.26^\circ)$$

$$= -7,070 \text{ psi} \quad \leftarrow$$

Minus sign means that τ_θ acts clockwise on the plane for which $\theta = 35.26^\circ$.



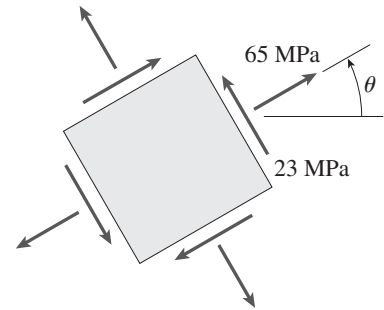
NOTE: All stresses have units of psi.

- (b) MAXIMUM NORMAL AND SHEAR STRESSES

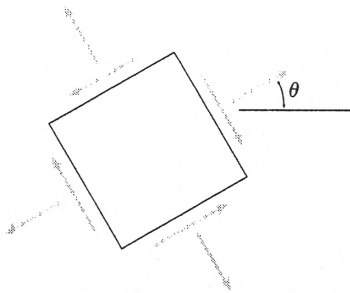
$$\sigma_{\max} = \sigma_x = 15,000 \text{ psi} \quad \leftarrow$$

$$\tau_{\max} = \frac{\sigma_x}{2} = 7,500 \text{ psi} \quad \leftarrow$$

Problem 2.6-16 A prismatic bar is subjected to an axial force that produces a tensile stress $\sigma_\theta = 65$ MPa and a shear stress $\tau_\theta = 23$ MPa on a certain inclined plane (see figure). Determine the stresses acting on all faces of a stress element oriented at $\theta = 30^\circ$ and show the stresses on a sketch of the element.



Solution 2.6-16



Find θ and σ_x for stress state shown in figure.

$$\sigma_\theta = \sigma_x \cos(\theta)^2 \quad \cos(\theta) = \sqrt{\frac{\sigma_\theta}{\sigma_x}}$$

$$\text{so} \quad \sin(\theta) = \sqrt{1 - \frac{\sigma_\theta}{\sigma_x}}$$

$$\tau_\theta = -\sigma_x \sin(\theta) \cos(\theta)$$

$$\frac{\tau_\theta}{\sigma_x} = -\sqrt{1 - \frac{\sigma_\theta}{\sigma_x}} \sqrt{\frac{\sigma_\theta}{\sigma_x}}$$

$$\left(\frac{\tau_\theta}{\sigma_x}\right)^2 = \frac{\sigma_\theta}{\sigma_x} - \left(\frac{\sigma_\theta}{\sigma_x}\right)$$

$$\left(\frac{23}{\sigma_x}\right)^2 = \frac{65}{\sigma_x} - \left(\frac{65}{\sigma_x}\right)$$

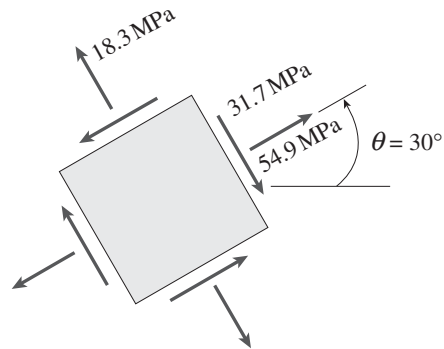
$$\left(\frac{65}{\sigma_x}\right)^2 - \left(\frac{65}{\sigma_x}\right) + \left(\frac{23}{\sigma_x}\right)^2 = 0$$

$$\frac{-(-4754 + 65\sigma_x)}{\sigma_x^2} = 0$$

$$\sigma_x = \frac{4754}{65}$$

$$\sigma_x = 73.1 \text{ MPa} \quad \sigma_\theta = 65 \text{ MPa}$$

$$\theta = \arccos\left(\sqrt{\frac{\sigma_\theta}{\sigma_x}}\right) \quad \theta = 19.5^\circ$$



Now find σ_θ and τ_θ for $\theta = 30^\circ$:

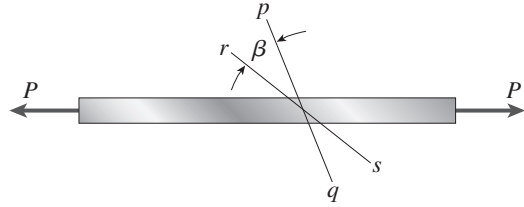
$$\sigma_{\theta 1} = \sigma_x \cos(\theta)^2 \quad \sigma_{\theta 1} = 54.9 \text{ MPa} \quad \leftarrow$$

$$\tau_\theta = -\sigma_x \sin(\theta) \cos(\theta) \quad \tau_\theta = -31.7 \text{ MPa} \quad \leftarrow$$

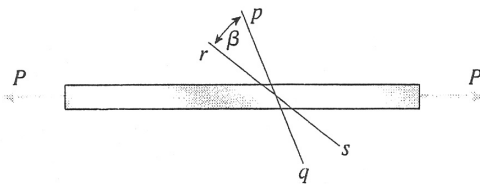
$$\sigma_{\theta 2} = \sigma_x \cos\left(\theta + \frac{\pi}{2}\right)^2 \quad \sigma_{\theta 2} = 18.3 \text{ MPa} \quad \leftarrow$$

Problem 2.6-17 The normal stress on plane pq of a prismatic bar in tension (see figure) is found to be 7500 psi. On plane rs , which makes an angle $\beta = 30^\circ$ with plane pq , the stress is found to be 2500 psi.

Determine the maximum normal stress σ_{\max} and maximum shear stress τ_{\max} in the bar.



Solution 2.6-17 Bar in tension



Eq. (2-31a):

$$\sigma_\theta = \sigma_x \cos^2 \theta$$

$$\beta = 30^\circ$$

$$\text{PLANE } pq: \sigma_1 = \sigma_x \cos^2 \theta_1 \quad \sigma_1 = 7500 \text{ psi}$$

$$\text{PLANE } rs: \sigma_2 = \sigma_x \cos^2(\theta_1 + \beta) \quad \sigma_2 = 2500 \text{ psi}$$

Equate σ_x from σ_1 and σ_2 :

$$\sigma_x = \frac{\sigma_1}{\cos^2 \theta_1} = \frac{\sigma_2}{\cos^2(\theta_1 + \beta)} \quad (\text{Eq. 1})$$

or

$$\frac{\cos^2 \theta_1}{\cos^2(\theta_1 + \beta)} = \frac{\sigma_1}{\sigma_2} \frac{\cos \theta_1}{\cos(\theta_1 + \beta)} = \sqrt{\frac{\sigma_1}{\sigma_2}} \quad (\text{Eq. 2})$$

SUBSTITUTE NUMERICAL VALUES INTO EQ. (2):

$$\frac{\cos \theta_1}{\cos(\theta_1 + 30^\circ)} = \sqrt{\frac{7500 \text{ psi}}{2500 \text{ psi}}} = \sqrt{3} = 1.7321$$

Solve by iteration or a computer program:

$$\theta_1 = 30^\circ$$

MAXIMUM NORMAL STRESS (FROM EQ. 1)

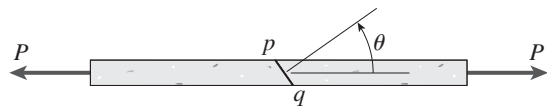
$$\begin{aligned} \sigma_{\max} = \sigma_x &= \frac{\sigma_1}{\cos^2 \theta_1} = \frac{7500 \text{ psi}}{\cos^2 30^\circ} \\ &= 10,000 \text{ psi} \quad \leftarrow \end{aligned}$$

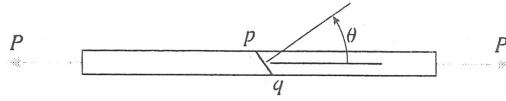
MAXIMUM SHEAR STRESS

$$\tau_{\max} = \frac{\sigma_x}{2} = 5,000 \text{ psi} \quad \leftarrow$$

Problem 2.6-18 A tension member is to be constructed of two pieces of plastic glued along plane pq (see figure). For purposes of cutting and gluing, the angle θ must be between 25° and 45° . The allowable stresses on the glued joint in tension and shear are 5.0 MPa and 3.0 MPa, respectively.

- Determine the angle θ so that the bar will carry the largest load P . (Assume that the strength of the glued joint controls the design.)
- Determine the maximum allowable load P_{\max} if the cross-sectional area of the bar is 225 mm^2 .



Solution 2.6-18 Bar in tension with glued joint

$$25^\circ < \theta < 45^\circ$$

$$A = 225 \text{ mm}^2$$

$$\text{On glued joint: } \sigma_{\text{allow}} = 5.0 \text{ MPa}$$

$$\tau_{\text{allow}} = 3.0 \text{ MPa}$$

ALLOWABLE STRESS σ_x IN TENSION

$$\sigma_\theta = \sigma_x \cos^2 \theta \quad \sigma_x = \frac{\sigma_\theta}{\cos^2 \theta} = \frac{5.0 \text{ MPa}}{\cos^2 \theta} \quad (1)$$

$$\tau_\theta = -\sigma_x \sin \theta \cos \theta$$

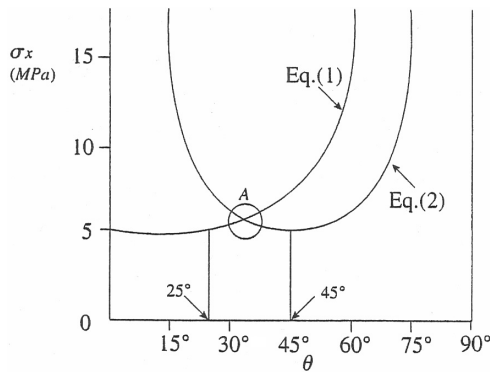
Since the direction of τ_θ is immaterial, we can write:

$$|\tau_\theta| = \sigma_x \sin \theta \cos \theta$$

or

$$\sigma_x = \frac{|\tau_\theta|}{\sin \theta \cos \theta} = \frac{3.0 \text{ MPa}}{\sin \theta \cos \theta} \quad (2)$$

GRAPH OF EQS. (1) AND (2)

**(a) DETERMINE ANGLE θ FOR LARGEST LOAD**

Point A gives the largest value of σ_x and hence the largest load. To determine the angle θ corresponding to point A, we equate Eqs. (1) and (2).

$$\frac{5.0 \text{ MPa}}{\cos^2 \theta} = \frac{3.0 \text{ MPa}}{\sin \theta \cos \theta}$$

$$\tan \theta = \frac{3.0}{5.0} \quad \theta = 30.96^\circ \quad \leftarrow$$

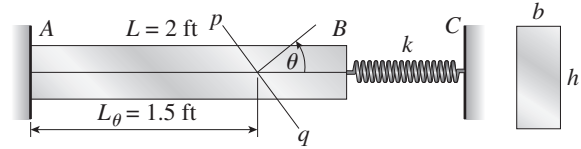
(b) DETERMINE THE MAXIMUM LOAD

From Eq. (1) or Eq. (2):

$$\sigma_x = \frac{5.0 \text{ MPa}}{\cos^2 \theta} = \frac{3.0 \text{ MPa}}{\sin \theta \cos \theta} = 6.80 \text{ MPa}$$

$$P_{\text{max}} = \sigma_x A = (6.80 \text{ MPa})(225 \text{ mm}^2) \\ = 1.53 \text{ kN} \quad \leftarrow$$

Problem 2.6-19 Plastic bar AB of rectangular cross section ($b = 0.75$ in. and $h = 1.5$ in.) and length $L = 2$ ft. is fixed at A and has a spring support ($k = 18$ k/in.) at C (see figure). Initially, the bar and spring have no stress. When the temperature of the bar is raised by 100°F , the compressive stress on an inclined plane pq at $L_\theta = 1.5$ ft becomes 950 psi. Assume the spring is massless and is unaffected by the temperature change. Let $\alpha = 55 \times 10^{-6}/^\circ\text{F}$ and $E = 400$ ksi.



- What is the shear stress τ_θ on plane pq ? What is angle θ ?
- Draw a stress element oriented to plane pq , and show the stresses acting on all faces of this element.
- If the allowable normal stress is ± 1000 psi and the allowable shear stress is ± 560 psi, what is the maximum permissible value of spring constant k if allowable stress values in the bar are not to be exceeded?
- What is the maximum permissible length L of the bar if allowable stress values in the bar are not to be exceeded? (Assume $k = 18$ k/in.)
- What is the maximum permissible temperature increase (ΔT) in the bar if allowable stress values in the bar are not to be exceeded? (Assume $L = 2$ ft and $k = 18$ k/in.)

Solution 2.6-19

NUMERICAL DATA

$$\alpha = 55 (10^{-6}) \quad E = 400 \text{ ksi} \quad L = 2 \text{ ft} \quad \Delta T = 100 \quad k = 18 \text{ k/in.} \quad b = 0.75 \text{ in.} \quad h = 1.5 \text{ in.}$$

$$\sigma_\theta = -950 \text{ psi} \quad \sigma_a = -1000 \text{ psi} \quad \tau_a = -560 \text{ psi} \quad L_\theta = 1.5 \text{ ft} \quad A = bh \quad f = \frac{1}{k} = 5.556 \times 10^{-5} \text{ in./lb}$$

- (a) FIND θ AND T_θ

$$R_2 = \text{redundant} \quad R_2 = \frac{-\alpha \Delta T L}{\left(\frac{L}{EA}\right) + f} = -1.212 \times 10^3 \text{ lb} \quad \sigma_x = \frac{R_2}{A} = -1077.551 \text{ psi} \quad \sqrt{\frac{\sigma_\theta}{\sigma_x}} = 0.939$$

$$\theta = \arccos\left(\sqrt{\frac{\sigma_\theta}{\sigma_x}}\right) = 0.351 \quad \cos(2\theta) = 0.763 \quad \theta = 20.124^\circ$$

$$\sigma_x \cos(\theta)^2 = -950 \text{ psi} \quad \text{or} \quad \frac{\sigma_x}{2}(1 + \cos(2\theta)) = -950 \text{ psi} \quad \sigma_y = \sigma_x \cos\left(\theta + \frac{\pi}{2}\right)^2 = -127.551 \text{ psi}$$

$$\theta = 0.351 \quad \theta = 20.124^\circ \quad \sigma_x = -1077.551 \text{ psi} \quad 2\theta = 0.702$$

$$\tau_\theta = -\sigma_x \sin(\theta) \cos(\theta) = 348.1 \text{ psi} \quad \text{or} \quad \tau_\theta = \frac{-\sigma_x}{2} \sin(2\theta) = 348.1 \text{ psi}$$

$$\boxed{\tau_\theta = 348 \text{ psi}} \quad \boxed{\theta = 20.1^\circ}$$

- (b) FIND σ_{x1} AND σ_{y1}

$$\sigma_{x1} = \sigma_x \cos(\theta)^2 \quad \sigma_{y1} = \sigma_x \cos\left(\theta + \frac{\pi}{2}\right)^2$$

$$\boxed{\sigma_{x1} = -950 \text{ psi}} \quad \boxed{\sigma_{y1} = -127.6 \text{ psi}}$$

(c) GIVEN $L = 2$ ft, FIND k_{\max}

$$k_{\max 1} = \frac{\sigma_a A}{-\alpha \Delta T L - \sigma_a A \left(\frac{L}{EA} \right)} = 15625 \text{ lb/in.} < \text{controls (based on } \sigma_{\text{allow}})$$

$$\text{or } k_{\max 2} = \frac{2\tau_a A}{-\alpha \Delta T L - 2\tau_a A \left(\frac{L}{EA} \right)} = 19444.444 \text{ lb/in.} < \text{based on allowable shear stress}$$

$$\boxed{k_{\max} = 15625 \text{ lb/in.}}$$

(d) GIVEN ALLOWABLE NORMAL AND SHEAR STRESSES, FIND L_{\max}

$$k = 18000 \text{ lb/in.}$$

$$\sigma_x = \frac{R_2}{A} \quad \sigma_a A = \frac{-\alpha \Delta T L}{\left(\frac{L}{EA} \right) + f} \quad L_{\max 1} = \frac{\sigma_a A (f)}{-\left(\alpha \Delta T + \frac{\sigma_a}{E} \right)} = 1.736 \text{ ft} < \text{controls (based on } \sigma_{\text{allow}})$$

$$\text{or } L_{\max 2} = \frac{2\tau_a A (f)}{-\left(\alpha \Delta T + \frac{2\tau_a}{E} \right)} = 2.16 \text{ ft} < \text{based on } T_{\text{allow}}$$

$$\boxed{L_{\max} = 1.736 \text{ ft}}$$

(e) FIND ΔT_{\max} GIVEN L , k , AND ALLOWABLE STRESSES $k = 18000 \text{ lb/in.}$ $L = 2$ ft $\sigma_a = -1000$ psi
 $\tau_a = -560$ psi

$$\Delta T_{\max 1} = \frac{\left(\frac{L}{EA} + f \right) \sigma_a A}{-\alpha L} = 92.803^\circ\text{F} < \text{based on } \sigma_{\text{allow}} \quad \Delta T = 100$$

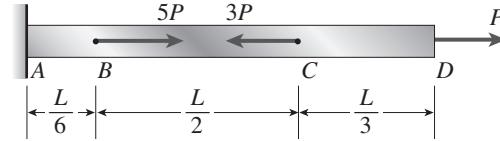
$$\Delta T_{\max 2} = \frac{\left(\frac{L}{EA} + f \right) 2\tau_a A}{-\alpha L} = 103.939^\circ\text{F} < \text{based on } T_{\text{allow}}$$

$$\boxed{\Delta T_{\max} = 92.8^\circ\text{F}}$$

Strain Energy

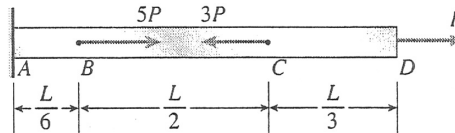
When solving the problems for Section 2.7, assume that the material behaves linearly elastically.

Problem 2.7-1 A prismatic bar AD of length L , cross-sectional area A , and modulus of elasticity E is subjected to loads $5P$, $3P$, and P acting at points B , C , and D , respectively (see figure). Segments AB , BC , and CD have lengths $L/6$, $L/2$, and $L/3$, respectively.



- Obtain a formula for the strain energy U of the bar.
- Calculate the strain energy if $P = 6$ k, $L = 52$ in., $A = 2.76$ in.², and the material is aluminum with $E = 10.4 \times 10^6$ psi.

Solution 2.7-1 Bar with three loads



$$P = 6 \text{ k}$$

$$L = 52 \text{ in.}$$

$$E = 10.4 \times 10^6 \text{ psi}$$

$$A = 2.76 \text{ in.}^2$$

INTERNAL AXIAL FORCES

$$N_{AB} = 3P \quad N_{BC} = -2P \quad N_{CD} = P$$

LENGTHS

$$L_{AB} = \frac{L}{6} \quad L_{BC} = \frac{L}{2} \quad L_{CD} = \frac{L}{3}$$

(a) STRAIN ENERGY OF THE BAR (EQ. 2-40)

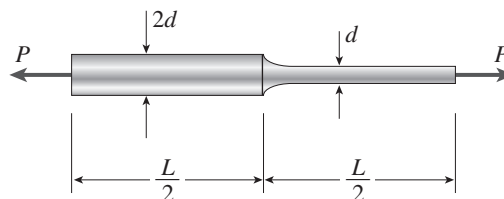
$$\begin{aligned} U &= \sum \frac{N_i^2 L_i}{2E_i A_i} \\ &= \frac{1}{2EA} \left[(3P)^2 \left(\frac{L}{6} \right) + (-2P)^2 \left(\frac{L}{2} \right) + (P)^2 \left(\frac{L}{3} \right) \right] \\ &= \frac{P^2 L}{2EA} \left(\frac{23}{6} \right) = \frac{23P^2 L}{12EA} \quad \leftarrow \end{aligned}$$

(b) SUBSTITUTE NUMERICAL VALUES:

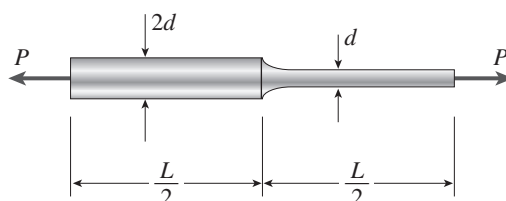
$$\begin{aligned} U &= \frac{23(6 \text{ k})^2(52 \text{ in.})}{12(10.4 \times 10^6 \text{ psi})(2.76 \text{ in.}^2)} \\ &= 125 \text{ in.-lb} \quad \leftarrow \end{aligned}$$

Problem 2.7-2 A bar of circular cross section having two different diameters d and $2d$ is shown in the figure. The length of each segment of the bar is $L/2$ and the modulus of elasticity of the material is E .

- Obtain a formula for the strain energy U of the bar due to the load P .
- Calculate the strain energy if the load $P = 27$ kN, the length $L = 600$ mm, the diameter $d = 40$ mm, and the material is brass with $E = 105$ GPa.



Solution 2.7-2 Bar with two segments



(a) STRAIN ENERGY OF THE BAR

Add the strain energies of the two segments of the bar (see Eq. 2-42).

$$U = \sum_{i=1}^2 \frac{N_i^2 L_i}{2 E_i A_i} = \frac{P^2 (L/2)}{2E} \left[\frac{1}{\frac{\pi}{4}(2d)^2} + \frac{1}{\frac{\pi}{4}(d^2)} \right]$$

$$= \frac{P^2 L}{\pi E} \left(\frac{1}{4d^2} + \frac{1}{d^2} \right) = \frac{5P^2 L}{4\pi E d^2} \quad \leftarrow$$

(b) SUBSTITUTE NUMERICAL VALUES:

$$P = 27 \text{ kN} \quad L = 600 \text{ mm}$$

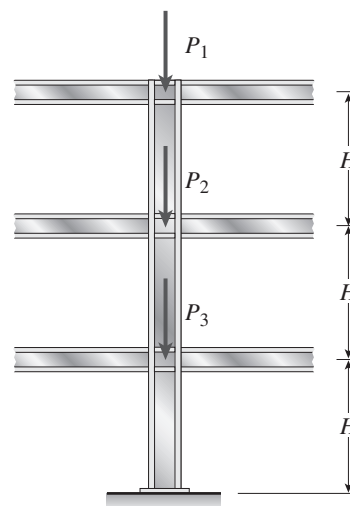
$$d = 40 \text{ mm} \quad E = 105 \text{ GPa}$$

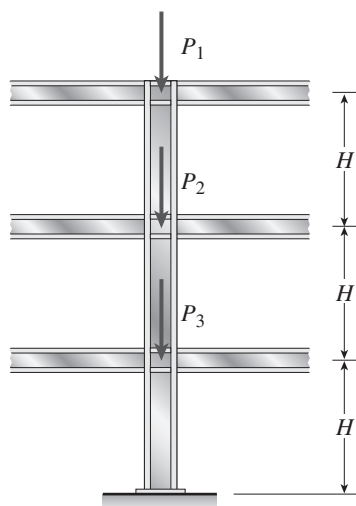
$$U = \frac{5(27 \text{ kN})^2(600 \text{ mm})}{4\pi(105 \text{ GPa})(40 \text{ mm})^2}$$

$$= 1.036 \text{ N} \cdot \text{m} = 1.036 \text{ J} \quad \leftarrow$$

Problem 2.7-3 A three-story steel column in a building supports roof and floor loads as shown in the figure. The story height H is 10.5 ft, the cross-sectional area A of the column is 15.5 in.^2 , and the modulus of elasticity E of the steel is 30×10^6 psi.

Calculate the strain energy U of the column assuming $P_1 = 40$ k and $P_2 = P_3 = 60$ k.



Solution 2.7-3 Three-story column

$$H = 10.5 \text{ ft} \quad E = 30 \times 10^6 \text{ psi}$$

$$A = 15.5 \text{ in.}^2 \quad P_1 = 40 \text{ k}$$

$$P_2 = P_3 = 60 \text{ k}$$

To find the strain energy of the column, add the strain energies of the three segments (see Eq. 2-42).

$$\text{Upper segment: } N_1 = -P_1$$

$$\text{Middle segment: } N_2 = -(P_1 + P_2)$$

$$\text{Lower segment: } N_3 = -(P_1 + P_2 + P_3)$$

STRAIN ENERGY

$$\begin{aligned} U &= \sum \frac{N_i^2 L_i}{2E_i A_i} \\ &= \frac{H}{2EA} [P_1^2 + (P_1 + P_2)^2 + (P_1 + P_2 + P_3)^2] \\ &= \frac{H}{2EA} [Q] \end{aligned}$$

$$[Q] = (40 \text{ k})^2 + (100 \text{ k})^2 + (160 \text{ k})^2 = 37,200 \text{ k}^2$$

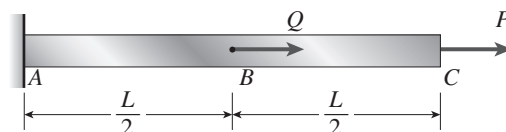
$$2EA = 2(30 \times 10^6 \text{ psi})(15.5 \text{ in.}^2) = 930 \times 10^6 \text{ lb}$$

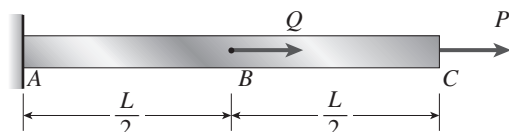
$$U = \frac{(10.5 \text{ ft})(12 \text{ in./ft})}{930 \times 10^6 \text{ lb}} [37,200 \text{ k}^2]$$

$$= 5040 \text{ in.-lb} \quad \leftarrow$$

Problem 2.7-4 The bar ABC shown in the figure is loaded by a force P acting at end C and by a force Q acting at the midpoint B . The bar has constant axial rigidity EA .

- Determine the strain energy U_1 of the bar when the force P acts alone ($Q = 0$).
- Determine the strain energy U_2 when the force Q acts alone ($P = 0$).
- Determine the strain energy U_3 when the forces P and Q act simultaneously upon the bar.



Solution 2.7-4 Bar with two loads(a) FORCE P ACTS ALONE ($Q = 0$)

$$U_1 = \frac{P^2 L}{2EA} \quad \leftarrow$$

(b) FORCE Q ACTS ALONE ($P = 0$)

$$U_2 = \frac{Q^2(L/2)}{2EA} = \frac{Q^2 L}{4EA} \quad \leftarrow$$

(c) FORCES P AND Q ACT SIMULTANEOUSLY

$$\text{Segment } BC: U_{BC} = \frac{P^2(L/2)}{2EA} = \frac{P^2 L}{4EA}$$

$$\begin{aligned} \text{Segment } AB: U_{AB} &= \frac{(P+Q)^2(L/2)}{2EA} \\ &= \frac{P^2 L}{4EA} + \frac{PQL}{2EA} + \frac{Q^2 L}{4EA} \end{aligned}$$

$$U_3 = U_{BC} + U_{AB} = \frac{P^2 L}{2EA} + \frac{PQL}{2EA} + \frac{Q^2 L}{4EA} \quad \leftarrow$$

(Note that U_3 is *not* equal to $U_1 + U_2$. In this case, $U_3 > U_1 + U_2$. However, if Q is reversed in direction, $U_3 < U_1 + U_2$. Thus, U_3 may be larger or smaller than $U_1 + U_2$.)

Problem 2.7-5 Determine the strain energy per unit volume (units of psi) and the strain energy per unit weight (units of in.) that can be stored in each of the materials listed in the accompanying table, assuming that the material is stressed to the proportional limit.

DATA FOR PROBLEM 2.7-5

Material	Weight density (lb/in. ³)	Modulus of elasticity (ksi)	Proportional limit (psi)
Mild steel	0.284	30,000	36,000
Tool steel	0.284	30,000	75,000
Aluminum	0.0984	10,500	60,000
Rubber (soft)	0.0405	0.300	300

Solution 2.7-5 Strain-energy density**DATA:**

Material	Weight density (lb/in. ³)	Modulus of elasticity (ksi)	Proportional limit (psi)
Mild steel	0.284	30,000	36,000
Tool steel	0.284	30,000	75,000
Aluminum	0.0984	10,500	60,000
Rubber (soft)	0.0405	0.300	300

STRAIN ENERGY PER UNIT VOLUME

$$U = \frac{P^2 L}{2EA} \quad \text{Volume } V = AL$$

$$\text{Stress } \sigma = \frac{P}{A}$$

$$u = \frac{U}{V} = \frac{\sigma_{PL}^2}{2E}$$

At the proportional limit:

$u = u_R = \text{modulus of resistance}$

$$u_R = \frac{\sigma_{PL}^2}{2E} \quad (\text{Eq. 1})$$

STRAIN ENERGY PER UNIT WEIGHT

$$U = \frac{P^2 L}{2EA} \quad \text{Weight } W = \gamma AL$$

$\gamma = \text{weight density}$

$$u_W = \frac{U}{W} = \frac{\sigma^2}{2\gamma E}$$

At the proportional limit:

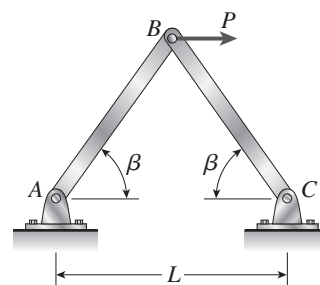
$$u_W = \frac{\sigma_{PL}^2}{2\gamma E} \quad (\text{Eq. 2})$$

RESULTS

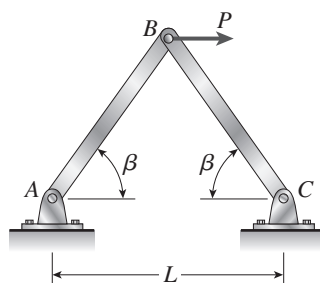
	u_R (psi)	u_W (in.)
Mild steel	22	76
Tool steel	94	330
Aluminum	171	1740
Rubber (soft)	150	3700

Problem 2.7-6 The truss ABC shown in the figure is subjected to a horizontal load P at joint B . The two bars are identical with cross-sectional area A and modulus of elasticity E .

- Determine the strain energy U of the truss if the angle $\beta = 60^\circ$.
- Determine the horizontal displacement δ_B of joint B by equating the strain energy of the truss to the work done by the load.



Solution 2.7-6 Truss subjected to a load P



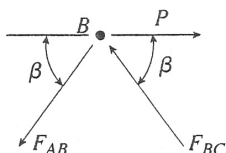
$$\beta = 60^\circ$$

$$L_{AB} = L_{BC} = L$$

$$\sin \beta = \sqrt{3}/2$$

$$\cos \beta = 1/2$$

FREE-BODY DIAGRAM OF JOINT B



$$\Sigma F_{\text{vert}} = 0 \quad \uparrow + \quad \downarrow -$$

$$-F_{AB} \sin \beta + F_{BC} \sin \beta = 0$$

$$F_{AB} = F_{BC} \quad (\text{Eq. 1})$$

$$\Sigma F_{\text{horiz}} = 0 \rightarrow \leftarrow$$

$$-F_{AB} \cos \beta - F_{BC} \cos \beta + P = 0$$

$$F_{AB} = F_{BC} = \frac{P}{2 \cos \beta} = \frac{P}{2(1/2)} = P \quad (\text{Eq. 2})$$

Axial forces: $N_{AB} = P$ (tension)

$N_{BC} = -P$ (compression)

(a) STRAIN ENERGY OF TRUSS (EQ. 2-42)

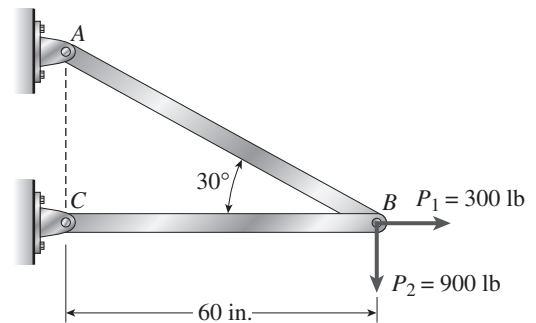
$$U = \sum \frac{N_i^2 L_i}{2E_i A_i} = \frac{(N_{AB})^2 L}{2EA} + \frac{(N_{BC})^2 L}{2EA} = \frac{P^2 L}{EA} \leftarrow$$

(b) HORIZONTAL DISPLACEMENT OF JOINT B (EQ. 2-44)

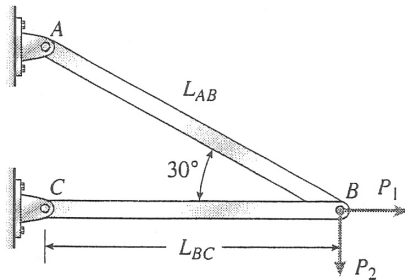
$$\delta_B = \frac{2U}{P} = \frac{2}{P} \left(\frac{P^2 L}{EA} \right) = \frac{2PL}{EA} \leftarrow$$

Problem 2.7-7 The truss ABC shown in the figure supports a horizontal load $P_1 = 300$ lb and a vertical load $P_2 = 900$ lb. Both bars have cross-sectional area $A = 2.4$ in.² and are made of steel with $E = 30 \times 10^6$ psi.

- Determine the strain energy U_1 of the truss when the load P_1 acts alone ($P_2 = 0$).
- Determine the strain energy U_2 when the load P_2 acts alone ($P_1 = 0$).
- Determine the strain energy U_3 when both loads act simultaneously.



Solution 2.7-7 Truss with two loads



$$P_1 = 300 \text{ lb}$$

$$P_2 = 900 \text{ lb}$$

$$A = 2.4 \text{ in.}^2$$

$$E = 30 \times 10^6 \text{ psi}$$

$$L_{BC} = 60 \text{ in.}$$

$$\beta = 30^\circ$$

$$\sin \beta = \sin 30^\circ = \frac{1}{2}$$

$$\cos \beta = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$L_{AB} = \frac{L_{BC}}{\cos 30^\circ} = \frac{120}{\sqrt{3}} \text{ in.} = 69.282 \text{ in.}$$

$$2EA = 2(30 \times 10^6 \text{ psi})(2.4 \text{ in.}^2) = 144 \times 10^6 \text{ lb}$$

FORCES F_{AB} AND F_{BC} IN THE BARS

From equilibrium of joint B :

$$F_{AB} = 2P_2 = 1800 \text{ lb}$$

$$F_{BC} = P_1 - P_2\sqrt{3} = 300 \text{ lb} - 1558.8 \text{ lb}$$

Force	P_1 alone	P_2 alone	P_1 and P_2
F_{AB}	0	1800 lb	1800 lb
F_{BC}	300 lb	-1558.8 lb	-1258.8 lb

(a) LOAD P_1 ACTS ALONE

$$U_1 = \frac{(F_{BC})^2 L_{BC}}{2EA} = \frac{(300 \text{ lb})^2 (60 \text{ in.})}{144 \times 10^6 \text{ lb}}$$

$$= 0.0375 \text{ in.-lb} \leftarrow$$

(b) LOAD P_2 ACTS ALONE

$$\begin{aligned}
 U_2 &= \frac{1}{2EA} \left[(F_{AB})^2 L_{AB} + (F_{BC})^2 L_{BC} \right] \\
 &= \frac{1}{2EA} \left[(1800 \text{ lb})^2 (69.282 \text{ in.}) \right. \\
 &\quad \left. + (-1558.8 \text{ lb})^2 (60 \text{ in.}) \right] \\
 &= \frac{370.265 \times 10^6 \text{ lb}^2\text{-in.}}{144 \times 10^6 \text{ lb}} = 2.57 \text{ in.-lb} \leftarrow
 \end{aligned}$$

(c) LOADS P_1 AND P_2 ACT SIMULTANEOUSLY

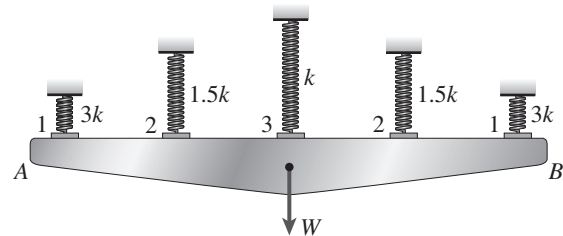
$$U_3 = \frac{1}{2EA} \left[(F_{AB})^2 L_{AB} + (F_{BC})^2 L_{BC} \right]$$

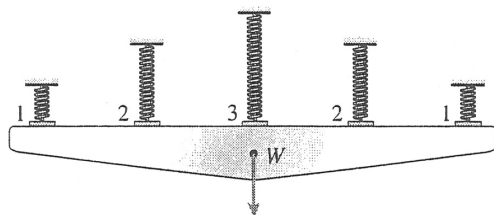
$$\begin{aligned}
 &= \frac{1}{2EA} \left[(1800 \text{ lb})^2 (69.282 \text{ in.}) \right. \\
 &\quad \left. + (-1258.8 \text{ lb})^2 (60 \text{ in.}) \right] \\
 &= \frac{319.548 \times 10^6 \text{ lb}^2\text{-in.}}{144 \times 10^6 \text{ lb}} \\
 &= 2.22 \text{ in.-lb} \leftarrow
 \end{aligned}$$

NOTE: The strain energy U_3 is *not* equal to $U_1 + U_2$.

Problem 2.7-8 The statically indeterminate structure shown in the figure consists of a horizontal rigid bar AB supported by five equally spaced springs. Springs 1, 2, and 3 have stiffnesses $3k$, $1.5k$, and k , respectively. When unstressed, the lower ends of all five springs lie along a horizontal line. Bar AB , which has weight W , causes the springs to elongate by an amount δ .

- Obtain a formula for the total strain energy U of the springs in terms of the downward displacement δ of the bar.
- Obtain a formula for the displacement δ by equating the strain energy of the springs to the work done by the weight W .
- Determine the forces F_1 , F_2 , and F_3 in the springs.
- Evaluate the strain energy U , the displacement δ , and the forces in the springs if $W = 600 \text{ N}$ and $k = 7.5 \text{ N/mm}$.



Solution 2.7-8 Rigid bar supported by springs

$$k_1 = 3k$$

$$k_2 = 1.5k$$

$$k_3 = k$$

δ = downward displacement of rigid bar

For a spring: $U = \frac{k\delta^2}{2}$ Eq. (2-40b)

(a) STRAIN ENERGY U OF ALL SPRINGS

$$U = 2\left(\frac{3k\delta^2}{2}\right) + 2\left(\frac{1.5k\delta^2}{2}\right) + \frac{k\delta^2}{2} = 5k\delta^2 \quad \leftarrow$$

(b) DISPLACEMENT δ

Work done by the weight W equals $\frac{W\delta}{2}$

Strain energy of the springs equals $5k\delta^2$

$$\therefore \frac{W\delta}{2} = 5k\delta^2 \quad \text{and} \quad \delta = \frac{W}{10k} \quad \leftarrow$$

(c) FORCES IN THE SPRINGS

$$F_1 = 3k\delta = \frac{3W}{10} \quad F_2 = 1.5k\delta = \frac{3W}{20} \quad \leftarrow$$

$$F_3 = k\delta = \frac{W}{10} \quad \leftarrow$$

(d) NUMERICAL VALUES

$$W = 600 \text{ N} \quad k = 7.5 \text{ N/mm} = 7500 \text{ N/mm}$$

$$U = 5k\delta^2 = 5k\left(\frac{W}{10k}\right)^2 = \frac{W^2}{20k}$$

$$= 2.4 \text{ N}\cdot\text{m} = 2.4 \text{ J} \quad \leftarrow$$

$$\delta = \frac{W}{10k} = 8.0 \text{ mm} \quad \leftarrow$$

$$F_1 = \frac{3W}{10} = 180 \text{ N} \quad \leftarrow$$

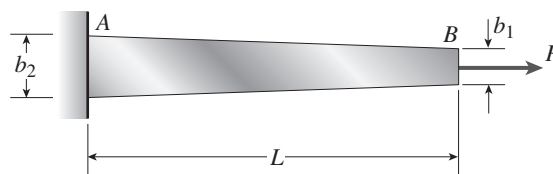
$$F_2 = \frac{3W}{20} = 90 \text{ N} \quad \leftarrow$$

$$F_3 = \frac{W}{10} = 60 \text{ N} \quad \leftarrow$$

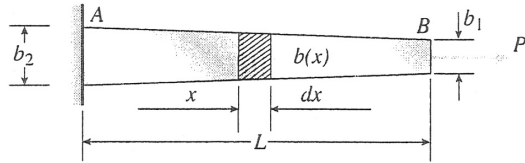
NOTE: $W = 2F_1 + 2F_2 + F_3 = 600 \text{ N}$ (Check)

Problem 2.7-9 A slightly tapered bar AB of rectangular cross section and length L is acted upon by a force P (see figure). The width of the bar varies uniformly from b_2 at end A to b_1 at end B . The thickness t is constant.

- Determine the strain energy U of the bar.
- Determine the elongation δ of the bar by equating the strain energy to the work done by the force P .



Solution 2.7-9 Tapered bar of rectangular cross section



$$b(x) = b_2 - \frac{(b_2 - b_1)x}{L}$$

$$A(x) = tb(x)$$

$$= t \left[b_2 - \frac{(b_2 - b_1)x}{L} \right]$$

(a) STRAIN ENERGY OF THE BAR

$$U = \int_0^L \frac{[N(x)]^2 dx}{2EA(x)} \quad (\text{Eq. 2-43})$$

$$= \int_0^L \frac{P^2 dx}{2Et b(x)} = \frac{P^2}{2Et} \int_0^L \frac{dx}{b_2 - (b_2 - b_1)\frac{x}{L}} \quad (1)$$

$$\text{From Appendix C: } \int \frac{dx}{a + bx} = \frac{1}{b} \ln(a + bx)$$

Apply this integration formula to Eq. (1):

$$U = \frac{P^2}{2Et} \left[\frac{1}{-(b_2 - b_1)(\frac{1}{L})} \ln \left[b_2 - \frac{(b_2 - b_1)x}{L} \right] \right]_0^L$$

$$= \frac{P^2}{2Et} \left[\frac{-L}{(b_2 - b_1)} \ln b_1 - \frac{-L}{(b_2 - b_1)} \ln b_2 \right]$$

$$U = \frac{P^2 L}{2Et(b_2 - b_1)} \ln \frac{b_2}{b_1} \quad \leftarrow$$

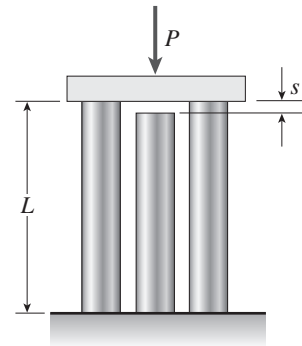
(b) ELONGATION OF THE BAR (Eq. 2-44)

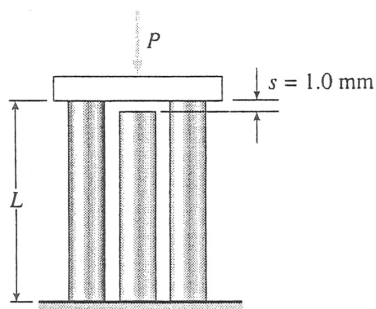
$$\delta = \frac{2U}{P} = \frac{PL}{Et(b_2 - b_1)} \ln \frac{b_2}{b_1} \quad \leftarrow$$

NOTE: This result agrees with the formula derived in Prob. 2.3-13.

Problem 2.7-10 A compressive load P is transmitted through a rigid plate to three magnesium-alloy bars that are identical except that initially the middle bar is slightly shorter than the other bars (see figure). The dimensions and properties of the assembly are as follows: length $L = 1.0$ m, cross-sectional area of each bar $A = 3000 \text{ mm}^2$, modulus of elasticity $E = 45 \text{ GPa}$, and the gap $s = 1.0$ mm.

- Calculate the load P_1 required to close the gap.
- Calculate the downward displacement δ of the rigid plate when $P = 400 \text{ kN}$.
- Calculate the total strain energy U of the three bars when $P = 400 \text{ kN}$.
- Explain why the strain energy U is *not* equal to $P\delta/2$.
(Hint: Draw a load-displacement diagram.)



Solution 2.7-10 Three bars in compression

$$s = 1.0 \text{ mm}$$

$$L = 1.0 \text{ m}$$

For each bar:

$$A = 3000 \text{ mm}^2$$

$$E = 45 \text{ GPa}$$

$$\frac{EA}{L} = 135 \times 10^6 \text{ N/m}$$

(a) LOAD P_1 REQUIRED TO CLOSE THE GAP

$$\text{In general, } \delta = \frac{PL}{EA} \text{ and } P = \frac{EA\delta}{L}$$

For two bars, we obtain:

$$P_1 = 2\left(\frac{EAs}{L}\right) = 2(135 \times 10^6 \text{ N/m})(1.0 \text{ mm})$$

$$P_1 = 270 \text{ kN} \quad \leftarrow$$

(b) DISPLACEMENT δ FOR $P = 400 \text{ kN}$

Since $P > P_1$, all three bars are compressed.

The force P equals P_1 plus the additional force required to compress all three bars by the amount $\delta - s$.

$$P = P_1 + 3\left(\frac{EA}{L}\right)(\delta - s)$$

$$\text{or } 400 \text{ kN} = 270 \text{ kN} + 3(135 \times 10^6 \text{ N/m})(\delta - 0.001 \text{ m})$$

$$\text{Solving, we get } \delta = 1.321 \text{ mm} \quad \leftarrow$$

(c) STRAIN ENERGY U FOR $P = 400 \text{ kN}$

$$U = \sum \frac{EA\delta^2}{2L}$$

$$\text{Outer bars: } \delta = 1.321 \text{ mm}$$

$$\begin{aligned} \text{Middle bar: } \delta &= 1.321 \text{ mm} - s \\ &= 0.321 \text{ mm} \end{aligned}$$

$$U = \frac{EA}{2L}[2(1.321 \text{ mm})^2 + (0.321 \text{ mm})^2]$$

$$= \frac{1}{2}(135 \times 10^6 \text{ N/m})(3.593 \text{ mm}^2)$$

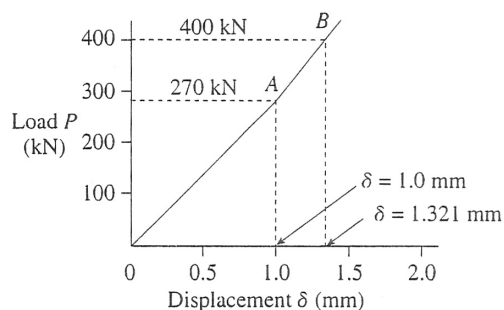
$$= 243 \text{ N}\cdot\text{m} = 243 \text{ J} \quad \leftarrow$$

(d) LOAD-DISPLACEMENT DIAGRAM

$$U = 243 \text{ J} = 243 \text{ N}\cdot\text{m}$$

$$\frac{P\delta}{2} = \frac{1}{2}(400 \text{ kN})(1.321 \text{ mm}) = 264 \text{ N}\cdot\text{m}$$

The strain energy U is *not* equal to $\frac{P\delta}{2}$ = because the load-displacement relation is not linear.

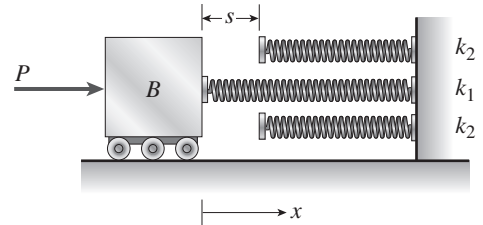


$$U = \text{area under line } OAB.$$

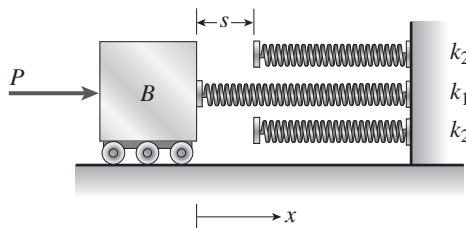
$$\frac{P\delta}{2} = \text{area under a straight line from } O \text{ to } B, \text{ which is larger than } U.$$

Problem 2.7-11 A block B is pushed against three springs by a force P (see figure). The middle spring has stiffness k_1 and the outer springs each have stiffness k_2 . Initially, the springs are unstressed and the middle spring is longer than the outer springs (the difference in length is denoted s).

- Draw a force-displacement diagram with the force P as ordinate and the displacement x of the block as abscissa.
- From the diagram, determine the strain energy U_1 of the springs when $x = 2s$.
- Explain why the strain energy U_1 is not equal to $P\delta/2$, where $\delta = 2s$.



Solution 2.7-11 Block pushed against three springs



Force P_0 required to close the gap:

$$P_0 = k_1 s \quad (1)$$

FORCE-DISPLACEMENT RELATION BEFORE GAP IS CLOSED

$$P = k_1 x \quad (0 \leq x \leq s) \quad (0 \leq P \leq P_0) \quad (2)$$

FORCE-DISPLACEMENT RELATION AFTER GAP IS CLOSED

All three springs are compressed. Total stiffness equals $k_1 + 2k_2$. Additional displacement equals $x - s$. Force P equals P_0 plus the force required to compress all three springs by the amount $x - s$.

$$P = P_0 + (k_1 + 2k_2)(x - s)$$

$$= k_1 s + (k_1 + 2k_2)x - k_1 s - 2k_2 s$$

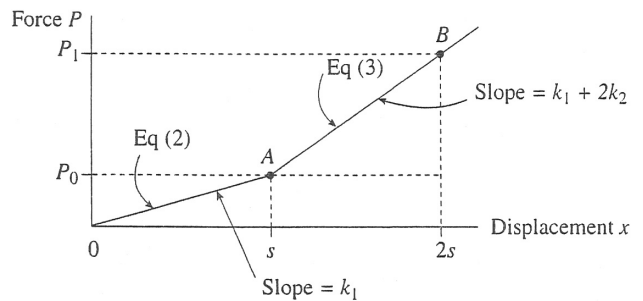
$$P = (k_1 + 2k_2)x - 2k_2 s \quad (x \geq s); \quad (P \geq P_0) \quad (3)$$

$$P_1 = \text{force } P \text{ when } x = 2s$$

Substitute $x = 2s$ into Eq. (3):

$$P_1 = 2(k_1 + k_2)s \quad (4)$$

(a) FORCE-DISPLACEMENT DIAGRAM



(b) STRAIN ENERGY U_1 WHEN $x = 2s$

$U_1 = \text{Area below force-displacement curve}$

$$= \triangle + \square + \triangle$$

$$= \frac{1}{2}P_0 s + P_0 s + \frac{1}{2}(P_1 - P_0)s = P_0 s + \frac{1}{2}P_1 s$$

$$= k_1 s^2 + (k_1 + k_2)s^2$$

$$U_1 = (2k_1 + k_2)s^2 \quad \leftarrow \quad (5)$$

(c) STRAIN ENERGY U_1 IS NOT EQUAL TO $\frac{P\delta}{2}$

$$\text{For } \delta = 2s: \frac{P\delta}{2} = \frac{1}{2} P_1(2s) = P_1s = 2(k_1 + k_2)s^2$$

(This quantity is greater than U_1 .)

U_1 = area under line OAB .

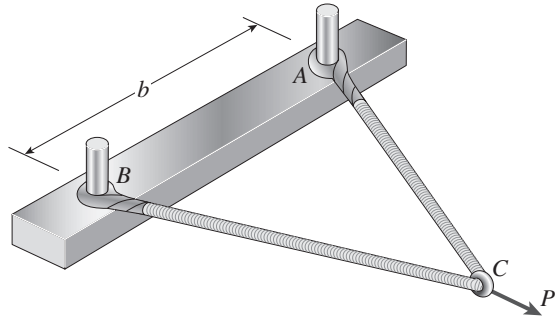
$\frac{P\delta}{2}$ = area under a straight line from O to B , which is larger than U_1 .

Thus, $\frac{P\delta}{2}$ is *not* equal to the strain energy because the force-displacement relation is not linear.

Problem 2.7-12 A bungee cord that behaves linearly elastically has an unstressed length $L_0 = 760$ mm and a stiffness $k = 140$ N/m. The cord is attached to two pegs, distance $b = 380$ mm apart, and pulled at its midpoint by a force $P = 80$ N (see figure).

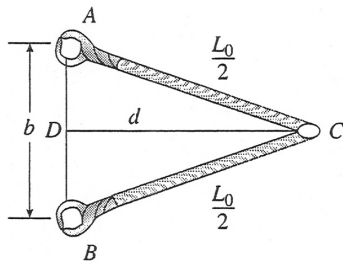
- How much strain energy U is stored in the cord?
- What is the displacement δ_C of the point where the load is applied?
- Compare the strain energy U with the quantity $P\delta_C/2$.

(Note: The elongation of the cord is *not* small compared to its original length.)



Solution 2.7-12 Bungee cord subjected to a load P .

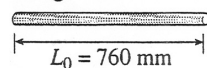
DIMENSIONS BEFORE THE LOAD P IS APPLIED



$$L_0 = 760 \text{ mm} \quad \frac{L_0}{2} = 380 \text{ mm}$$

$$b = 380 \text{ mm}$$

Bungee cord:

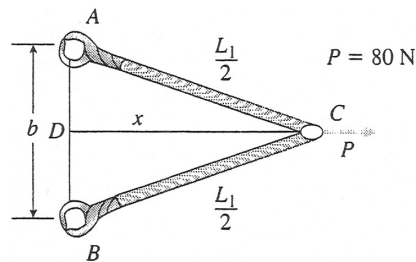


$$k = 140 \text{ N/m}$$

From triangle ACD :

$$d = \frac{1}{2} \sqrt{L_0^2 - b^2} = 329.09 \text{ mm} \quad (1)$$

DIMENSIONS AFTER THE LOAD P IS APPLIED



Let x = distance CD

Let L_1 = stretched length of bungee cord

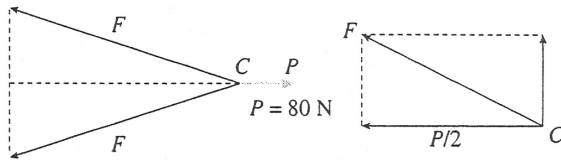
From triangle ACD :

$$\frac{L_1}{2} = \sqrt{\left(\frac{b}{2}\right)^2 + x^2} \quad (2)$$

$$L_1 = \sqrt{b^2 + 4x^2} \quad (3)$$

EQUILIBRIUM AT POINT C

Let F = tensile force in bungee cord



$$\begin{aligned} \frac{F}{P/2} &= \frac{L_1/2}{x} \quad F = \left(\frac{P}{2}\right)\left(\frac{L_1}{2}\right)\left(\frac{1}{x}\right) \\ &= \frac{P}{2}\sqrt{1 + \left(\frac{b}{2x}\right)^2} \end{aligned} \quad (4)$$

ELONGATION OF BUNGEE CORD

Let δ = elongation of the entire bungee cord

$$\delta = \frac{F}{k} = \frac{P}{2k}\sqrt{1 + \frac{b^2}{4x^2}} \quad (5)$$

Final length of bungee cord = original length + δ

$$L_1 = L_0 + \delta = L_0 + \frac{P}{2k}\sqrt{1 + \frac{b^2}{4x^2}} \quad (6)$$

SOLUTION OF EQUATIONS

Combine Eqs. (6) and (3):

$$L_1 = L_0 + \frac{P}{2k}\sqrt{1 + \frac{b^2}{4x^2}} = \sqrt{b^2 + 4x^2}$$

$$\text{or } L_1 = L_0 + \frac{P}{4kx}\sqrt{b^2 + 4x^2} = \sqrt{b^2 + 4x^2}$$

$$L_0 = \left(1 - \frac{P}{4kx}\right)\sqrt{b^2 + 4x^2} \quad (7)$$

This equation can be solved for x .

SUBSTITUTE NUMERICAL VALUES INTO EQ. (7):

$$\begin{aligned} 760 \text{ mm} &= \left[1 - \frac{(80 \text{ N})(1000 \text{ mm/m})}{4(140 \text{ N/m})x}\right] \\ &\quad \times \sqrt{(380 \text{ mm})^2 + 4x^2} \end{aligned} \quad (8)$$

$$760 = \left(1 - \frac{142.857}{x}\right)\sqrt{144,400 + 4x^2} \quad (9)$$

Units: x is in millimeters

Solve for x (Use trial-and-error or a computer program):

$$x = 497.88 \text{ mm}$$

(a) STRAIN ENERGY U OF THE BUNGEE CORD

$$U = \frac{k\delta^2}{2} \quad k = 140 \text{ N/m} \quad P = 80 \text{ N}$$

From Eq. (5):

$$\delta = \frac{P}{2k}\sqrt{1 + \frac{b^2}{4x^2}} = 305.81 \text{ mm}$$

$$U = \frac{1}{2}(140 \text{ N/m})(305.81 \text{ mm})^2 = 6.55 \text{ N}\cdot\text{m}$$

$$U = 6.55 \text{ J} \quad \leftarrow$$

(b) DISPLACEMENT δ_C OF POINT C

$$\begin{aligned} \delta_C &= x - d = 497.88 \text{ mm} - 329.09 \text{ mm} \\ &= 168.8 \text{ mm} \quad \leftarrow \end{aligned}$$

- (c) COMPARISON OF STRAIN ENERGY U WITH THE QUANTITY $P\delta_C/2$

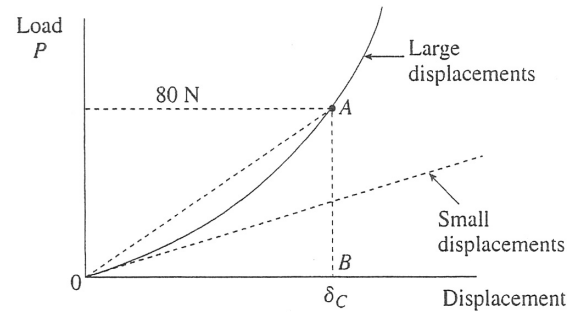
$$U = 6.55 \text{ J}$$

$$\frac{P\delta_C}{2} = \frac{1}{2}(80 \text{ N})(168.8 \text{ mm}) = 6.75 \text{ J}$$

The two quantities are not the same. The work done by the load P is *not* equal to $P\delta_C/2$ because the load-displacement relation (see below) is non-linear when the displacements are large. (The *work* done by the load P is equal to the strain energy because the bungee cord behaves elastically and there are no energy losses.)

$$U = \text{area } OAB \text{ under the curve } OA.$$

$$\frac{P\delta_C}{2} = \text{area of triangle } OAB, \text{ which is greater than } U.$$

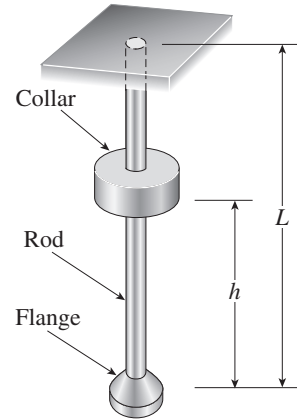


Impact Loading

The problems for Section 2.8 are to be solved on the basis of the assumptions and idealizations described in the text. In particular, assume that the material behaves linearly elastically and no energy is lost during the impact.

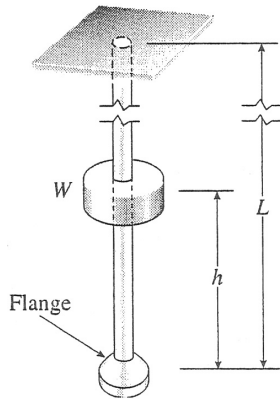
Problem 2.8-1 A sliding collar of weight $W = 150$ lb falls from a height $h = 2.0$ in. onto a flange at the bottom of a slender vertical rod (see figure). The rod has length $L = 4.0$ ft, cross-sectional area $A = 0.75$ in.², and modulus of elasticity $E = 30 \times 10^6$ psi.

Calculate the following quantities: (a) the maximum downward displacement of the flange, (b) the maximum tensile stress in the rod, and (c) the impact factor.



Probs. 2.8-1, 2.8-2, 2.8-3

Solution 2.8-1 Collar falling onto a flange



$$W = 150 \text{ lb}$$

$$h = 2.0 \text{ in.}$$

$$E = 30 \times 10^6 \text{ psi}$$

$$L = 4.0 \text{ ft} = 48 \text{ in.}$$

$$A = 0.75 \text{ in.}^2$$

(a) DOWNWARD DISPLACEMENT OF FLANGE

$$\delta_{st} = \frac{WL}{EA} = 0.00032 \text{ in.}$$

Eq. (2-55):

$$\begin{aligned} \delta_{\max} &= \delta_{st} \left[1 + \left(1 + \frac{2h}{\delta_{st}} \right)^{1/2} \right] \\ &= 0.0361 \text{ in.} \quad \leftarrow \end{aligned}$$

(b) MAXIMUM TENSILE STRESS (EQ. 2-57)

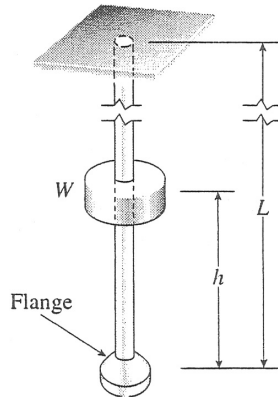
$$\sigma_{\max} = \frac{E\delta_{\max}}{L} = 22,600 \text{ psi} \quad \leftarrow$$

(c) IMPACT FACTOR (EQ. 2-63)

$$\begin{aligned} \text{Impact factor} &= \frac{\delta_{\max}}{\delta_{st}} = \frac{0.0361 \text{ in.}}{0.00032 \text{ in.}} \\ &= 113 \quad \leftarrow \end{aligned}$$

Problem 2.8-2 Solve the preceding problem if the collar has mass $M = 80$ kg, the height $h = 0.5$ m, the length $L = 3.0$ m, the cross-sectional area $A = 350$ mm², and the modulus of elasticity $E = 170$ GPa.

Solution 2.8-2 Collar falling onto a flange



$$\begin{aligned}
 M &= 80 \text{ kg} \\
 W &= Mg = (80 \text{ kg})(9.81 \text{ m/s}^2) \\
 &= 784.8 \text{ N} \\
 h &= 0.5 \text{ m} \quad L = 3.0 \text{ m} \\
 E &= 170 \text{ GPa} \quad A = 350 \text{ mm}^2
 \end{aligned}$$

(a) DOWNWARD DISPLACEMENT OF FLANGE

$$\delta_{st} = \frac{WL}{EA} = 0.03957 \text{ mm}$$

$$\begin{aligned}
 \text{Eq. (2-53): } \delta_{\max} &= \delta_{st} \left[1 + \left(1 + \frac{2h}{\delta_{st}} \right)^{1/2} \right] \\
 &= 6.33 \text{ mm} \quad \leftarrow
 \end{aligned}$$

(b) MAXIMUM TENSILE STRESS (EQ. 2-57)

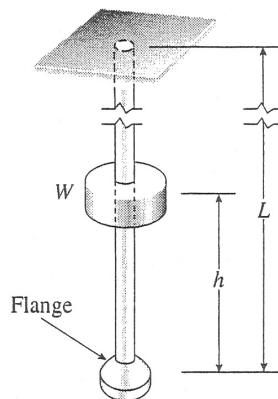
$$\sigma_{\max} = \frac{E\delta_{\max}}{L} = 359 \text{ MPa} \quad \leftarrow$$

(c) IMPACT FACTOR (EQ. 2-63)

$$\begin{aligned}
 \text{Impact factor} &= \frac{\delta_{\max}}{\delta_{st}} = \frac{6.33 \text{ mm}}{0.03957 \text{ mm}} \\
 &= 160 \quad \leftarrow
 \end{aligned}$$

Problem 2.8-3 Solve Problem 2.8-1 if the collar has weight $W = 50$ lb, the height $h = 2.0$ in., the length $L = 3.0$ ft, the cross-sectional area $A = 0.25$ in.², and the modulus of elasticity $E = 30,000$ ksi.

Solution 2.8-3 Collar falling onto a flange



$$W = 50 \text{ lb} \quad h = 2.0 \text{ in.}$$

$$L = 3.0 \text{ ft} = 36 \text{ in.}$$

$$E = 30,000 \text{ psi} \quad A = 0.25 \text{ in.}^2$$

(a) DOWNWARD DISPLACEMENT OF FLANGE

$$\delta_{st} = \frac{WL}{EA} = 0.00024 \text{ in.}$$

$$\begin{aligned}
 \text{Eq. (2-55): } \delta_{\max} &= \delta_{st} \left[1 + \left(1 + \frac{2h}{\delta_{st}} \right)^{1/2} \right] \\
 &= 0.0312 \text{ in.} \quad \leftarrow
 \end{aligned}$$

(b) MAXIMUM TENSILE STRESS (EQ. 2-57)

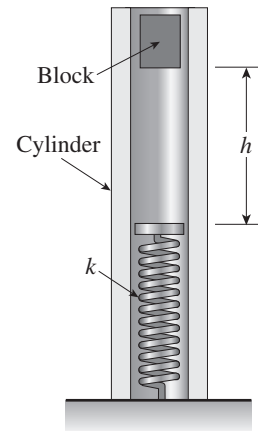
$$\sigma_{\max} = \frac{E\delta_{\max}}{L} = 26,000 \text{ psi} \quad \leftarrow$$

(c) IMPACT FACTOR (EQ. 2-63)

$$\text{Impact factor} = \frac{\delta_{\max}}{\delta_{st}} = \frac{0.0312 \text{ in.}}{0.00024 \text{ in.}} = 130 \quad \leftarrow$$

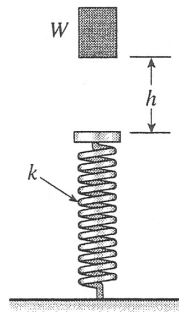
Problem 2.8-4 A block weighing $W = 5.0 \text{ N}$ drops inside a cylinder from a height $h = 200 \text{ mm}$ onto a spring having stiffness $k = 90 \text{ N/m}$ (see figure).

- (a) Determine the maximum shortening of the spring due to the impact and (b) determine the impact factor.



Prob. 2.8-4 and 2.8-5

Solution 2.8-4 Block dropping onto a spring



$$W = 5.0 \text{ N} \quad h = 200 \text{ mm} \quad k = 90 \text{ N/m}$$

(a) MAXIMUM SHORTENING OF THE SPRING

$$\delta_{st} = \frac{W}{k} = \frac{5.0 \text{ N}}{90 \text{ N/m}} = 55.56 \text{ mm}$$

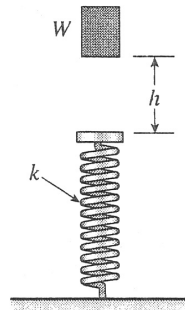
$$\begin{aligned} \text{Eq. (2-55): } \delta_{\max} &= \delta_{st} \left[1 + \left(1 + \frac{2h}{\delta_{st}} \right)^{1/2} \right] \\ &= 215 \text{ mm} \quad \leftarrow \end{aligned}$$

(b) IMPACT FACTOR (EQ. 2-63)

$$\begin{aligned} \text{Impact factor} &= \frac{\delta_{\max}}{\delta_{st}} = \frac{215 \text{ mm}}{55.56 \text{ mm}} \\ &= 3.9 \quad \leftarrow \end{aligned}$$

Problem 2.8-5 Solve the preceding problem if the block weighs $W = 1.0 \text{ lb}$, $h = 12 \text{ in.}$, and $k = 0.5 \text{ lb/in.}$

Solution 2.8-5 Block dropping onto a spring



$$W = 1.0 \text{ lb} \quad h = 12 \text{ in.} \quad k = 0.5 \text{ lb/in.}$$

(a) MAXIMUM SHORTENING OF THE SPRING

$$\delta_{st} = \frac{W}{k} = \frac{1.0 \text{ lb}}{0.5 \text{ lb/in.}} = 2.0 \text{ in.}$$

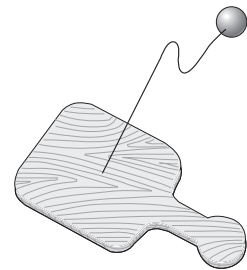
$$\begin{aligned} \text{Eq. (2-55): } \delta_{\max} &= \delta_{st} \left[1 + \left(1 + \frac{2h}{\delta_{st}} \right)^{1/2} \right] \\ &= 9.21 \text{ in.} \quad \leftarrow \end{aligned}$$

(b) IMPACT FACTOR (EQ. 2-63)

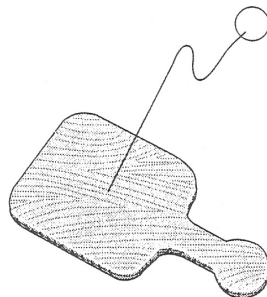
$$\begin{aligned} \text{Impact factor} &= \frac{\delta_{\max}}{\delta_{st}} = \frac{9.21 \text{ in.}}{2.0 \text{ in.}} \\ &= 4.6 \quad \leftarrow \end{aligned}$$

Problem 2.8-6 A small rubber ball (weight $W = 450 \text{ mN}$) is attached by a rubber cord to a wood paddle (see figure). The natural length of the cord is $L_0 = 200 \text{ mm}$, its cross-sectional area is $A = 1.6 \text{ mm}^2$, and its modulus of elasticity is $E = 2.0 \text{ MPa}$. After being struck by the paddle, the ball stretches the cord to a total length $L_1 = 900 \text{ mm}$.

What was the velocity v of the ball when it left the paddle? (Assume linearly elastic behavior of the rubber cord, and disregard the potential energy due to any change in elevation of the ball.)



Solution 2.8-6 Rubber ball attached to a paddle



$$\begin{aligned} g &= 9.81 \text{ m/s}^2 & E &= 2.0 \text{ MPa} \\ A &= 1.6 \text{ mm}^2 & L_0 &= 200 \text{ mm} \\ L_1 &= 900 \text{ mm} & W &= 450 \text{ mN} \end{aligned}$$

WHEN THE BALL LEAVES THE PADDLE

$$KE = \frac{Wv^2}{2g}$$

WHEN THE RUBBER CORD IS FULLY STRETCHED:

$$U = \frac{EA\delta^2}{2L_0} = \frac{EA}{2L_0}(L_1 - L_0)^2$$

CONSERVATION OF ENERGY

$$KE = U \quad \frac{Wv^2}{2g} = \frac{EA}{2L_0}(L_1 - L_0)^2$$

$$v^2 = \frac{gEA}{WL_0}(L_1 - L_0)^2$$

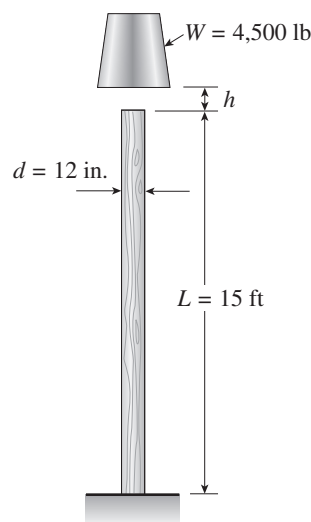
$$v = (L_1 - L_0) \sqrt{\frac{gEA}{WL_0}} \quad \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

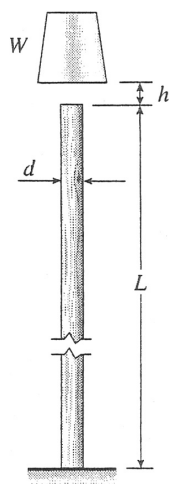
$$\begin{aligned} v &= (700 \text{ mm}) \sqrt{\frac{(9.81 \text{ m/s}^2)(2.0 \text{ MPa})(1.6 \text{ mm}^2)}{(450 \text{ mN})(200 \text{ mm})}} \\ &= 13.1 \text{ m/s} \quad \leftarrow \end{aligned}$$

Problem 2.8-7 A weight $W = 4500$ lb falls from a height h onto a vertical wood pole having length $L = 15$ ft, diameter $d = 12$ in., and modulus of elasticity $E = 1.6 \times 10^6$ psi (see figure).

If the allowable stress in the wood under an impact load is 2500 psi, what is the maximum permissible height h ?



Solution 2.8-7 Weight falling on a wood pole



$$W = 4500 \text{ lb} \quad d = 12 \text{ in.}$$

$$L = 15 \text{ ft} = 180 \text{ in.}$$

$$A = \frac{\pi d^2}{4} = 113.10 \text{ in.}^2$$

$$E = 1.6 \times 10^6 \text{ psi}$$

$$\sigma_{\text{allow}} = 2500 \text{ psi} (= \sigma_{\text{max}})$$

Find h_{max}

STATIC STRESS

$$\sigma_{st} = \frac{W}{A} = \frac{4500 \text{ lb}}{113.10 \text{ in.}^2} = 39.79 \text{ psi}$$

MAXIMUM HEIGHT h_{max}

$$\text{Eq. (2-61): } \sigma_{\text{max}} = \sigma_{st} \left[1 + \left(1 + \frac{2hE}{L\sigma_{st}} \right)^{1/2} \right]$$

or

$$\frac{\sigma_{\text{max}}}{\sigma_{st}} - 1 = \left(1 + \frac{2hE}{L\sigma_{st}} \right)^{1/2}$$

Square both sides and solve for h :

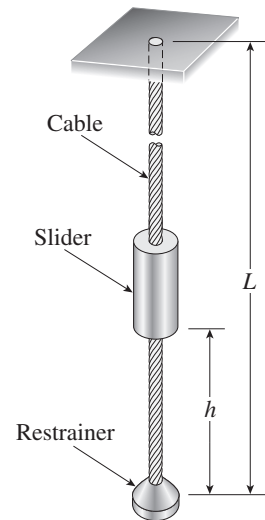
$$h = h_{\text{max}} = \frac{L\sigma_{\text{max}}}{2E} \left(\frac{\sigma_{\text{max}}}{\sigma_{st}} - 2 \right) \quad \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

$$\begin{aligned} h_{\text{max}} &= \frac{(180 \text{ in.})(2500 \text{ psi})}{2(1.6 \times 10^6 \text{ psi})} \left(\frac{2500 \text{ psi}}{39.79 \text{ psi}} - 2 \right) \\ &= 8.55 \text{ in.} \quad \leftarrow \end{aligned}$$

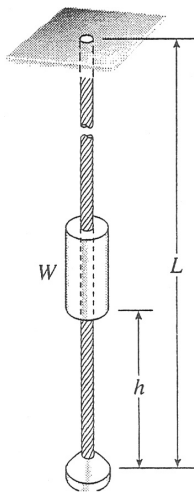
Problem 2.8-8 A cable with a restrainer at the bottom hangs vertically from its upper end (see figure). The cable has an effective cross-sectional area $A = 40 \text{ mm}^2$ and an effective modulus of elasticity $E = 130 \text{ GPa}$. A slider of mass $M = 35 \text{ kg}$ drops from a height $h = 1.0 \text{ m}$ onto the restrainer.

If the allowable stress in the cable under an impact load is 500 MPa , what is the minimum permissible length L of the cable?



Probs. 2.8-8, 2.8-2, 2.8-9

Solution 2.8-8 Slider on a cable



$$W = Mg = (35 \text{ kg})(9.81 \text{ m/s}^2) = 343.4 \text{ N}$$

$$A = 40 \text{ mm}^2 \quad E = 130 \text{ GPa}$$

$$h = 1.0 \text{ m} \quad \sigma_{\text{allow}} = \sigma_{\text{max}} = 500 \text{ MPa}$$

Find minimum length L_{min} .

STATIC STRESS

$$\sigma_{st} = \frac{W}{A} = \frac{343.4 \text{ N}}{40 \text{ mm}^2} = 8.585 \text{ MPa}$$

MINIMUM LENGTH L_{min}

$$\text{Eq. (2-61): } \sigma_{\text{max}} = \sigma_{st} \left[1 + \left(1 + \frac{2hE}{L\sigma_{st}} \right)^{1/2} \right]$$

or

$$\frac{\sigma_{\text{max}}}{\sigma_{st}} - 1 = \left(1 + \frac{2hE}{L\sigma_{st}} \right)^{1/2}$$

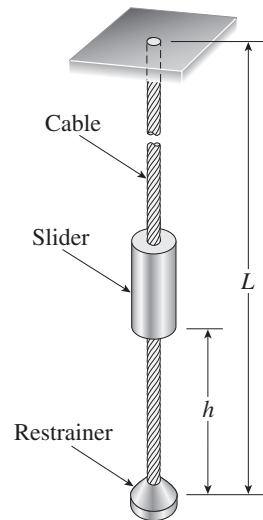
Square both sides and solve for L :

$$L = L_{\text{min}} = \frac{2Eh\sigma_{st}}{\sigma_{\text{max}}(\sigma_{\text{max}} - 2\sigma_{st})} \quad \leftarrow$$

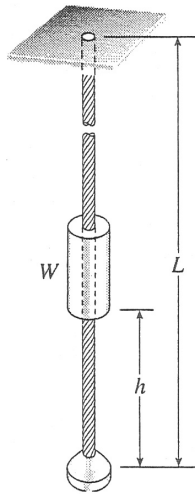
SUBSTITUTE NUMERICAL VALUES:

$$\begin{aligned} L_{\text{min}} &= \frac{2(130 \text{ GPa})(1.0 \text{ m})(8.585 \text{ MPa})}{(500 \text{ MPa})[500 \text{ MPa} - 2(8.585 \text{ MPa})]} \\ &= 9.25 \text{ mm} \quad \leftarrow \end{aligned}$$

Problem 2.8-9 Solve the preceding problem if the slider has weight $W = 100 \text{ lb}$, $h = 45 \text{ in.}$, $A = 0.080 \text{ in.}^2$, $E = 21 \times 10^6 \text{ psi}$, and the allowable stress is 70 ksi.



Solution 2.8-9 Slider on a cable



$$W = 100 \text{ lb}$$

$$A = 0.080 \text{ in.}^2 \quad E = 21 \times 10^6 \text{ psi}$$

$$h = 45 \text{ in.} \quad \sigma_{\text{allow}} = \sigma_{\text{max}} = 70 \text{ ksi}$$

Find minimum length L_{min} .

STATIC STRESS

$$\sigma_{st} = \frac{W}{A} = \frac{100 \text{ lb}}{0.080 \text{ in.}^2} = 1250 \text{ psi}$$

MINIMUM LENGTH L_{min}

$$\text{Eq. (2-61): } \sigma_{\text{max}} = \sigma_{st} \left[1 + \left(1 + \frac{2hE}{L\sigma_{st}} \right)^{1/2} \right]$$

or

$$\frac{\sigma_{\text{max}}}{\sigma_{st}} - 1 = \left(1 + \frac{2hE}{L\sigma_{st}} \right)^{1/2}$$

Square both sides and solve for L :

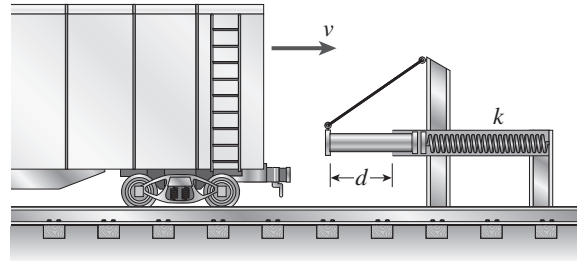
$$L = L_{\text{min}} = \frac{2Eh\sigma_{st}}{\sigma_{\text{max}}(\sigma_{\text{max}} - 2\sigma_{st})} \quad \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

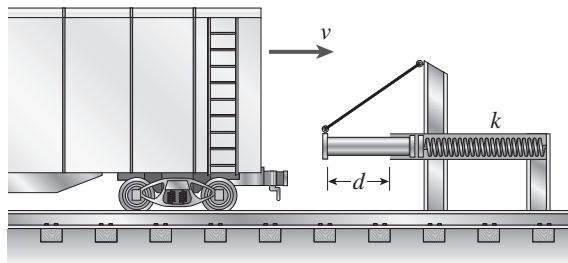
$$\begin{aligned} L_{\text{min}} &= \frac{2(21 \times 10^6 \text{ psi})(45 \text{ in.})(1250 \text{ psi})}{(70,000 \text{ psi})[70,000 \text{ psi} - 2(1250 \text{ psi})]} \\ &= 500 \text{ in.} \quad \leftarrow \end{aligned}$$

Problem 2.8-10 A bumping post at the end of a track in a railway yard has a spring constant $k = 8.0 \text{ MN/m}$ (see figure). The maximum possible displacement d of the end of the striking plate is 450 mm.

What is the maximum velocity v_{\max} that a railway car of weight $W = 545 \text{ kN}$ can have without damaging the bumping post when it strikes it?



Solution 2.8-10 Bumping post for a railway car



$$k = 8.0 \text{ MN/m} \quad W = 545 \text{ kN}$$

$d =$ maximum displacement of spring

$$d = \delta_{\max} = 450 \text{ mm}$$

Find v_{\max} .

KINETIC ENERGY BEFORE IMPACT

$$KE = \frac{Mv^2}{2} = \frac{Wv^2}{2g}$$

STRAIN ENERGY WHEN SPRING IS COMPRESSED TO THE MAXIMUM ALLOWABLE AMOUNT

$$U = \frac{k\delta_{\max}^2}{2} = \frac{kd^2}{2}$$

CONSERVATION OF ENERGY

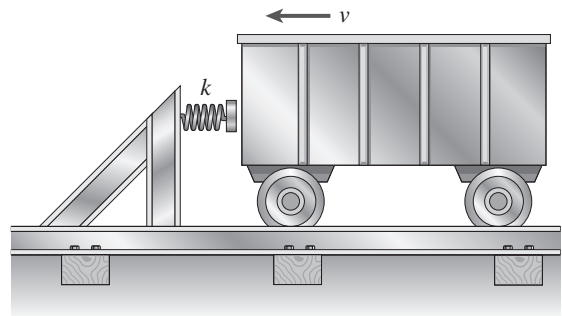
$$KE = U \quad \frac{Wv^2}{2g} = \frac{kd^2}{2} \quad v^2 = \frac{kd^2}{W/g}$$

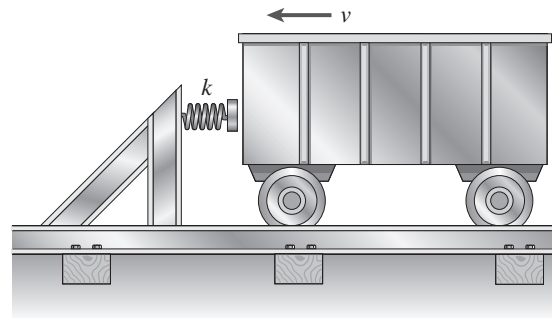
$$v = v_{\max} = d\sqrt{\frac{k}{W/g}} \quad \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

$$\begin{aligned} v_{\max} &= (450 \text{ mm})\sqrt{\frac{8.0 \text{ MN/m}}{(545 \text{ kN})/(9.81 \text{ m/s}^2)}} \\ &= 5400 \text{ mm/s} = 5.4 \text{ m/s} \quad \leftarrow \end{aligned}$$

Problem 2.8-11 A bumper for a mine car is constructed with a spring of stiffness $k = 1120 \text{ lb/in.}$ (see figure). If a car weighing 3450 lb is traveling at velocity $v = 7 \text{ mph}$ when it strikes the spring, what is the maximum shortening of the spring?



Solution 2.8-11 Bumper for a mine car

$$k = 1120 \text{ lb/in.} \quad W = 3450 \text{ lb}$$

$$v = 7 \text{ mph} = 123.2 \text{ in./sec}$$

$$g = 32.2 \text{ ft/sec}^2 = 386.4 \text{ in./sec}^2$$

Find the shortening δ_{\max} of the spring.

KINETIC ENERGY JUST BEFORE IMPACT

$$KE = \frac{Mv^2}{2} = \frac{Wv^2}{2g}$$

STRAIN ENERGY WHEN SPRING IS FULLY COMPRESSED

$$U = \frac{k\delta_{\max}^2}{2}$$

Conservation of energy

$$KE = U \quad \frac{Wv^2}{2g} = \frac{k\delta_{\max}^2}{2}$$

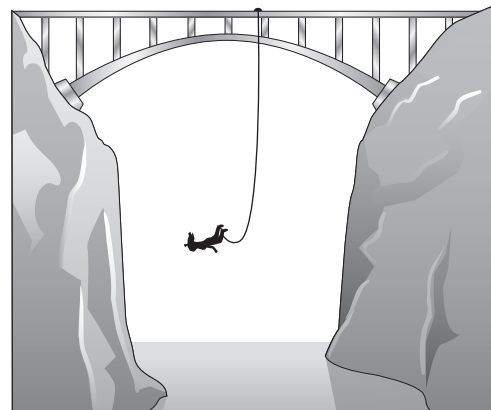
$$\text{Solve for } \delta_{\max}: \delta_{\max} = \sqrt{\frac{Wv^2}{gk}} \quad \leftarrow$$

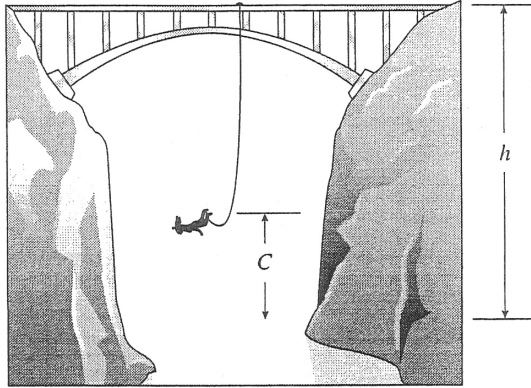
SUBSTITUTE NUMERICAL VALUES:

$$\begin{aligned} \delta_{\max} &= \sqrt{\frac{(3450 \text{ lb})(123.2 \text{ in./sec})^2}{(386.4 \text{ in./sec}^2)(1120 \text{ lb/in.})}} \\ &= 11.0 \text{ in.} \quad \leftarrow \end{aligned}$$

Problem 2.8-12 A bungee jumper having a mass of 55 kg leaps from a bridge, braking her fall with a long elastic shock cord having axial rigidity $EA = 2.3 \text{ kN}$ (see figure).

If the jumpoff point is 60 m above the water, and if it is desired to maintain a clearance of 10 m between the jumper and the water, what length L of cord should be used?



Solution 2.8-12 Bungee jumper

$$W = Mg = (55 \text{ kg})(9.81 \text{ m/s}^2) \\ = 539.55 \text{ N}$$

$$EA = 2.3 \text{ kN}$$

$$\text{Height: } h = 60 \text{ m}$$

$$\text{Clearance: } C = 10 \text{ m}$$

Find length L of the bungee cord.

$P.E.$ = Potential energy of the jumper at the top of bridge (with respect to lowest position)

$$= W(L + \delta_{\max})$$

U = strain energy of cord at lowest position

$$= \frac{EA\delta_{\max}^2}{2L}$$

CONSERVATION OF ENERGY

$$P.E. = U \quad W(L + \delta_{\max}) = \frac{EA\delta_{\max}^2}{2L}$$

$$\text{or} \quad \delta_{\max}^2 - \frac{2WL}{EA}\delta_{\max} - \frac{2WL^2}{EA} = 0$$

SOLVE QUADRATIC EQUATION FOR δ_{\max} :

$$\delta_{\max} = \frac{WL}{EA} + \left[\left(\frac{WL}{EA} \right)^2 + 2L \left(\frac{WL}{EA} \right) \right]^{1/2} \\ = \frac{WL}{EA} \left[1 + \left(1 + \frac{2EA}{W} \right)^{1/2} \right]$$

VERTICAL HEIGHT

$$h = C + L + \delta_{\max}$$

$$h - C = L + \frac{WL}{EA} \left[1 + \left(1 + \frac{2EA}{W} \right)^{1/2} \right]$$

SOLVE FOR L :

$$L = \frac{h - C}{1 + \frac{W}{EA} \left[1 + \left(1 + \frac{2EA}{W} \right)^{1/2} \right]} \quad \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

$$\frac{W}{EA} = \frac{539.55 \text{ N}}{2.3 \text{ kN}} = 0.234587$$

$$\text{Numerator} = h - C = 60 \text{ m} - 10 \text{ m} = 50 \text{ m}$$

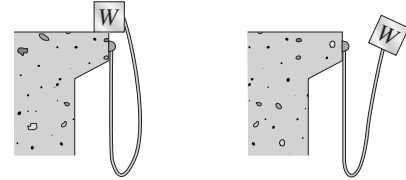
$$\text{Denominator} = 1 + (0.234587)$$

$$\times \left[1 + \left(1 + \frac{2}{0.234587} \right)^{1/2} \right]$$

$$= 1.9586$$

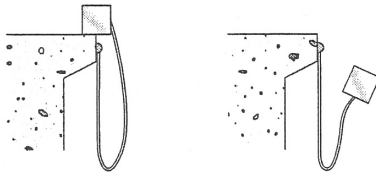
$$L = \frac{50 \text{ m}}{1.9586} = 25.5 \text{ m} \quad \leftarrow$$

Problem 2.8-13 A weight W rests on top of a wall and is attached to one end of a very flexible cord having cross-sectional area A and modulus of elasticity E (see figure). The other end of the cord is attached securely to the wall. The weight is then pushed off the wall and falls freely the full length of the cord.



- Derive a formula for the impact factor.
- Evaluate the impact factor if the weight, when hanging statically, elongates the band by 2.5% of its original length.

Solution 2.8-13 Weight falling off a wall



W = Weight

Properties of elastic cord:

E = modulus of elasticity

A = cross-sectional area

L = original length

δ_{\max} = elongation of elastic cord

$P.E.$ = potential energy of weight before fall (with respect to lowest position)

$P.E. = W(L + \delta_{\max})$

Let U = strain energy of cord at lowest position.

$$U = \frac{EA\delta_{\max}^2}{2L}$$

CONSERVATION OF ENERGY

$$P.E. = U \quad W(L + \delta_{\max}) = \frac{EA\delta_{\max}^2}{2L}$$

$$\text{or} \quad \delta_{\max}^2 - \frac{2WL}{EA}\delta_{\max} - \frac{2WL^2}{EA} = 0$$

SOLVE QUADRATIC EQUATION FOR δ_{\max} :

$$\delta_{\max} = \frac{WL}{EA} + \left[\left(\frac{WL}{EA} \right)^2 + 2L \left(\frac{WL}{EA} \right) \right]^{1/2}$$

STATIC ELONGATION

$$\delta_{st} = \frac{WL}{EA}$$

IMPACT FACTOR

$$\frac{\delta_{\max}}{\delta_{st}} = 1 + \left[1 + \frac{2EA}{W} \right]^{1/2} \leftarrow$$

NUMERICAL VALUES

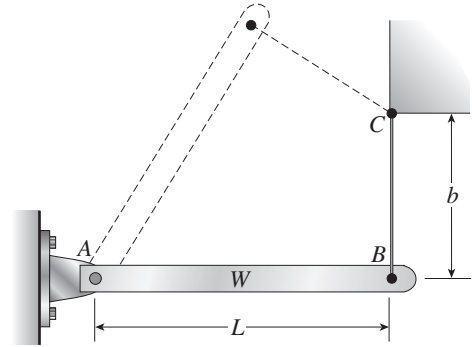
$$\delta_{st} = (2.5\%)(L) = 0.025L$$

$$\delta_{st} = \frac{WL}{EA} \quad \frac{W}{EA} = 0.025 \quad \frac{EA}{W} = 40$$

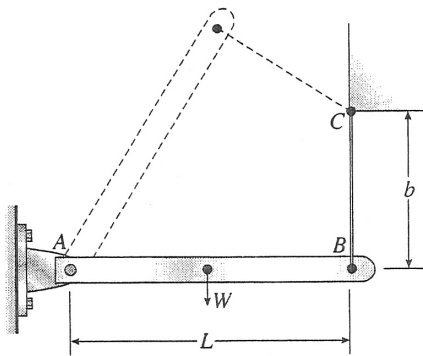
$$\text{Impact factor} = 1 + [1 + 2(40)]^{1/2} = 10 \leftarrow$$

Problem 2.8-14 A rigid bar AB having mass $M = 1.0$ kg and length $L = 0.5$ m is hinged at end A and supported at end B by a nylon cord BC (see figure). The cord has cross-sectional area $A = 30 \text{ mm}^2$, length $b = 0.25$ m, and modulus of elasticity $E = 2.1 \text{ GPa}$.

If the bar is raised to its maximum height and then released, what is the maximum stress in the cord?



Solution 2.8-14 Falling bar AB



RIGID BAR:

$$W = Mg = (1.0 \text{ kg})(9.81 \text{ m/s}^2)$$

$$= 9.81 \text{ N}$$

$$L = 0.5 \text{ m}$$

NYLON CORD:

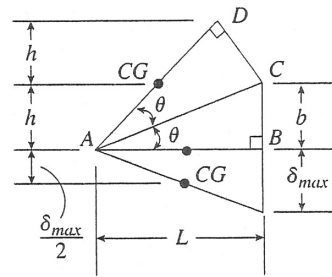
$$A = 30 \text{ mm}^2$$

$$b = 0.25 \text{ m}$$

$$E = 2.1 \text{ GPa}$$

Find maximum stress σ_{\max} in cord BC .

GEOMETRY OF BAR AB AND CORD BC



$$\overline{CD} = \overline{CB} = b$$

$$\overline{AD} = \overline{AB} = L$$

h = height of center of gravity of raised bar AD

δ_{\max} = elongation of cord

$$\text{From triangle } ABC: \sin \theta = \frac{b}{\sqrt{b^2 + L^2}}$$

$$\cos \theta = \frac{L}{\sqrt{b^2 + L^2}}$$

$$\text{From line } AD: \sin 2\theta = \frac{2h}{AD} = \frac{2h}{L}$$

$$\text{From Appendix D: } \sin 2\theta = 2 \sin \theta \cos \theta$$

$$\therefore \frac{2h}{L} = 2 \left(\frac{b}{\sqrt{b^2 + L^2}} \right) \left(\frac{L}{\sqrt{b^2 + L^2}} \right) = \frac{2bL}{b^2 + L^2}$$

$$\text{and } h = \frac{bL^2}{b^2 + L^2} \quad (\text{Eq. 1})$$

CONSERVATION OF ENERGY

$P.E.$ = potential energy of raised bar AD

$$= W \left(h + \frac{\delta_{\max}}{2} \right)$$

$$U = \text{strain energy of stretched cord} = \frac{EA\delta_{\max}^2}{2b}$$

$$P.E. = U \quad W \left(h + \frac{\delta_{\max}}{2} \right) = \frac{EA\delta_{\max}^2}{2b} \quad (\text{Eq. 2})$$

$$\text{For the cord: } \delta_{\max} = \frac{\sigma_{\max} b}{E}$$

Substitute into Eq. (2) and rearrange:

$$\sigma_{\max}^2 - \frac{W}{A} \sigma_{\max} - \frac{2WhE}{bA} = 0 \quad (\text{Eq. 3})$$

Substitute from Eq. (1) into Eq. (3):

$$\sigma_{\max}^2 - \frac{W}{A} \sigma_{\max} - \frac{2WL^2E}{A(b^2 + L^2)} = 0 \quad (\text{Eq. 4})$$

SOLVE FOR σ_{\max} :

$$\sigma_{\max} = \frac{W}{2A} \left[1 + \sqrt{1 + \frac{8L^2EA}{W(b^2 + L^2)}} \right] \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

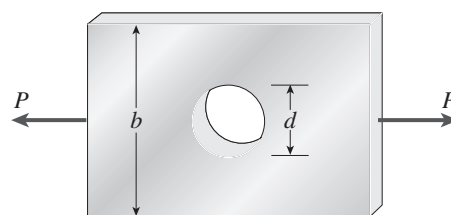
$$\sigma_{\max} = 33.3 \text{ MPa} \quad \leftarrow$$

Stress Concentrations

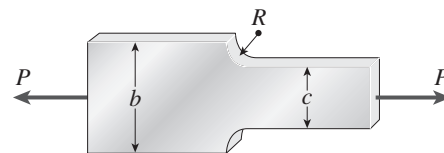
The problems for Section 2.10 are to be solved by considering the stress-concentration factors and assuming linearly elastic behavior.

Problem 2.10-1 The flat bars shown in parts (a) and (b) of the figure are subjected to tensile forces $P = 3.0 \text{ k}$. Each bar has thickness $t = 0.25 \text{ in.}$

- For the bar with a circular hole, determine the maximum stresses for hole diameters $d = 1 \text{ in.}$ and $d = 2 \text{ in.}$ if the width $b = 6.0 \text{ in.}$
- For the stepped bar with shoulder fillets, determine the maximum stresses for fillet radii $R = 0.25 \text{ in.}$ and $R = 0.5 \text{ in.}$ if the bar widths are $b = 4.0 \text{ in.}$ and $c = 2.5 \text{ in.}$

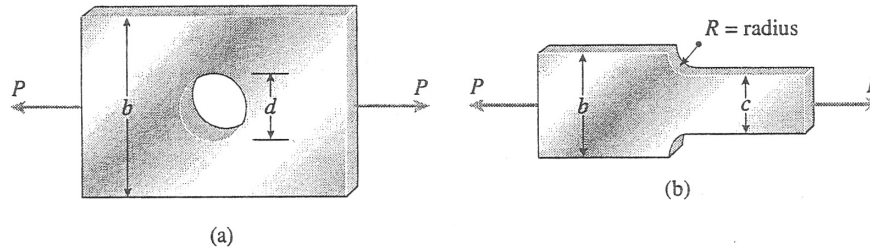


(a)



(b)

Probs. 2.10-1 and 2.10-2

Solution 2.10-1 Flat bars in tension

$$P = 3.0 \text{ k} \quad t = 0.25 \text{ in.}$$

(a) BAR WITH CIRCULAR HOLE ($b = 6 \text{ in.}$)

Obtain K from Fig. 2-63

FOR $d = 1 \text{ in.}$: $c = b - d = 5 \text{ in.}$

$$\sigma_{\text{nom}} = \frac{P}{ct} = \frac{3.0 \text{ k}}{(5 \text{ in.})(0.25 \text{ in.})} = 2.40 \text{ ksi}$$

$$d/b = \frac{1}{6} \quad K \approx 2.60$$

$$\sigma_{\text{max}} = K\sigma_{\text{nom}} \approx 6.2 \text{ ksi} \quad \leftarrow$$

FOR $d = 2 \text{ in.}$: $c = b - d = 4 \text{ in.}$

$$\sigma_{\text{nom}} = \frac{P}{ct} = \frac{3.0 \text{ k}}{(4 \text{ in.})(0.25 \text{ in.})} = 3.00 \text{ ksi}$$

$$d/b = \frac{2}{6} \quad K \approx 2.31$$

$$\sigma_{\text{max}} = K\sigma_{\text{nom}} \approx 6.9 \text{ ksi} \quad \leftarrow$$

(b) STEPPED BAR WITH SHOULDER FILLETS

$b = 4.0 \text{ in.}$ $c = 2.5 \text{ in.}$; Obtain k from Fig. 2-65

$$\sigma_{\text{nom}} = \frac{P}{ct} = \frac{3.0 \text{ k}}{(2.5 \text{ in.})(0.25 \text{ in.})} = 4.80 \text{ ksi}$$

FOR $R = 0.25 \text{ in.}$: $R/c = 0.1$ $b/c = 1.60$

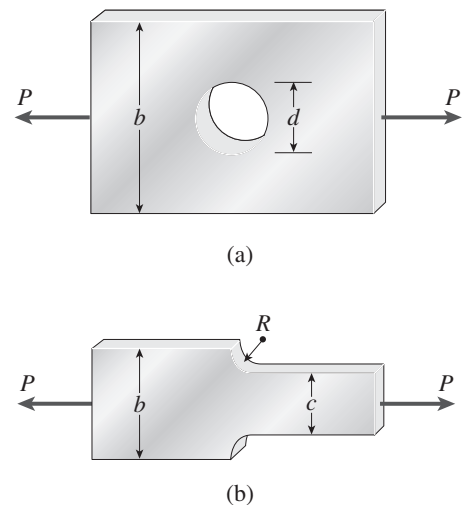
$$k \approx 2.30 \quad \sigma_{\text{max}} = K\sigma_{\text{nom}} \approx 11.0 \text{ ksi} \quad \leftarrow$$

FOR $R = 0.5 \text{ in.}$: $R/c = 0.2$ $b/c = 1.60$

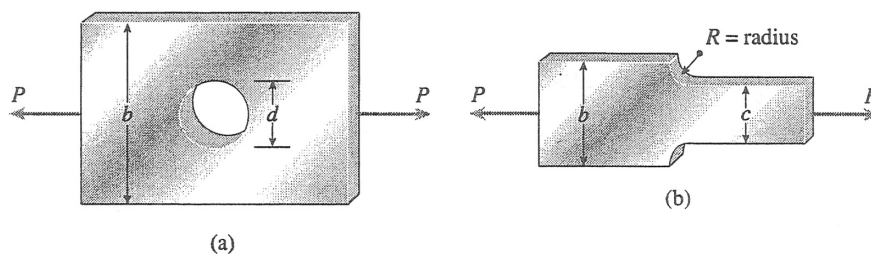
$$K \approx 1.87 \quad \sigma_{\text{max}} = K\sigma_{\text{nom}} \approx 9.0 \text{ ksi} \quad \leftarrow$$

Problem 2.10-2 The flat bars shown in parts (a) and (b) of the figure are subjected to tensile forces $P = 2.5 \text{ kN}$. Each bar has thickness $t = 5.0 \text{ mm}$.

- For the bar with a circular hole, determine the maximum stresses for hole diameters $d = 12 \text{ mm}$ and $d = 20 \text{ mm}$ if the width $b = 60 \text{ mm}$.
- For the stepped bar with shoulder fillets, determine the maximum stresses for fillet radii $R = 6 \text{ mm}$ and $R = 10 \text{ mm}$ if the bar widths are $b = 60 \text{ mm}$ and $c = 40 \text{ mm}$.



Solution 2.10-2 Flat bars in tension



$$P = 2.5 \text{ kN} \quad t = 5.0 \text{ mm}$$

(a) BAR WITH CIRCULAR HOLE ($b = 60 \text{ mm}$)

Obtain K from Fig. 2-63

FOR $d = 12 \text{ mm}$: $c = b - d = 48 \text{ mm}$

$$\sigma_{\text{nom}} = \frac{P}{ct} = \frac{2.5 \text{ kN}}{(48 \text{ mm})(5 \text{ mm})} = 10.42 \text{ MPa}$$

$$d/b = \frac{1}{5} \quad K \approx 2.51$$

$$\sigma_{\text{max}} = K\sigma_{\text{nom}} \approx 26 \text{ MPa} \quad \leftarrow$$

FOR $d = 20 \text{ mm}$: $c = b - d = 40 \text{ mm}$

$$\sigma_{\text{nom}} = \frac{P}{ct} = \frac{2.5 \text{ kN}}{(40 \text{ mm})(5 \text{ mm})} = 12.50 \text{ MPa}$$

$$d/b = \frac{1}{3} \quad K \approx 2.31$$

$$\sigma_{\text{max}} = K\sigma_{\text{nom}} \approx 29 \text{ MPa} \quad \leftarrow$$

(b) STEPPED BAR WITH SHOULDER FILLETS

$$b = 60 \text{ mm} \quad c = 40 \text{ mm};$$

Obtain K from Fig. 2-65

$$\sigma_{\text{nom}} = \frac{P}{ct} = \frac{2.5 \text{ kN}}{(40 \text{ mm})(5 \text{ mm})} = 12.50 \text{ MPa}$$

FOR $R = 6 \text{ mm}$: $R/c = 0.15 \quad b/c = 1.5$

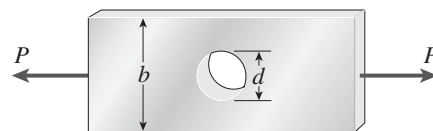
$$K \approx 2.00 \quad \sigma_{\text{max}} = K\sigma_{\text{nom}} \approx 25 \text{ MPa} \quad \leftarrow$$

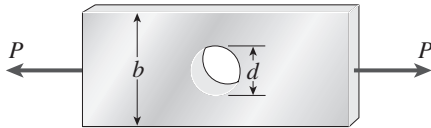
FOR $R = 10 \text{ mm}$: $R/c = 0.25 \quad b/c = 1.5$

$$K \approx 1.75 \quad \sigma_{\text{max}} = K\sigma_{\text{nom}} \approx 22 \text{ MPa} \quad \leftarrow$$

Problem 2.10-3 A flat bar of width b and thickness t has a hole of diameter d drilled through it (see figure). The hole may have any diameter that will fit within the bar.

What is the maximum permissible tensile load P_{max} if the allowable tensile stress in the material is σ_t ?



Solution 2.10-3 Flat bar in tension

t = thickness

σ_t = allowable tensile stress

Find P_{\max}

Find K from Fig. 2-63

$$P_{\max} = \sigma_{\text{nom}} ct = \frac{\sigma_{\max}}{K} ct = \frac{\sigma_t}{K} (b - d)t$$

$$= \frac{\sigma_t}{K} bt \left(1 - \frac{d}{b} \right)$$

Because σ_t , b , and t are constants, we write:

$$P^* = \frac{P_{\max}}{\sigma_t bt} = \frac{1}{K} \left(1 - \frac{d}{b} \right)$$

$\frac{d}{b}$	K	P^*
0	3.00	0.333
0.1	2.73	0.330
0.2	2.50	0.320
0.3	2.35	0.298
0.4	2.24	0.268

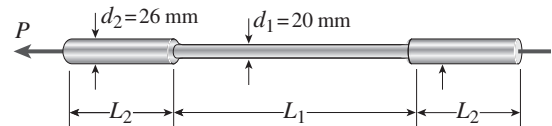
We observe that P_{\max} decreases as d/b increases. Therefore, the maximum load occurs when the hole becomes very small.

$$\left(\frac{d}{b} \rightarrow 0 \quad \text{and} \quad K \rightarrow 3 \right)$$

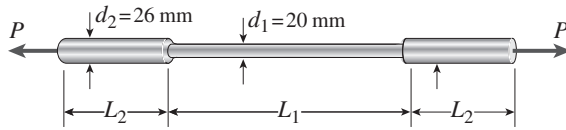
$$P_{\max} = \frac{\sigma_t bt}{3} \leftarrow$$

Problem 2.10-4 A round brass bar of diameter $d_1 = 20$ mm has upset ends of diameter $d_2 = 26$ mm (see figure). The lengths of the segments of the bar are $L_1 = 0.3$ m and $L_2 = 0.1$ m. Quarter-circular fillets are used at the shoulders of the bar, and the modulus of elasticity of the brass is $E = 100$ GPa.

If the bar lengthens by 0.12 mm under a tensile load P , what is the maximum stress σ_{\max} in the bar?



Probs. 2.10-4 and 2.10-5

Solution 2.10-4 Round brass bar with upset ends

$$E = 100 \text{ GPa}$$

$$\delta = 0.12 \text{ mm}$$

$$L_2 = 0.1 \text{ m}$$

$$L_1 = 0.3 \text{ m}$$

$$R = \text{radius of fillets} = \frac{26 \text{ mm} - 20 \text{ mm}}{2} = 3 \text{ mm}$$

$$\delta = 2\left(\frac{PL_2}{EA_2}\right) + \frac{PL_1}{EA_1}$$

$$\text{Solve for } P: \quad P = \frac{\delta EA_1 A_2}{2L_2 A_1 + L_1 A_2}$$

Use Fig. 2-66 for the stress-concentration factor:

$$\begin{aligned} \sigma_{\text{nom}} &= \frac{P}{A_1} = \frac{\delta EA_2}{2L_2 A_1 + L_1 A_2} = \frac{\delta E}{2L_2 \left(\frac{A_1}{A_2}\right) + L_1} \\ &= \frac{\delta E}{2L_2 \left(\frac{d_1}{d_2}\right)^2 + L_1} \end{aligned}$$

SUBSTITUTE NUMERICAL VALUES:

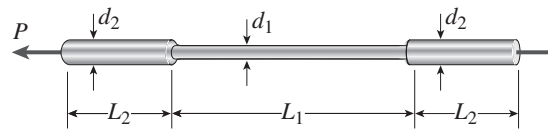
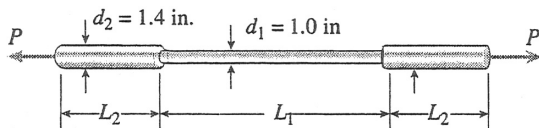
$$\sigma_{\text{nom}} = \frac{(0.12 \text{ mm})(100 \text{ GPa})}{2(0.1 \text{ m}) \left(\frac{20}{26}\right)^2 + 0.3 \text{ m}} = 28.68 \text{ MPa}$$

$$\frac{R}{D_1} = \frac{3 \text{ mm}}{20 \text{ mm}} = 0.15$$

Use the dashed curve in Fig. 2-66. $K \approx 1.6$

$$\begin{aligned} \sigma_{\text{max}} &= K\sigma_{\text{nom}} \approx (1.6)(28.68 \text{ MPa}) \\ &\approx 46 \text{ MPa} \quad \leftarrow \end{aligned}$$

Problem 2.10-5 Solve the preceding problem for a bar of monel metal having the following properties: $d_1 = 1.0 \text{ in.}$, $d_2 = 1.4 \text{ in.}$, $L_1 = 20.0 \text{ in.}$, $L_2 = 5.0 \text{ in.}$, and $E = 25 \times 10^6 \text{ psi}$. Also, the bar lengthens by 0.0040 in. when the tensile load is applied.

**Solution 2.10-5 Round bar with upset ends**

$$E = 25 \times 10^6 \text{ psi}$$

$$\delta = 0.0040 \text{ in.}$$

$$L_1 = 20 \text{ in.}$$

$$L_2 = 5 \text{ in.}$$

$$R = \text{radius of fillets} \quad R = \frac{1.4 \text{ in.} - 1.0 \text{ in.}}{2}$$

$$= 0.2 \text{ in.}$$

$$\delta = 2\left(\frac{PL_2}{EA_2}\right) + \frac{PL_1}{EA_1}$$

$$\text{Solve for } P: \quad P = \frac{\delta EA_1 A_2}{2L_2 A_1 + L_1 A_2}$$

Use Fig. 2-66 for the stress-concentration factor.

$$\begin{aligned} \sigma_{\text{nom}} &= \frac{P}{A_1} = \frac{\delta EA_2}{2L_2 A_1 + L_1 A_2} = \frac{\delta E}{2L_2 \left(\frac{A_1}{A_2}\right) + L_1} \\ &= \frac{\delta E}{2L_2 \left(\frac{d_1}{d_2}\right)^2 + L_1} \end{aligned}$$

SUBSTITUTE NUMERICAL VALUES:

$$\sigma_{\text{nom}} = \frac{(0.0040 \text{ in.})(25 \times 10^6 \text{ psi})}{2(5 \text{ in.})\left(\frac{1.0}{1.4}\right)^2 + 20 \text{ in.}} = 3,984 \text{ psi}$$

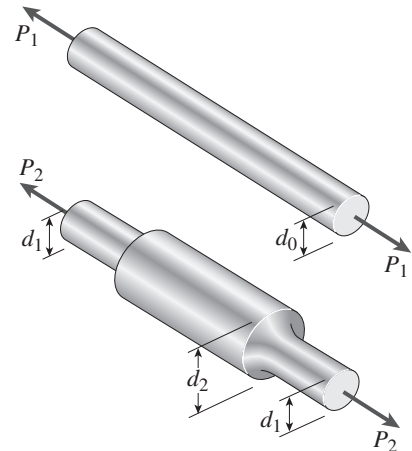
$$\frac{R}{D_1} = \frac{0.2 \text{ in.}}{1.0 \text{ in.}} = 0.2$$

Use the dashed curve in Fig. 2-66. $K \approx 1.53$

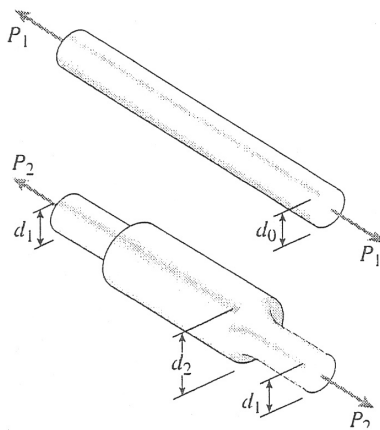
$$\sigma_{\text{max}} = K\sigma_{\text{nom}} \approx (1.53)(3984 \text{ psi}) \approx 6100 \text{ psi} \quad \leftarrow$$

Problem 2.10-6 A prismatic bar of diameter $d_0 = 20 \text{ mm}$ is being compared with a stepped bar of the same diameter ($d_1 = 20 \text{ mm}$) that is enlarged in the middle region to a diameter $d_2 = 25 \text{ mm}$ (see figure). The radius of the fillets in the stepped bar is 2.0 mm .

- Does enlarging the bar in the middle region make it stronger than the prismatic bar? Demonstrate your answer by determining the maximum permissible load P_1 for the prismatic bar and the maximum permissible load P_2 for the enlarged bar, assuming that the allowable stress for the material is 80 MPa .
- What should be the diameter d_0 of the prismatic bar if it is to have the same maximum permissible load as does the stepped bar?



Solution 2.10-6 Prismatic bar and stepped bar



$$d_0 = 20 \text{ mm}$$

$$d_1 = 20 \text{ mm}$$

$$d_2 = 25 \text{ mm}$$

Fillet radius: $R = 2 \text{ mm}$

Allowable stress: $\sigma_t = 80 \text{ MPa}$

(a) COMPARISON OF BARS

$$\begin{aligned} \text{Prismatic bar: } P_1 &= \sigma_t A_0 = \sigma_t \left(\frac{\pi d_0^2}{4} \right) \\ &= (80 \text{ MPa}) \left(\frac{\pi}{4} \right) (20 \text{ mm})^2 = 25.1 \text{ kN} \quad \leftarrow \end{aligned}$$

Stepped bar: See Fig. 2-66 for the stress-concentration factor.

$$\begin{array}{lll} R = 2.0 \text{ mm} & D_1 = 20 \text{ mm} & D_2 = 25 \text{ mm} \\ R/D_1 = 0.10 & D_2/D_1 = 1.25 & K \approx 1.75 \end{array}$$

$$\sigma_{\text{nom}} = \frac{P_2}{\frac{\pi d_1^2}{4}} = \frac{P_2}{A_1} \quad \sigma_{\text{nom}} = \frac{\sigma_{\text{max}}}{K}$$

$$\begin{aligned}
 P_2 &= \sigma_{\text{nom}} A_1 = \frac{\sigma_{\text{max}}}{K} A_1 = \frac{\sigma_t}{K} A_1 \\
 &= \left(\frac{80 \text{ MPa}}{1.75} \right) \left(\frac{\pi}{4} \right) (20 \text{ mm})^2 \\
 &\approx 14.4 \text{ kN} \quad \leftarrow
 \end{aligned}$$

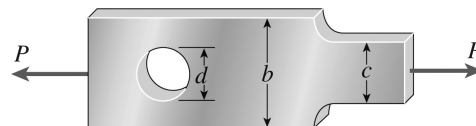
Enlarging the bar makes it *weaker*, not stronger. The ratio of loads is $P_1/P_2 = K = 1.75$

(b) DIAMETER OF PRISMATIC BAR FOR THE SAME ALLOWABLE LOAD

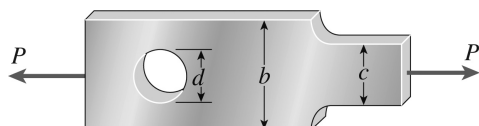
$$\begin{aligned}
 P_1 &= P_2 \quad \sigma_t \left(\frac{\pi d_0^2}{4} \right) = \frac{\sigma_t}{K} \left(\frac{\pi d_1^2}{4} \right) \quad d_0^2 = \frac{d_1^2}{K} \\
 d_0 &= \frac{d_1}{\sqrt{K}} \approx \frac{20 \text{ mm}}{\sqrt{1.75}} \approx 15.1 \text{ mm} \quad \leftarrow
 \end{aligned}$$

Problem 2.10-7 A stepped bar with a hole (see figure) has widths $b = 2.4$ in. and $c = 1.6$ in. The fillets have radii equal to 0.2 in.

What is the diameter d_{max} of the largest hole that can be drilled through the bar without reducing the load-carrying capacity?



Solution 2.10-7 Stepped bar with a hole



$$b = 2.4 \text{ in.}$$

$$c = 1.6 \text{ in.}$$

$$\text{Fillet radius: } R = 0.2 \text{ in.}$$

Find d_{max}

BASED UPON FILLETS (Use Fig. 2-65)

$$b = 2.4 \text{ in.} \quad c = 1.6 \text{ in.} \quad R = 0.2 \text{ in.}$$

$$R/c = 0.125 \quad b/c = 1.5 \quad K \approx 2.10$$

$$\begin{aligned}
 P_{\text{max}} &= \sigma_{\text{nom}} c t = \frac{\sigma_{\text{max}}}{K} c t = \frac{\sigma_{\text{max}}}{K} \left(\frac{c}{b} \right) (b t) \\
 &\approx 0.317 b t \sigma_{\text{max}}
 \end{aligned}$$

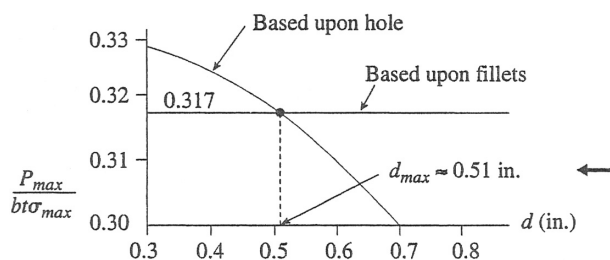
BASED UPON HOLE (Use Fig. 2-63)

$$b = 2.4 \text{ in.} \quad d = \text{diameter of the hole (in.)}$$

$$c_1 = b - d$$

$$\begin{aligned}
 P_{\text{max}} &= \sigma_{\text{nom}} c_1 t = \frac{\sigma_{\text{max}}}{K} (b - d) t \\
 &= \frac{1}{K} \left(1 - \frac{d}{b} \right) b t \sigma_{\text{max}}
 \end{aligned}$$

$d(\text{in.})$	d/b	K	$P_{\text{max}}/b t \sigma_{\text{max}}$
0.3	0.125	2.66	0.329
0.4	0.167	2.57	0.324
0.5	0.208	2.49	0.318
0.6	0.250	2.41	0.311
0.7	0.292	2.37	0.299



Nonlinear Behavior (Changes in Lengths of Bars)

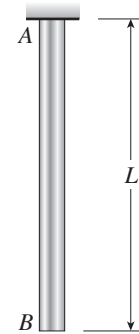
Problem 2.11-1 A bar AB of length L and weight density γ hangs vertically under its own weight (see figure). The stress-strain relation for the material is given by the Ramberg-Osgood equation (Eq. 2-73):

$$\epsilon = \frac{\sigma}{E} + \frac{\sigma_0 \alpha}{E} \left(\frac{\sigma}{\sigma_0} \right)^m$$

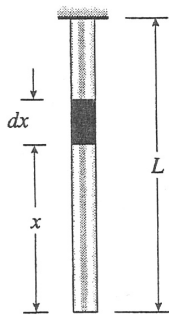
Derive the following formula

$$\delta = \frac{\gamma L^2}{2E} + \frac{\sigma_0 \alpha L}{(m+1)E} \left(\frac{\gamma L}{\sigma_0} \right)^m$$

for the elongation of the bar.



Solution 2.11-1 Bar hanging under its own weight



Let A = cross-sectional area

Let N = axial force at distance x

$$N = \gamma Ax$$

$$\sigma = \frac{N}{A} = \gamma x$$

STRAIN AT DISTANCE x

$$\epsilon = \frac{\sigma}{E} + \frac{\sigma_0 \alpha}{E} \left(\frac{\sigma}{\sigma_0} \right)^m = \frac{\gamma x}{E} + \frac{\sigma_0 \alpha}{E} \left(\frac{\gamma x}{\sigma_0} \right)^m$$

ELONGATION OF BAR

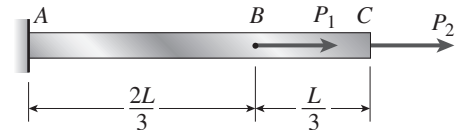
$$\begin{aligned} \delta &= \int_0^L \epsilon dx = \int_0^L \frac{\gamma x}{E} dx + \frac{\sigma_0 \alpha}{E} \int_0^L \left(\frac{\gamma x}{\sigma_0} \right)^m dx \\ &= \frac{\gamma L^2}{2E} + \frac{\sigma_0 \alpha L}{(m+1)E} \left(\frac{\gamma L}{\sigma_0} \right)^m \quad \text{Q.E.D.} \quad \leftarrow \end{aligned}$$

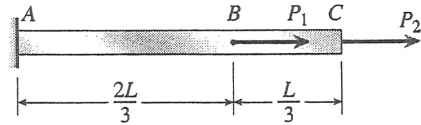
Problem 2.11-2 A prismatic bar of length $L = 1.8$ m and cross-sectional area $A = 480 \text{ mm}^2$ is loaded by forces $P_1 = 30$ kN and $P_2 = 60$ kN (see figure). The bar is constructed of magnesium alloy having a stress-strain curve described by the following Ramberg-Osgood equation:

$$\epsilon = \frac{\sigma}{45,000} + \frac{1}{618} \left(\frac{\sigma}{170} \right)^{10} \quad (\sigma = \text{MPa})$$

in which σ has units of megapascals.

- Calculate the displacement δ_C of the end of the bar when the load P_1 acts alone.
- Calculate the displacement when the load P_2 acts alone.
- Calculate the displacement when both loads act simultaneously.



Solution 2.11-2 Axially loaded bar

$$L = 1.8 \text{ m} \quad A = 480 \text{ mm}^2$$

$$P_1 = 30 \text{ kN} \quad P_2 = 60 \text{ kN}$$

Ramberg–Osgood equation:

$$\epsilon = \frac{\sigma}{45,000} + \frac{1}{618} \left(\frac{\sigma}{170} \right)^{10} \quad (\sigma = \text{MPa})$$

Find displacement at end of bar.

(a) P_1 ACTS ALONE

$$AB: \sigma = \frac{P_1}{A} = \frac{30 \text{ kN}}{480 \text{ mm}^2} = 62.5 \text{ MPa}$$

$$\epsilon = 0.001389$$

$$\delta_c = \epsilon \left(\frac{2L}{3} \right) = 1.67 \text{ mm} \quad \leftarrow$$

(b) P_2 ACTS ALONE

$$ABC: \sigma = \frac{P_2}{A} = \frac{60 \text{ kN}}{480 \text{ mm}^2} = 125 \text{ MPa}$$

$$\epsilon = 0.002853$$

$$\delta_c = \epsilon L = 5.13 \text{ mm} \quad \leftarrow$$

(c) BOTH P_1 AND P_2 ARE ACTING

$$AB: \sigma = \frac{P_1 + P_2}{A} = \frac{90 \text{ kN}}{480 \text{ mm}^2} = 187.5 \text{ MPa}$$

$$\epsilon = 0.008477$$

$$\delta_{AB} = \epsilon \left(\frac{2L}{3} \right) = 10.17 \text{ mm}$$

$$BC: \sigma = \frac{P_2}{A} = \frac{60 \text{ kN}}{480 \text{ mm}^2} = 125 \text{ MPa}$$

$$\epsilon = 0.002853$$

$$\delta_{BC} = \epsilon \left(\frac{L}{3} \right) = 1.71 \text{ mm}$$

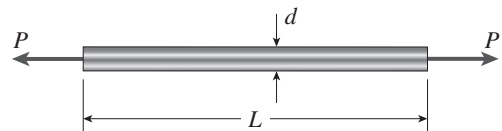
$$\delta_c = \delta_{AB} + \delta_{BC} = 11.88 \text{ mm} \quad \leftarrow$$

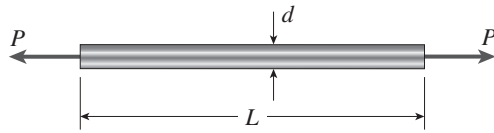
(Note that the displacement when both loads act simultaneously is *not* equal to the sum of the displacements when the loads act separately.)

Problem 2.11-3 A circular bar of length $L = 32$ in. and diameter $d = 0.75$ in. is subjected to tension by forces P (see figure). The wire is made of a copper alloy having the following *hyperbolic stress-strain relationship*:

$$\sigma = \frac{18,000\epsilon}{1 + 300\epsilon} \quad 0 \leq \epsilon \leq 0.03 \quad (\sigma = \text{ksi})$$

- Draw a stress-strain diagram for the material.
- If the elongation of the wire is limited to 0.25 in. and the maximum stress is limited to 40 ksi, what is the allowable load P ?



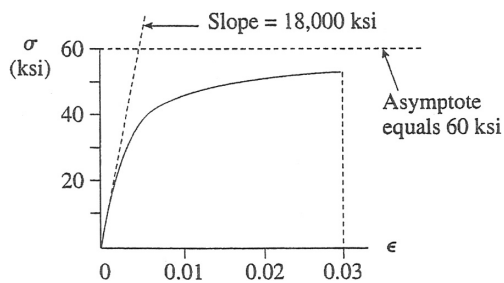
Solution 2.11-3 Copper bar in tension

$$L = 32 \text{ in.} \quad d = 0.75 \text{ in.}$$

$$A = \frac{\pi d^2}{4} = 0.4418 \text{ in.}^2$$

(a) STRESS-STRAIN DIAGRAM

$$\sigma = \frac{18,000\varepsilon}{1 + 300\varepsilon} \quad 0 \leq \varepsilon \leq 0.03 \quad (\sigma = \text{ksi})$$

(b) ALLOWABLE LOAD P Maximum elongation $\delta_{\max} = 0.25 \text{ in.}$ Maximum stress $\sigma_{\max} = 40 \text{ ksi}$

Based upon elongation:

$$\varepsilon_{\max} = \frac{\delta_{\max}}{L} = \frac{0.25 \text{ in.}}{32 \text{ in.}} = 0.007813$$

$$\sigma_{\max} = \frac{18,000\varepsilon_{\max}}{1 + 300\varepsilon_{\max}} = 42.06 \text{ ksi}$$

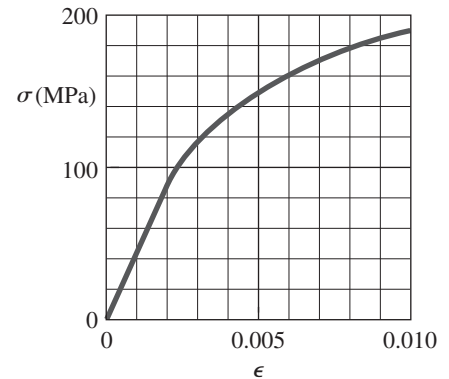
BASED UPON STRESS:

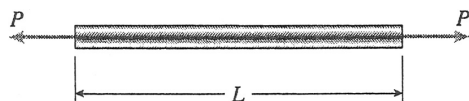
$$\sigma_{\max} = 40 \text{ ksi}$$

$$\text{Stress governs. } P = \sigma_{\max} A = (40 \text{ ksi})(0.4418 \text{ in.}^2) = 17.7 \text{ k} \quad \leftarrow$$

Problem 2.11-4 A prismatic bar in tension has length $L = 2.0 \text{ m}$ and cross-sectional area $A = 249 \text{ mm}^2$. The material of the bar has the stress-strain curve shown in the figure.

Determine the elongation δ of the bar for each of the following axial loads: $P = 10 \text{ kN}$, 20 kN , 30 kN , 40 kN , and 45 kN . From these results, plot a diagram of load P versus elongation δ (load-displacement diagram).



Solution 2.11-4 Bar in tension

$$L = 2.0 \text{ m}$$

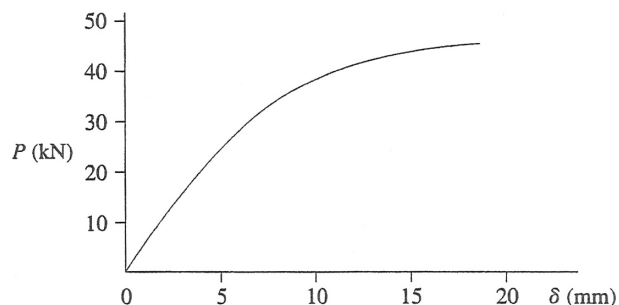
$$A = 249 \text{ mm}^2$$

STRESS-STRAIN DIAGRAM

(See the problem statement for the diagram)

LOAD-DISPLACEMENT DIAGRAM

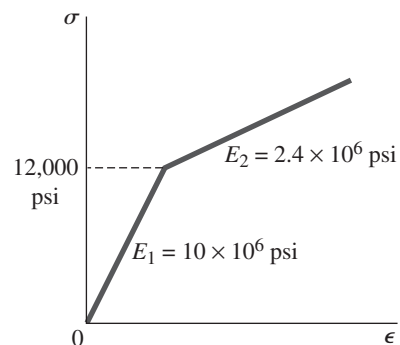
P (kN)	$\sigma = P/A$ (MPa)	ϵ (from diagram)	$\delta = \epsilon L$ (mm)
10	40	0.0009	1.8
20	80	0.0018	3.6
30	120	0.0031	6.2
40	161	0.0060	12.0
45	181	0.0081	16.2

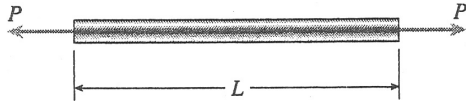


NOTE: The load-displacement curve has the same shape as the stress-strain curve.

Problem 2.11-5 An aluminum bar subjected to tensile forces P has length $L = 150 \text{ in.}$ and cross-sectional area $A = 2.0 \text{ in.}^2$. The stress-strain behavior of the aluminum may be represented approximately by the bilinear stress-strain diagram shown in the figure.

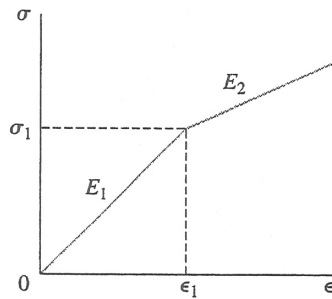
Calculate the elongation δ of the bar for each of the following axial loads: $P = 8 \text{ k}, 16 \text{ k}, 24 \text{ k}, 32 \text{ k},$ and 40 k . From these results, plot a diagram of load P versus elongation δ (load-displacement diagram).



Solution 2.11-5 Aluminum bar in tension

$$L = 150 \text{ in.}$$

$$A = 2.0 \text{ in.}^2$$

STRESS-STRAIN DIAGRAM

$$E_1 = 10 \times 10^6 \text{ psi}$$

$$E_2 = 2.4 \times 10^6 \text{ psi}$$

$$\sigma_1 = 12,000 \text{ psi}$$

$$\begin{aligned} \epsilon_1 &= \frac{\sigma_1}{E_1} = \frac{12,000 \text{ psi}}{10 \times 10^6 \text{ psi}} \\ &= 0.0012 \end{aligned}$$

For $0 \leq \sigma \leq \sigma_1$:

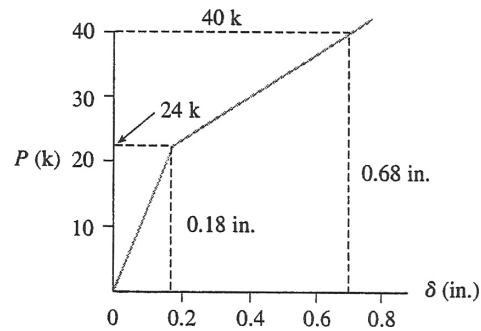
$$\epsilon = \frac{\sigma}{E_1} = \frac{\sigma}{10 \times 10^6 \text{ psi}} \quad (\sigma = \text{psi}) \quad \text{Eq. (1)}$$

For $\sigma \geq \sigma_1$:

$$\begin{aligned} \epsilon &= \epsilon_1 + \frac{\sigma - \sigma_1}{E_2} = 0.0012 + \frac{\sigma - 12,000}{2.4 \times 10^6} \\ &= \frac{\sigma}{2.4 \times 10^6} - 0.0038 \quad (\sigma = \text{psi}) \quad \text{Eq. (2)} \end{aligned}$$

LOAD-DISPLACEMENT DIAGRAM

P (k)	$\sigma = P/A$ (psi)	ϵ (from Eq. 1 or Eq. 2)	$\delta = \epsilon L$ (in.)
8	4,000	0.00040	0.060
16	8,000	0.00080	0.120
24	12,000	0.00120	0.180
32	16,000	0.00287	0.430
40	20,000	0.00453	0.680

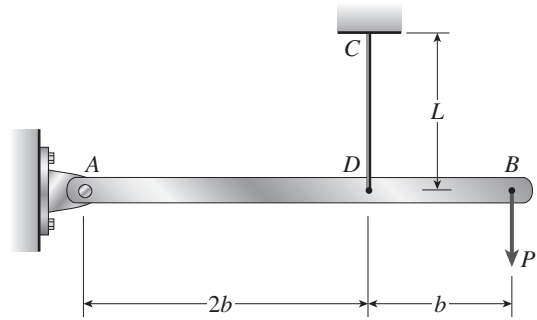


Problem 2.11-6 A rigid bar AB , pinned at end A , is supported by a wire CD and loaded by a force P at end B (see figure). The wire is made of high-strength steel having modulus of elasticity $E = 210$ GPa and yield stress $\sigma_Y = 820$ MPa. The length of the wire is $L = 1.0$ m and its diameter is $d = 3$ mm. The stress-strain diagram for the steel is defined by the *modified power law*, as follows:

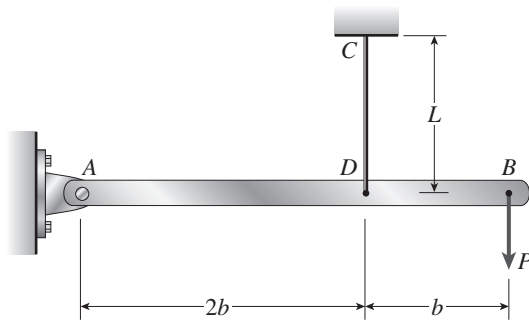
$$\sigma = E\epsilon \quad 0 \leq \sigma \leq \sigma_Y$$

$$\sigma = \sigma_Y \left(\frac{E\epsilon}{\sigma_Y} \right)^n \quad \sigma \geq \sigma_Y$$

- (a) Assuming $n = 0.2$, calculate the displacement δ_B at the end of the bar due to the load P . Take values of P from 2.4 kN to 5.6 kN in increments of 0.8 kN.
- (b) Plot a load-displacement diagram showing P versus δ_B .



Solution 2.11-6 Rigid bar supported by a wire



Wire: $E = 210$ GPa

$\sigma_Y = 820$ MPa

$L = 1.0$ m

$d = 3$ mm

$$A = \frac{\pi d^2}{4} = 7.0686 \text{ mm}^2$$

STRESS-STRAIN DIAGRAM

$$\sigma = E\epsilon \quad (0 \leq \sigma \leq \sigma_Y) \quad (1)$$

$$\sigma = \sigma_Y \left(\frac{E\epsilon}{\sigma_Y} \right)^n \quad (\sigma \geq \sigma_Y) \quad (n = 0.2) \quad (2)$$

(a) DISPLACEMENT δ_B AT END OF BAR

$$\delta = \text{elongation of wire} \quad \delta_B = \frac{3}{2}\delta = \frac{3}{2}\epsilon L \quad (3)$$

Obtain ϵ from stress-strain equations:

$$\text{From Eq. (1): } \epsilon = \frac{\sigma E}{\sigma_Y} \quad (0 \leq \sigma \leq \sigma_Y) \quad (4)$$

$$\text{From Eq. (2): } \epsilon = \frac{\sigma_Y}{E} \left(\frac{\sigma}{\sigma_Y} \right)^{1/n} \quad (5)$$

$$\text{Axial force in wire: } F = \frac{3P}{2}$$

$$\text{Stress in wire: } \sigma = \frac{F}{A} = \frac{3P}{2A} \quad (6)$$

PROCEDURE: Assume a value of P

Calculate σ from Eq. (6)

Calculate ϵ from Eq. (4) or (5)

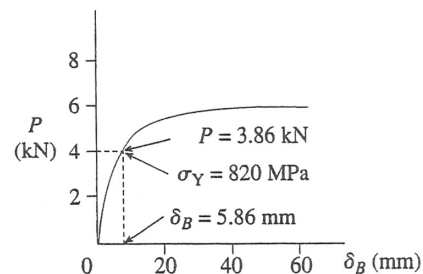
Calculate δ_B from Eq. (3)

P (kN)	σ (MPa) Eq. (6)	ϵ Eq. (4) or (5)	δ_B (mm) Eq. (3)
2.4	509.3	0.002425	3.64
3.2	679.1	0.003234	4.85
4.0	848.8	0.004640	6.96
4.8	1018.6	0.01155	17.3
5.6	1188.4	0.02497	37.5

For $\sigma = \sigma_Y = 820$ MPa:

$$\epsilon = 0.0039048 \quad P = 3.864 \text{ kN} \quad \delta_B = 5.86 \text{ mm}$$

(b) LOAD-DISPLACEMENT DIAGRAM

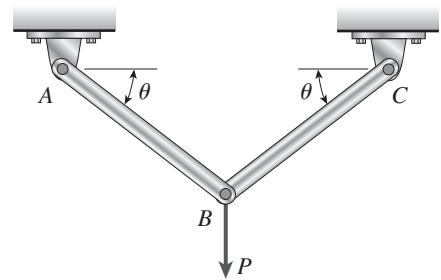


Elastoplastic Analysis

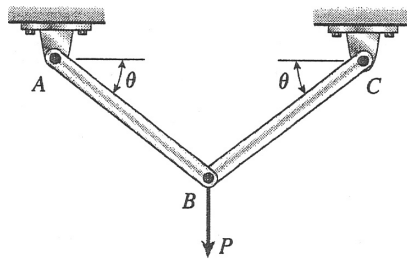
The problems for Section 2.12 are to be solved assuming that the material is elastoplastic with yield stress σ_Y , yield strain ϵ_Y , and modulus of elasticity E in the linearly elastic region (see Fig. 2-72).

Problem 2.12-1 Two identical bars AB and BC support a vertical load P (see figure). The bars are made of steel having a stress-strain curve that may be idealized as elastoplastic with yield stress σ_Y . Each bar has cross-sectional area A .

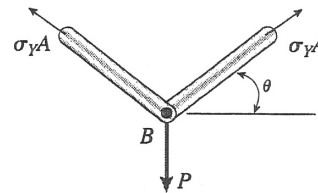
Determine the yield load P_Y and the plastic load P_P .



Solution 2.12-1 Two bars supporting a load P



Structure is statically determinate. The yield load P_Y and the plastic load P_P occur at the same time, namely, when both bars reach the yield stress.



JOINT B

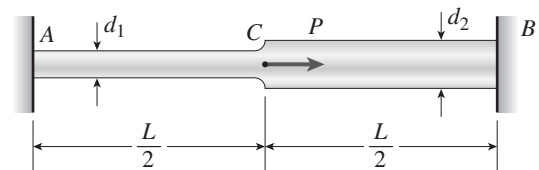
$$\Sigma F_{\text{vert}} = 0$$

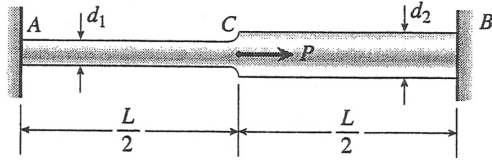
$$(2\sigma_Y A) \sin \theta = P$$

$$P_Y = P_P = 2\sigma_Y A \sin \theta \quad \leftarrow$$

Problem 2.12-2 A stepped bar ACB with circular cross sections is held between rigid supports and loaded by an axial force P at midlength (see figure). The diameters for the two parts of the bar are $d_1 = 20$ mm and $d_2 = 25$ mm, and the material is elastoplastic with yield stress $\sigma_Y = 250$ MPa.

Determine the plastic load P_P .



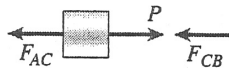
Solution 2.12-2 Bar between rigid supports

$$d_1 = 20 \text{ mm} \quad d_2 = 25 \text{ mm} \quad \sigma_Y = 250 \text{ MPa}$$

DETERMINE THE PLASTIC LOAD P_P :

At the plastic load, all parts of the bar are stressed to the yield stress.

Point C:



$$F_{AC} = \sigma_Y A_1 \quad F_{CB} = \sigma_Y A_2$$

$$P = F_{AC} + F_{CB}$$

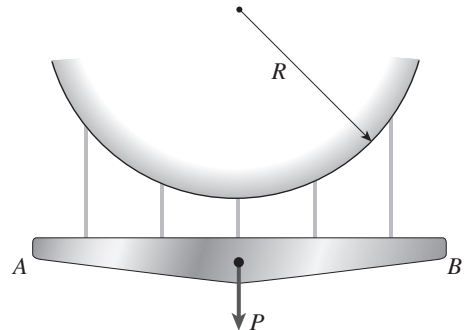
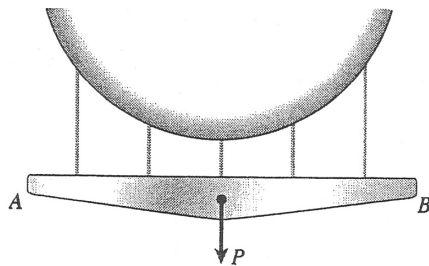
$$P_P = \sigma_Y A_1 + \sigma_Y A_2 = \sigma_Y (A_1 + A_2) \quad \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

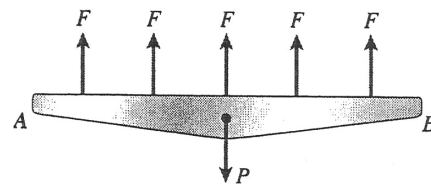
$$\begin{aligned} P_P &= (250 \text{ MPa}) \left(\frac{\pi}{4} \right) (d_1^2 + d_2^2) \\ &= (250 \text{ MPa}) \left(\frac{\pi}{4} \right) [(20 \text{ mm})^2 + (25 \text{ mm})^2] \\ &= 201 \text{ kN} \quad \leftarrow \end{aligned}$$

Problem 2.12-3 A horizontal rigid bar AB supporting a load P is hung from five symmetrically placed wires, each of cross-sectional area A (see figure). The wires are fastened to a curved surface of radius R .

- Determine the plastic load P_P if the material of the wires is elastoplastic with yield stress σ_Y .
- How is P_P changed if bar AB is flexible instead of rigid?
- How is P_P changed if the radius R is increased?

**Solution 2.12-3 Rigid bar supported by five wires**

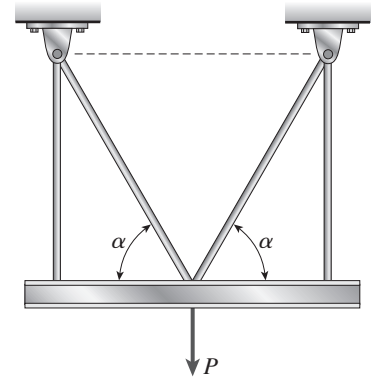
- PLASTIC LOAD P_P**
At the plastic load, each wire is stressed to the yield stress. $\therefore P_P = 5\sigma_Y A \quad \leftarrow$
 $F = \sigma_Y A$



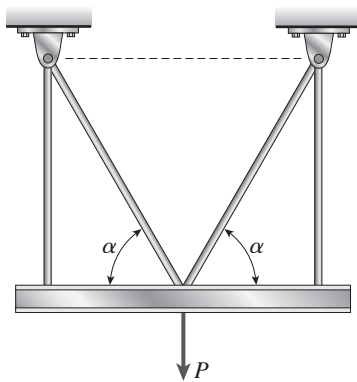
- BAR AB IS FLEXIBLE**
At the plastic load, each wire is stressed to the yield stress, so the plastic load is not changed. \leftarrow
- RADIUS R IS INCREASED**
Again, the forces in the wires are not changed, so the plastic load is not changed. \leftarrow

Problem 2.12-4 A load P acts on a horizontal beam that is supported by four rods arranged in the symmetrical pattern shown in the figure. Each rod has cross-sectional area A and the material is elastoplastic with yield stress σ_Y .

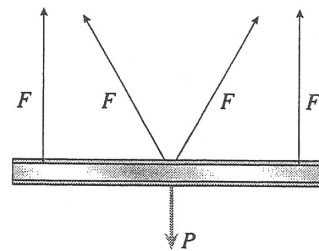
Determine the plastic load P_P .



Solution 2.12-4 Beam supported by four rods



At the plastic load, all four rods are stressed to the yield stress.



$$F = \sigma_Y A$$

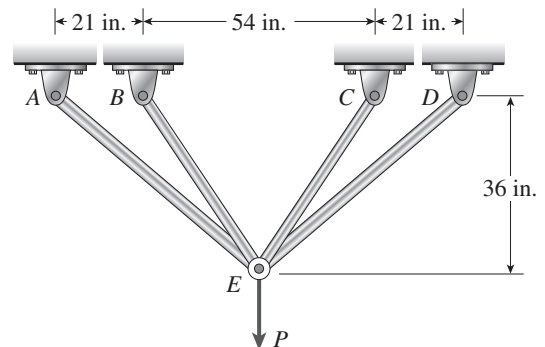
Sum forces in the vertical direction and solve for the load:

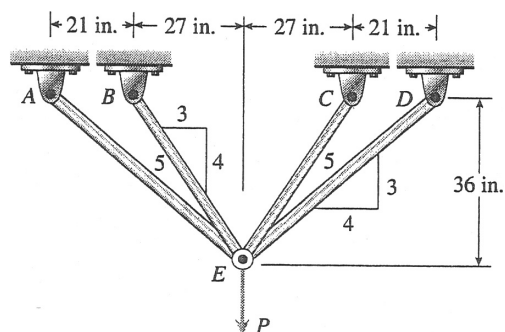
$$P_P = 2F + 2F \sin \alpha$$

$$P_P = 2\sigma_Y A (1 + \sin \alpha) \quad \leftarrow$$

Problem 2.12-5 The symmetric truss $ABCDE$ shown in the figure is constructed of four bars and supports a load P at joint E . Each of the two outer bars has a cross-sectional area of 0.307 in.^2 , and each of the two inner bars has an area of 0.601 in.^2 . The material is elastoplastic with yield stress $\sigma_Y = 36 \text{ ksi}$.

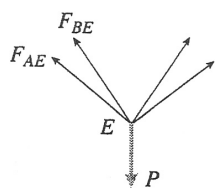
Determine the plastic load P_P .



Solution 2.12-5 Truss with four bars

$$L_{AE} = 60 \text{ in.} \quad L_{BE} = 45 \text{ in.}$$

JOINT E



Equilibrium:

$$2F_{AE}\left(\frac{3}{5}\right) + 2F_{BE}\left(\frac{4}{5}\right) = P$$

or

$$P = \frac{6}{5}F_{AE} + \frac{8}{5}F_{BE}$$

PLASTIC LOAD P_P

At the plastic load, all bars are stressed to the yield stress.

$$F_{AE} = \sigma_Y A_{AE} \quad F_{BE} = \sigma_Y A_{BE}$$

$$P_P = \frac{6}{5}\sigma_Y A_{AE} + \frac{8}{5}\sigma_Y A_{BE} \quad \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

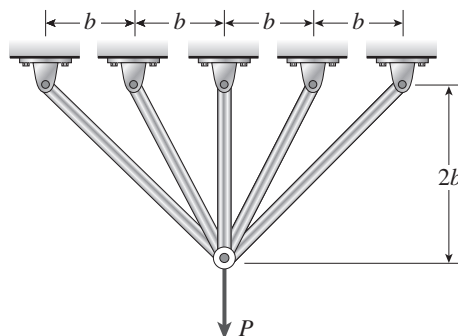
$$A_{AE} = 0.307 \text{ in.}^2 \quad A_{BE} = 0.601 \text{ in.}^2$$

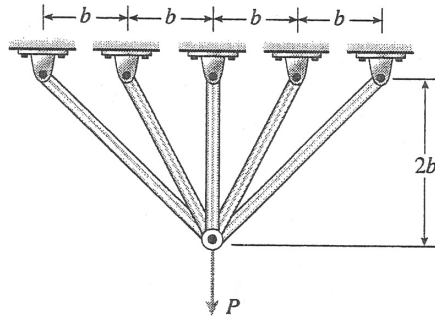
$$\sigma_Y = 36 \text{ ksi}$$

$$P_P = \frac{6}{5}(36 \text{ ksi})(0.307 \text{ in.}^2) + \frac{8}{5}(36 \text{ ksi})(0.601 \text{ in.}^2)$$

$$= 13.26 \text{ k} + 34.62 \text{ k} = 47.9 \text{ k} \quad \leftarrow$$

Problem 2.12-6 Five bars, each having a diameter of 10 mm, support a load P as shown in the figure. Determine the plastic load P_P if the material is elastoplastic with yield stress $\sigma_Y = 250 \text{ MPa}$.

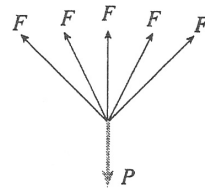


Solution 2.12-6 Truss consisting of five bars

$$d = 10 \text{ mm}$$

$$A = \frac{\pi d^2}{4} = 78.54 \text{ mm}^2$$

$$\sigma_Y = 250 \text{ MPa}$$



At the plastic load, all five bars are stressed to the yield stress

$$F = \sigma_Y A$$

Sum forces in the vertical direction and solve for the load:

$$P_P = 2F\left(\frac{1}{\sqrt{2}}\right) + 2F\left(\frac{2}{\sqrt{5}}\right) + F$$

$$= \frac{\sigma_Y A}{5}(5\sqrt{2} + 4\sqrt{5} + 5)$$

$$= 4.2031\sigma_Y A \quad \leftarrow$$

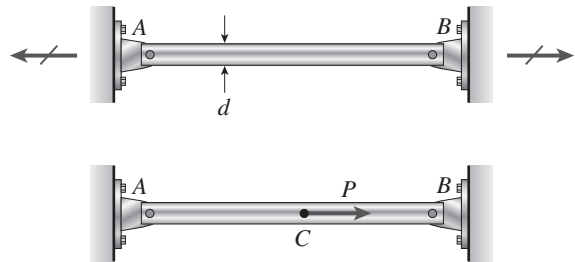
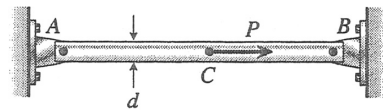
Substitute numerical values:

$$P_P = (4.2031)(250 \text{ MPa})(78.54 \text{ mm}^2)$$

$$= 82.5 \text{ kN} \quad \leftarrow$$

Problem 2.12-7 A circular steel rod AB of diameter $d = 0.60$ in. is stretched tightly between two supports so that initially the tensile stress in the rod is 10 ksi (see figure). An axial force P is then applied to the rod at an intermediate location C .

- Determine the plastic load P_P if the material is elastoplastic with yield stress $\sigma_Y = 36$ ksi.
- How is P_P changed if the initial tensile stress is doubled to 20 ksi?

**Solution 2.12-7 Bar held between rigid supports**

$$d = 0.6 \text{ in.}$$

$$\sigma_Y = 36 \text{ ksi}$$

Initial tensile stress = 10 ksi

(a) PLASTIC LOAD P_P

The presence of the initial tensile stress does not affect the plastic load. Both parts of the bar must yield in order to reach the plastic load.

POINT C:

$$\leftarrow \sigma_Y A \quad C \quad \rightarrow \sigma_Y A$$

$$P_P = 2\sigma_Y A = (2)(36 \text{ ksi})\left(\frac{\pi}{4}\right)(0.60 \text{ in.})^2$$

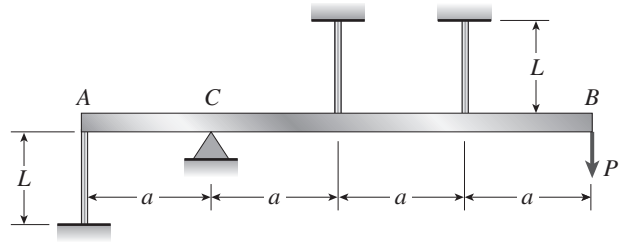
$$= 20.4 \text{ k} \quad \leftarrow$$

(B) INITIAL TENSILE STRESS IS DOUBLED

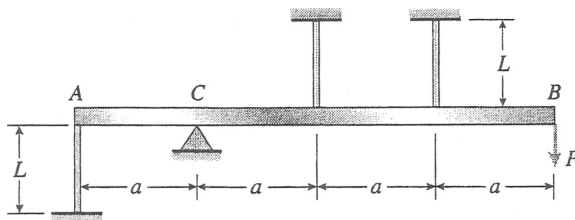
P_P is not changed. \leftarrow

Problem 2.12-8 A rigid bar ACB is supported on a fulcrum at C and loaded by a force P at end B (see figure). Three identical wires made of an elastoplastic material (yield stress σ_Y and modulus of elasticity E) resist the load P . Each wire has cross-sectional area A and length L .

- Determine the yield load P_Y and the corresponding yield displacement δ_Y at point B .
- Determine the plastic load P_P and the corresponding displacement δ_P at point B when the load just reaches the value P_P .
- Draw a load-displacement diagram with the load P as ordinate and the displacement δ_B of point B as abscissa.

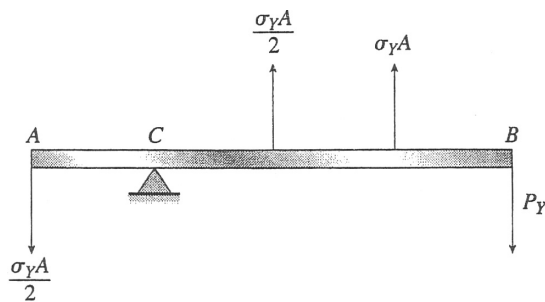


Solution 2.12-8 Rigid bar supported by wires



(a) YIELD LOAD P_Y

Yielding occurs when the most highly stressed wire reaches the yield stress σ_Y



$$\Sigma M_C = 0$$

$$P_Y = \sigma_Y A \quad \leftarrow$$

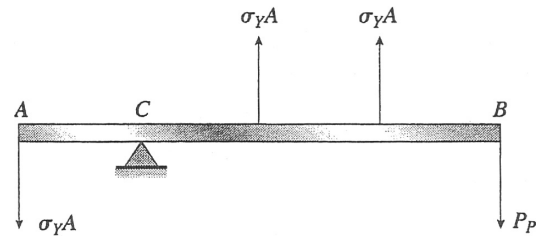
At point A:

$$\delta_A = \left(\frac{\sigma_Y A}{2} \right) \left(\frac{L}{EA} \right) = \frac{\sigma_Y L}{2E}$$

At point B:

$$\delta_B = 3\delta_A = \delta_Y = \frac{3\sigma_Y L}{2E} \quad \leftarrow$$

(b) PLASTIC LOAD P_P



At the plastic load, all wires reach the yield stress.

$$\Sigma M_C = 0$$

$$P_P = \frac{4\sigma_Y A}{3} \quad \leftarrow$$

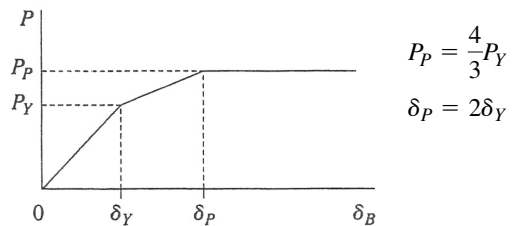
At point A:

$$\delta_A = (\sigma_Y A) \left(\frac{L}{EA} \right) = \frac{\sigma_Y L}{E}$$

At point B:

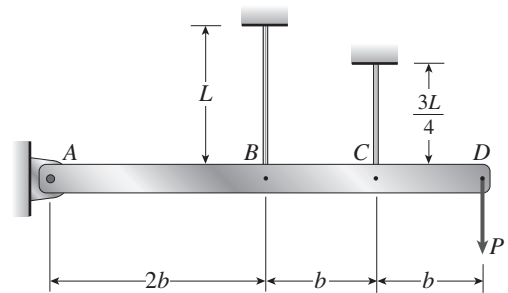
$$\delta_B = 3\delta_A = \delta_P = \frac{3\sigma_Y L}{E} \quad \leftarrow$$

(c) LOAD-DISPLACEMENT DIAGRAM

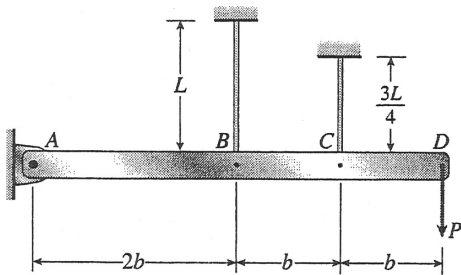


Problem 2.12-9 The structure shown in the figure consists of a horizontal rigid bar $ABCD$ supported by two steel wires, one of length L and the other of length $3L/4$. Both wires have cross-sectional area A and are made of elastoplastic material with yield stress σ_Y and modulus of elasticity E . A vertical load P acts at end D of the bar.

- Determine the yield load P_Y and the corresponding yield displacement δ_Y at point D .
- Determine the plastic load P_P and the corresponding displacement δ_P at point D when the load just reaches the value P_P .
- Draw a load-displacement diagram with the load P as ordinate and the displacement δ_D of point D as abscissa.



Solution 2.12-9 Rigid bar supported by two wires

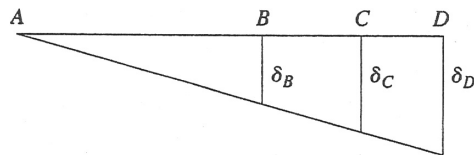


A = cross-sectional area

σ_Y = yield stress

E = modulus of elasticity

DISPLACEMENT DIAGRAM

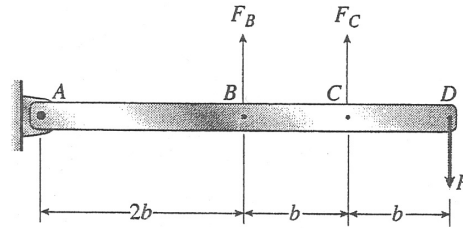


COMPATIBILITY:

$$\delta_C = \frac{3}{2}\delta_B \quad (1)$$

$$\delta_D = 2\delta_B \quad (2)$$

FREE-BODY DIAGRAM



EQUILIBRIUM:

$$\sum M_A = 0 \quad F_B(2b) + F_C(3b) = P(4b) \quad (3)$$

$$2F_B + 3F_C = 4P$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_B = \frac{F_B L}{EA} \quad \delta_C = \frac{F_C \left(\frac{3L}{4}\right)}{EA} \quad (4, 5)$$

Substitute into Eq. (1):

$$\frac{3F_C L}{4EA} = \frac{3F_B L}{2EA} \quad (6)$$

$$F_C = 2F_B$$

STRESSES

$$\sigma_B = \frac{F_B}{A} \quad \sigma_C = \frac{F_C}{A} \quad \sigma_C = 2\sigma_B \quad (7)$$

Wire C has the larger stress. Therefore, it will yield first.

(a) YIELD LOAD

$$\sigma_C = \sigma_Y \quad \sigma_B = \frac{\sigma_C}{2} = \frac{\sigma_Y}{2} \quad (\text{From Eq. 7})$$

$$F_C = \sigma_Y A \quad F_B = \frac{1}{2} \sigma_Y A$$

From Eq. (3):

$$2\left(\frac{1}{2}\sigma_Y A\right) + 3(\sigma_Y A) = 4P$$

$$P = P_Y = \sigma_Y A \quad \leftarrow$$

From Eq. (4):

$$\delta_B = \frac{F_B L}{EA} = \frac{\sigma_Y L}{2E}$$

From Eq. (2):

$$\delta_D = \delta_Y = 2\delta_B = \frac{\sigma_Y L}{E} \quad \leftarrow$$

(b) PLASTIC LOAD

At the plastic load, both wires yield.

$$\sigma_B = \sigma_Y = \sigma_C \quad F_B = F_C = \sigma_Y A$$

From Eq. (3):

$$2(\sigma_Y A) + 3(\sigma_Y A) = 4P$$

$$P = P_P = \frac{5}{4}\sigma_Y A \quad \leftarrow$$

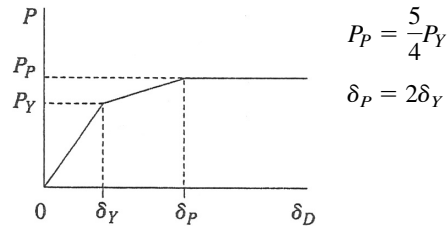
From Eq. (4):

$$\delta_B = \frac{F_B L}{EA} = \frac{\sigma_Y L}{E}$$

From Eq. (2):

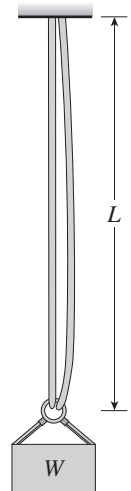
$$\delta_D = \delta_P = 2\delta_B = \frac{2\sigma_Y L}{E} \quad \leftarrow$$

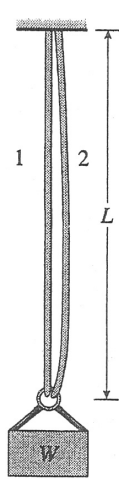
(c) LOAD-DISPLACEMENT DIAGRAM



Problem 2.12-10 Two cables, each having a length L of approximately 40 m, support a loaded container of weight W (see figure). The cables, which have effective cross-sectional area $A = 48.0 \text{ mm}^2$ and effective modulus of elasticity $E = 160 \text{ GPa}$, are identical except that one cable is longer than the other when they are hanging separately and unloaded. The difference in lengths is $d = 100 \text{ mm}$. The cables are made of steel having an elastoplastic stress-strain diagram with $\sigma_Y = 500 \text{ MPa}$. Assume that the weight W is initially zero and is slowly increased by the addition of material to the container.

- Determine the weight W_Y that first produces yielding of the shorter cable. Also, determine the corresponding elongation δ_Y of the shorter cable.
- Determine the weight W_P that produces yielding of both cables. Also, determine the elongation δ_P of the shorter cable when the weight W just reaches the value W_P .
- Construct a load-displacement diagram showing the weight W as ordinate and the elongation δ of the shorter cable as abscissa. (*Hint:* The load displacement diagram is not a single straight line in the region $0 \leq W \leq W_Y$.)



Solution 2.12-10 Two cables supporting a load

$$L = 40 \text{ m} \quad A = 48.0 \text{ mm}^2$$

$$E = 160 \text{ GPa}$$

$$d = \text{difference in length} = 100 \text{ mm}$$

$$\sigma_Y = 500 \text{ MPa}$$

INITIAL STRETCHING OF CABLE 1

Initially, cable 1 supports all of the load.

Let W_1 = load required to stretch cable 1 to the same length as cable 2

$$W_1 = \frac{EA}{L}d = 19.2 \text{ kN}$$

$$\delta_1 = 100 \text{ mm (elongation of cable 1)}$$

$$\sigma_1 = \frac{W_1}{A} = \frac{Ed}{L} = 400 \text{ MPa} \quad (\sigma_1 < \sigma_Y \therefore \text{OK})$$

(a) YIELD LOAD W_Y

Cable 1 yields first. $F_1 = \sigma_Y A = 24 \text{ kN}$

δ_{1Y} = total elongation of cable 1

δ_{1Y} = total elongation of cable 1

$$\delta_{1Y} = \frac{F_1 L}{EA} = \frac{\sigma_Y L}{E} = 0.125 \text{ m} = 125 \text{ mm}$$

$$\delta_Y = \delta_{1Y} = 125 \text{ mm} \quad \leftarrow$$

δ_{2Y} = elongation of cable 2

$$= \delta_{1Y} - d = 25 \text{ mm}$$

$$F_2 = \frac{EA}{L} \delta_{2Y} = 4.8 \text{ kN}$$

$$W_Y = F_1 + F_2 = 24 \text{ kN} + 4.8 \text{ kN}$$

$$= 28.8 \text{ kN} \quad \leftarrow$$

(b) PLASTIC LOAD W_P

$$F_1 = \sigma_Y A \quad F_2 = \sigma_Y A$$

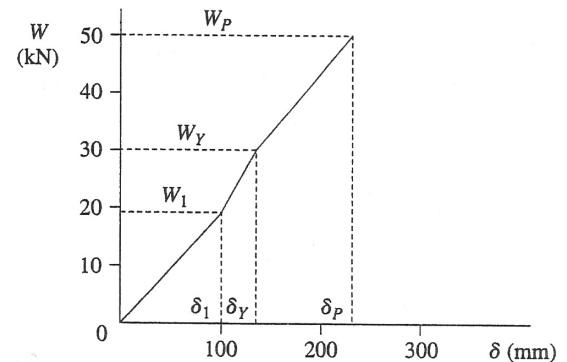
$$W_P = 2\sigma_Y A = 48 \text{ kN} \quad \leftarrow$$

δ_{2P} = elongation of cable 2

$$= F_2 \left(\frac{L}{EA} \right) = \frac{\sigma_Y L}{E} = 0.125 \text{ m} = 125 \text{ mm}$$

$$\delta_{1P} = \delta_{2P} + d = 225 \text{ mm}$$

$$\delta_P = \delta_{1P} = 225 \text{ mm} \quad \leftarrow$$

(c) LOAD-DISPLACEMENT DIAGRAM

$$\frac{W_Y}{W_1} = 1.5 \quad \frac{\delta_Y}{\delta_1} = 1.25$$

$$\frac{W_P}{W_Y} = 1.667 \quad \frac{\delta_P}{\delta_Y} = 1.8$$

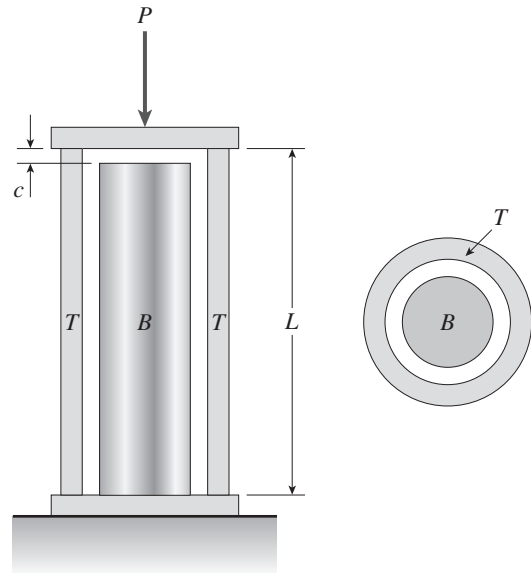
$$0 < W < W_1: \text{slope} = 192,000 \text{ N/m}$$

$$W_1 < W < W_Y: \text{slope} = 384,000 \text{ N/m}$$

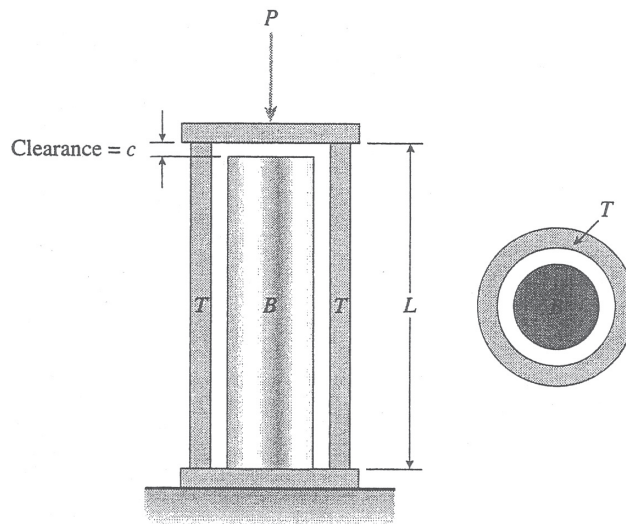
$$W_Y < W < W_P: \text{slope} = 192,000 \text{ N/m}$$

Problem 2.12-11 A hollow circular tube T of length $L = 15$ in. is uniformly compressed by a force P acting through a rigid plate (see figure). The outside and inside diameters of the tube are 3.0 and 2.75 in., respectively. A concentric solid circular bar B of 1.5 in. diameter is mounted inside the tube. When no load is present, there is a clearance $c = 0.010$ in. between the bar B and the rigid plate. Both bar and tube are made of steel having an elastoplastic stress-strain diagram with $E = 29 \times 10^3$ ksi and $\sigma_Y = 36$ ksi.

- Determine the yield load P_Y and the corresponding shortening δ_Y of the tube.
- Determine the plastic load P_P and the corresponding shortening δ_P of the tube.
- Construct a load-displacement diagram showing the load P as ordinate and the shortening δ of the tube as abscissa. (*Hint:* The load-displacement diagram is not a single straight line in the region $0 \leq P \leq P_Y$.)



Solution 2.12-11 Tube and bar supporting a load



$$L = 15 \text{ in.}$$

$$c = 0.010 \text{ in.}$$

$$E = 29 \times 10^3 \text{ ksi}$$

$$\sigma_Y = 36 \text{ ksi}$$

TUBE:

$$d_2 = 3.0 \text{ in.}$$

$$d_1 = 2.75 \text{ in.}$$

$$A_T = \frac{\pi}{4} (d_2^2 - d_1^2) = 1.1290 \text{ in.}^2$$

BAR

$$d = 1.5 \text{ in.}$$

$$A_B = \frac{\pi d^2}{4} = 1.7671 \text{ in.}^2$$

INITIAL SHORTENING OF TUBE T

Initially, the tube supports all of the load.

Let P_1 = load required to close the clearance

$$P_1 = \frac{EA_T}{L}c = 21,827 \text{ lb}$$

Let δ_1 = shortening of tube $\delta_1 = c = 0.010 \text{ in.}$

$$\sigma_1 = \frac{P_1}{A_T} = 19,330 \text{ psi} \quad (\sigma_1 < \sigma_Y \therefore \text{OK})$$

(a) YIELD LOAD P_Y

Because the tube and bar are made of the same material, and because the strain in the tube is larger than the strain in the bar, the tube will yield first.

$$F_T = \sigma_Y A_T = 40,644 \text{ lb}$$

δ_{TY} = shortening of tube at the yield stress

$$\sigma_{TY} = \frac{F_T L}{EA_T} = \frac{\sigma_Y L}{E} = 0.018621 \text{ in.}$$

$$\delta_Y = \delta_{TY} = 0.018621 \text{ in.} \quad \leftarrow$$

δ_{BY} = shortening of bar

$$= \delta_{TY} - c = 0.008621 \text{ in.}$$

$$F_B = \frac{EA_B}{L} \delta_{BY} = 29,453 \text{ lb}$$

$$P_Y = F_T + F_B = 40,644 \text{ lb} + 29,453 \text{ lb}$$

$$= 70,097 \text{ lb}$$

$$P_Y = 70,100 \text{ lb} \quad \leftarrow$$

(b) PLASTIC LOAD P_P

$$F_T = \sigma_Y A_T \quad F_B = \sigma_Y A_B$$

$$P_P = F_T + F_B = \sigma_Y (A_T + A_B)$$

$$= 104,300 \text{ lb} \quad \leftarrow$$

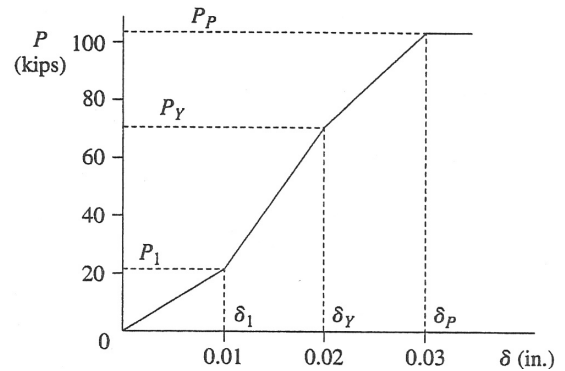
δ_{BP} = shortening of bar

$$= F_B \left(\frac{L}{EA_B} \right) = \frac{\sigma_Y L}{E} = 0.018621 \text{ in.}$$

$$\delta_{TP} = \delta_{BP} + c = 0.028621 \text{ in.}$$

$$\delta_P = \delta_{TP} = 0.02862 \text{ in.} \quad \leftarrow$$

(c) LOAD-DISPLACEMENT DIAGRAM



$$\frac{P_Y}{P_1} = 3.21 \quad \frac{\delta_Y}{\delta_1} = 1.86$$

$$\frac{P_P}{P_Y} = 1.49 \quad \frac{\delta_P}{\delta_Y} = 1.54$$

$$0 < P < P_1: \text{ slope} = 2180 \text{ k/in.}$$

$$P_1 < P < P_Y: \text{ slope} = 5600 \text{ k/in.}$$

$$P_Y < P < P_P: \text{ slope} = 3420 \text{ k/in.}$$

