

CHAPTER 2

2.2-1. The rectangular rules for numerical integration are illustrated in Fig. P2.2-1. The left-side rule is depicted in Fig. P2.2-1(a), and the right-side rule is depicted in Fig. P2.2-1(b). The integral of $x(t)$ is approximated by the sum of the rectangular areas shown for each rule. Let $y(kT)$ be the numerical integral of $x(t)$, $0 \leq t \leq kT$.

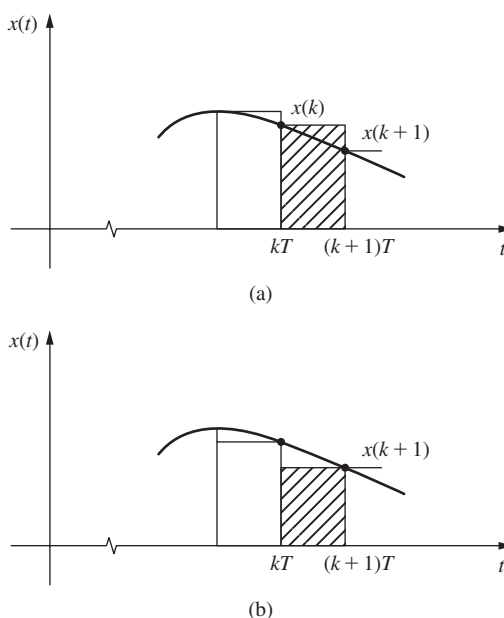


FIGURE P2.2-1 Rectangular rules for integration: (a) left side; (b) right side.

- (a) Write the difference equation relating $y(k+1)$, $y(k)$, and $x(k)$ for the left-side rule.
- (b) Find the transfer function $Y(z)/X(z)$ for part (a).
- (c) Write the difference equation relating $y(k+1)$, $y(k)$, and $x(k+1)$ for the right-side rule.
- (d) Find the transfer function $Y(z)/X(z)$ for part (c).
- (e) Express $y(k)$ as a summation on $x(k)$ for the left-side rule.
- (f) Express $y(k)$ as a summation on $x(k)$ for the right-side rule.

Solution:

(a) $y(k+1) = y(k) + Tx(k)$

(b) $zY(z) = Y(z) + TX(z) \Rightarrow \frac{Y(z)}{X(z)} = \frac{T}{z-1}$

(c) $y(k+1) = y(k) + Tx(k+1)$

(d) $zY(z) = Y(z) + TzX(z) \Rightarrow \frac{Y(z)}{X(z)} = \frac{Tz}{z-1}$

(e) $y(1) = y(0) + Tx(0)$

$$y(2) = y(1) + Tx(1) = y(0) + T(x(0) + x(1))$$

$$y(3) = y(2) + Tx(2) = y(0) + T[x(0) + x(1) + x(2)]$$

$$\therefore y(k) = y(0) + T \sum_{n=0}^{k-1} x(n)$$

(f) $y(1) = y(0) + Tx(1)$

$$y(2) = y(1) + Tx(2) = y(0) + T[x(1) + x(2)]$$

$$\therefore y(k) = y(0) + T \sum_{n=1}^k x(n)$$

2.2-2. The trapezoidal rule (modified Euler method) for numerical integration approximates the integral of a function $x(t)$ by summing trapezoid areas as shown in Fig. P2.2-2. Let $y(t)$ be the integral of $x(t)$.

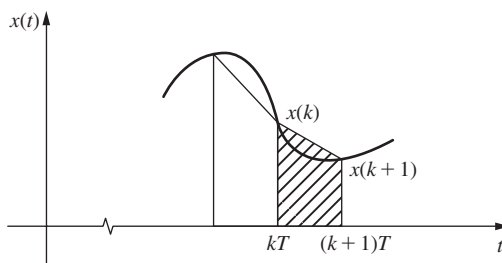


FIGURE P2.2-2 Trapezoidal rule for numerical integration.

(a) Write the difference equation relating $y[(k+1)T]$, $y(kT)$, $x[(k+1)T]$, and $x(kT)$ for this rule.

(b) Show that the transfer function for this integrator is given by

$$\frac{Y(z)}{X(z)} = \frac{(T/2)(z+1)}{z-1}$$

Solution:

$$(a) \ y(k+1) = y(k) + T \frac{x(k) + x(k+1)}{2}$$

$$(b) \ zY(z) = Y(z) + \frac{T}{2} [X(z) + zX(z)] \Rightarrow Y(z) = \frac{T}{2} \frac{z+1}{z-1} X(z)$$

- 2.2-3.** (a) The transfer function for the right-side rectangular-rule integrator was found in Problem 2.2-1 to be $Y(z)/X(z) = Tz/(z-1)$. We would suspect that the reciprocal of this transfer function should yield an approximation to a differentiator. That is, if $w(kT)$ is a numerical derivative of $x(t)$ at $t = kT$,

$$\frac{W(z)}{X(z)} = \frac{z-1}{Tz}$$

Write the difference equation describing this differentiator.

(b) Draw a figure similar to those in Fig. P2.2-1 illustrating the approximate differentiation.

(c) Repeat part (a) for the left-side rule, where $W(z)/X(z) = T/(z-1)$.

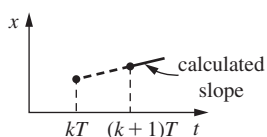
(d) Repeat part (b) for the differentiator of part (c).

Solution:

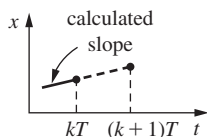
$$(a) \ Tz W(z) = zX(z) - X(z)$$

$$w(k+1) = \frac{1}{T} [x(k+1) - x(k)]$$

(b)



$$(c) \ TW(z) = zX(z) - X(z)$$



$$w(k) = \frac{1}{T} [x(k+1) - x(k)]$$

2.3-1. Find the z -transform of the number sequence generated by sampling the time function $e(t) = t$ every T seconds, beginning at $t = 0$. Can you express this transform in closed form?

Solution: $e(t) = t$; $E(z) = 0 + Tz^{-1} + 2Tz^{-2} + \dots = \frac{Tz}{(z-1)^2}$

2.3-2. (a) Write, as a series, the z -transform of the number sequence generated by sampling the time function $e(t) = \epsilon^{-t}$ every T seconds, beginning at $t = 0$. Can you express this transform in closed form?

(b) Evaluate the coefficients in the series of part (a) for the case that $T = 0.05$ s.

(c) The exponential $e(t) = \epsilon^{-bt}$ is sampled every $T = 0.2$ s, yielding the z -transform

$$E(z) = 1 + \left(\frac{1}{2}\right)z^{-1} + \left(\frac{1}{2}\right)^2 z^{-2} + \left(\frac{1}{2}\right)^3 z^{-3} + \dots$$

Evaluate b .

Solution:

(a) $E(z) = 1 + \epsilon^{-T} z^{-1} + \epsilon^{-2T} z^{-2} + \dots$

$$= 1 + (\epsilon^{-T} z^{-1})^1 + (\epsilon^{-T} z^{-1})^2 + \dots = \frac{1}{1 - \epsilon^{-T} z^{-1}} = \frac{z}{z - \epsilon^{-T}}$$

(b) $E(z) = 1 + (0.9512z^{-1})^1 + (0.9512z^{-1})^2 + \dots = \frac{z}{z - 0.9512}$

(c) $\epsilon^{-bT} \Big|_{T=0.2} = \epsilon^{-0.2b} = 0.5$

$$\therefore -0.2b = \ln(0.5) = -0.6931 \Rightarrow b = -3.466$$

2.3-3. Find the z -transforms of the number sequences generated by sampling the following time functions every T seconds, beginning at $t = 0$. Express these transforms in closed form.

(a) $e(t) = \epsilon^{-at}$

(b) $e(t) = \epsilon^{-(t-T)}u(t-T)$

(c) $e(t) = \epsilon^{-(t-5T)}u(t-5T)$

Solution:

(a) $e(t) = \epsilon^{-at} \Rightarrow E(z) = 1 + \epsilon^{-aT}z^{-1} + \epsilon^{-2aT}z^{-2} + \dots = \frac{z}{z - \epsilon^{-aT}}$ 2-3.

(b) $e(t) = \epsilon^{-(t-T)}u(t-T)$

$$E(z) = z^{-1} + \epsilon^{-T}z^{-2} + \epsilon^{-2T}z^{-3} + \dots = z^{-1} \left[\frac{z}{z - \epsilon^{-T}} \right] = \frac{1}{z - \epsilon^{-T}}$$

(c) $e(t) = \epsilon^{-(t-5T)}u(t-5T)$

$$E(z) = z^{-5} + \epsilon^{-T}z^{-6} + \epsilon^{-2T}z^{-7} + \dots = z^{-5} \left[\frac{z}{z - \epsilon^{-T}} \right] = \frac{1}{z^4(z - \epsilon^{-T})}$$

2.4-1. A function $e(t)$ is sampled, and the resultant sequence has the z -transform

$$E(z) = \frac{z^3 - 2z}{z^4 - 0.9z^2 + 0.8}$$

Solve this problem using $E(z)$ and the properties of the z -transform.

(a) Find the z -transform of $e(t-2T)u(t-2T)$.

(b) Find the z -transform of $e(t+2)u(t)$.

(c) Find the z -transform of $e(t-T)u(t-2T)$.

Solution:

$$(a) \mathfrak{Z}[e(t-2T)u(t-2T)] = \frac{(z^3 - 2z)z^{-2}}{z^4 - 0.9z^2 + 0.8}$$

$$(b) e(0) = 0, e(1) = 1$$

$$\therefore \mathfrak{Z}[e(t+T)u(t)] = z[E(z) - e(0) - e(1)z^{-1}]$$

$$= z \left[\frac{z^3 - 2z}{z^4 - 0.9z^2 + 0.8} - \frac{1}{z} \right] = \frac{-1.1z^2 + 0.8}{z^4 - 0.9z^2 + 0.8}$$

$$(c) \mathfrak{Z}[e(t-T)u(t-2T)] = e(T)z^{-2} + e(2T)z^{-3} + \dots$$

$$= z^{-1}[E(z) - e(0)] = z^{-1}E(z), \text{ since } e(0) = 0$$

$$= \frac{z^2 - z}{z^4 - 0.9z^2 + 0.8}$$

2.4-2. A function $e(t)$ is sampled, and the resultant sequence has the z -transform

$$E(z) = \frac{z - b}{z^2 - cz^2 + d}$$

Find the z -transform of $\epsilon^{akT}e(kT)$. Solve this problem using $E(z)$ and the properties of the z -transform.

Solution:

By complex translation

$$\mathfrak{Z}[\epsilon^{akT}e(kT)] = E(z\epsilon^{-aT}) = \frac{z\epsilon^{-aT} - b}{z^2\epsilon^{-2aT} - cz^2\epsilon^{-2aT} + d}$$

2.5-1. From Table 2-3,

$$\mathfrak{Z}[\cos akT] = \frac{z(z - \cos aT)}{z^2 - 2z \cos aT + 1}$$

- (a) Find the conditions on the parameter a such that $\mathcal{Z}[\cos akT]$ is first order (pole-zero cancellation occurs).
- (b) Give the first-order transfer function in part (a).
- (c) Find a such that $\mathcal{Z}[\cos akT] = \mathcal{Z}[u(kT)]$, where $u(kT)$ is the unit step function.

Solution:

$$(a) \text{ poles: } z = \frac{z \cos a \pm \sqrt{4 \cos^2 a - 4}}{2} = \cos(a) \pm j \sin(a)$$

$$\therefore \text{ pole} = \cos a, \text{ provided } \sin a = 0 \Rightarrow a = 0, \pm\pi, \pm2\pi, \dots, \pm n\pi$$

$$\text{Then } \cos a = (-1)^n \therefore \text{ poles} = \cos a$$

$$(b) E(z) = \frac{z(z - \cos a)}{(z - \cos a)(z - \cos a)} = \frac{z}{z - \cos a}, \quad a = \pm n\pi, n = 0, 1, \dots$$

$$(c) E(z) = \frac{z}{z - \cos a} = \frac{z}{z - 1}, \quad \therefore \cos a = 1, \quad a = 0, \pm2\pi, \pm4\pi, \dots$$

2.5-2. Find the z -transform, in closed form, of the number sequence generated by sampling the time function $e(t)$ every T seconds beginning at $t = 0$. The function $e(t)$ is specified by its Laplace transform,

$$E(s) = \frac{2(1 - e^{-5s})}{s(s + 2)}, \quad T = 1s$$

Solution:

$$E_1(s) = \frac{2}{s(s + 2)} = \frac{1}{s} + \frac{-1}{s + 2}$$

$$\therefore e_1(t) = (1 - e^{-2t})u(t) \Rightarrow e_1(kT) = (1 - e^{-2kT})u(kT)$$

$$\therefore E_1(z) = (1 + z^{-1} + z^{-2} + \dots) - (1 - e^{-2T}z^{-1} + e^{-4T}z^{-2} + \dots)$$

$$= \frac{1}{1-z^{-1}} - \frac{1}{1-\epsilon^{-2}z^{-1}} = \frac{z}{z-1} - \frac{z}{z-\epsilon^{-2}} = \frac{(1-\epsilon^{-2})z}{(z-1)(z-\epsilon^{-2})}, T=1$$

$$E(z) = E_1(z) - z^{-5}E_1(z) = \frac{(1-\epsilon^{-2})(z^5-1)}{z^4(z-1)(z-\epsilon^{-2})} = \frac{0.8647(z^5-1)}{z^4(z-1)(z-0.1353)}$$

2.6-1. Solve the given difference equation for $x(k)$ using:

$$x(k) - 3x(k-1) + 2x(k-2) = e(k), \quad e(k) = \begin{cases} 1, & k = 0, 1 \\ 0, & k \geq 2 \end{cases}$$

$$x(-2) = x(-1) = 0$$

(a) The sequential technique.

(b) The z -transform.

(c) Will the final-value theorem give the correct value of $x(k)$ as $k \rightarrow \infty$?

Solution:

(a) $x(0) = e(0) = 1$

$$x(1) = e(1) + 3x(0) = 4$$

$$x(2) = e(2) + 3x(1) - 2x(0) = 10$$

$$x(3) = 0 + 3(10) - 2(4) = 22$$

$$x(4) = 0 + 3(22) - 2(10) = 46$$

(b) $[1 - 3z^{-1} + 2z^{-2}]X(z) = E(z) = 1 + z^{-1} = \frac{z+1}{z}$

$$X(z) = \frac{z^2}{(z-1)(z-2)} \times \frac{z+1}{z} = \frac{z(z+1)}{(z-1)(z-2)} = z \left[\frac{-2}{z-1} + \frac{3}{z-2} \right]$$

$$\therefore x(k) = -2 + 3(2)^k$$

(c) No, since the final value does not exist.

2.6-2. Given the difference equation

$$y(k+2) - \frac{3}{4}y(k+1) + \frac{1}{8}y(k) = e(k)$$

where $y(0) = y(1) = 0$, $e(0) = 0$, and $e(k) = 1$, $k = 1, 2, \dots$.

(a) Solve for $y(k)$ as a function of k , and give the numerical values of $y(k)$, $0 \leq k \leq 4$.

(b) Solve the difference equation directly for $y(k)$, $0 \leq k \leq 4$, to verify the results of part (a).

(c) Repeat parts (a) and (b) for $e(k) = 0$ for all k , and $y(0) = 1$, $y(1) = -2$.

Solution:

$$(a) \quad E(z) = \mathcal{Z}[u(k-1)] = z^{-1} \left[\frac{z}{z-1} \right] = \frac{1}{z-1}$$

$$\left[z^2 - \frac{3}{4}z + \frac{1}{8} \right] Y(z) = E(z)$$

$$\frac{Y(z)}{z} = \frac{1}{z \left(z - \frac{1}{2} \right) \left(z - \frac{1}{4} \right)} \cdot \frac{1}{z-1} = \frac{-8}{z} + \frac{8/3}{z-1} + \frac{-16}{z-1/2} + \frac{64/3}{z-1/4}$$

$$\therefore y(k) = -8\delta(0) + \frac{8}{3} - 16 \left(\frac{1}{2} \right)^k + \frac{64}{3} \left(\frac{1}{4} \right)^k$$

$$\therefore y(0) = 0; \quad y(1) = 0; \quad y(2) = 0; \quad y(3) = 1; \quad y(4) = \frac{7}{4}$$

$$(b) \quad y(k+2) = e(k) + \frac{3}{4}y(k+1) - \frac{1}{8}y(k)$$

$$y(2) = 0 + \frac{3}{4}(0) - \frac{1}{8}(0) = 0$$

$$y(3) = 1 + \frac{3}{4}(0) - \frac{1}{8}(0) = 1$$

$$y(4) = 1 + \frac{3}{4}(1) - \frac{1}{8}(0) = 7/4$$

$$(c) \quad (a) \quad y(k+2) - \frac{3}{4}y(k+1) + \frac{1}{8}y(k) = 0$$

$$\therefore z^2[Y(z) - y(0) - y(1)z^{-1}] - \frac{3}{4}z[Y(z) - y(0)] + \frac{1}{8}Y(z) = 0$$

$$\therefore \left[z^2 - \frac{3}{4}z + \frac{1}{8} \right] Y(z) = z^2 - 2z - \frac{3}{4}z$$

$$\therefore Y(z) = z \left[\frac{z - 1/4}{\left(z - 1/2\right)\left(z - 1/4\right)} \right] = z \left[\frac{-9}{z - 1/2} + \frac{10}{z - 1/4} \right] \Rightarrow y(k) = -9 \left(\frac{1}{2} \right)^k + 10 \left(\frac{1}{4} \right)^k$$

$$y(0) = 1, \quad y(1) = -2, \quad y(2) = -13/8, \quad y(3) = -31/32, \quad y(4) = -67/128$$

$$(b) \quad y(k+2) = \frac{3}{4}y(k+1) - \frac{1}{8}y(k)$$

$$y(2) = \frac{3}{4}(-2) - \frac{1}{8}(1) = -13/8$$

$$y(3) = \frac{3}{4} \left(-\frac{13}{8} \right) - \frac{1}{8}(-2) = -31/32$$

$$y(4) = \frac{3}{4} \left(-\frac{31}{32} \right) - \frac{1}{8} \left(-\frac{13}{8} \right) = -\frac{67}{128}$$

2.6-3. Given the difference equation

$$x(k) - x(k-1) + x(k-2) = e(k)$$

where $e(k) = 1$ for $k \geq 0$.

(a) Solve for $x(k)$ as a function of k , using the z -transform. Give the values of $x(0)$, $x(1)$, and $x(2)$.

(b) Verify the values $x(0)$, $x(1)$, and $x(2)$, using the power-series method.

(c) Verify the values $x(0)$, $x(1)$, and $x(2)$ by solving the difference equation directly.

(d) Will the final-value property give the correct value for $x(\infty)$?

Solution:

$$(a) [1 - z^{-1} + z^{-2}]X(z) = E(z) = \frac{z}{z-1}$$

$$X(z) = \frac{z^3}{(z-1)(z^2 - z + 1)}, \quad \text{poles: } z = \frac{1}{2} \pm j\frac{\sqrt{3}}{2} = 1 \angle \pm 60^\circ$$

$$\frac{X(z)}{z} = \frac{1}{z-1} + \frac{k_1}{z-p_1} + \frac{k_1^*}{z-p_1^*} \quad \text{with } p = 1 \angle 60^\circ$$

$$k_1 = \left. \frac{z^2}{(z-1)(z-1 \angle -60^\circ)} \right|_{z=1 \angle 60^\circ} = \frac{1 \angle 120^\circ}{(.5 + j.866 - 1)(.5 + j.866 - .5 + j.866)}$$

$$= \frac{1 \angle 120^\circ}{1 \angle 120^\circ [j2(0.866)]} = 0.5774 \angle -90^\circ$$

$$\therefore aT = \ln(|p_1|) = 0; \quad bT = \arg p_1 = \frac{\pi}{3}$$

$$A = 2|k_1| = 1.155; \quad \theta = \arg k_1 = -90^\circ$$

$$\therefore x(k) = 1 + 1.155 \cos\left(\frac{\pi}{3}k - 90^\circ\right) = 1 + 1.155 \sin\left(\frac{\pi}{3}k\right)$$

$$x(0) = 1, \quad x(1) = 2, \quad x(2) = 2$$

$$(b) \quad \begin{array}{l} \frac{1 + 2z^{-1} + 2z^{-2} + \dots}{z^3 - 2z^2 + 2z - 1} \Bigg) z^3 \\ \hline \frac{z^3 - 2z^2 + 2z - 1}{2z^2 - 2z + 1} \\ \hline \frac{2z^2 - 4z + 4 - 2z^{-1}}{2z + \dots} \end{array} \quad \begin{array}{l} \therefore x(0) = 1 \\ x(1) = 2 \\ x(2) = 2 \end{array}$$

$$(c) \quad x(k) = 1 + x(k-1) - x(k-2)$$

$$x(0) = 1 + 0 - 0 = 1$$

$$x(1) = 1 + 1 - 0 = 2$$

$$x(2) = 1 + 2 - 1 = 2$$

$$(d) \quad \text{No, 3 poles for } X(z) \text{ on the unit circle.}$$

2.6-4. Given the difference equation

$$x(k+2) + 3x(k+1) + 2x(k) = e(k)$$

where

$$e(k) = \begin{cases} 1, & k = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$x(0) = 1$$

$$x(1) = -1$$

(a) Solve for $x(k)$ as a function of k .

(b) Evaluate $x(0)$, $x(1)$, $x(2)$, and $x(3)$ in part (a).

(c) Verify the results in part (b) using the power-series method.

(d) Verify the results in part (b) by solving the difference equation directly.

Solution:

$$(a) \quad z^2[X(z) - x(0) - x(1)z^{-1}] + 3z[X(z) - x(0)] + 2X(z) = E(z) = 1$$

$$\therefore X(z) = \frac{1 + z^2 - z + 3z}{z^2 - 3z + 2} = \frac{z^2 + 2z + 1}{z^2 + 3z + 2} = \frac{z + 1}{z + 2}$$

$$\therefore X(z) = z \left[\frac{z + 1}{z(z + 2)} \right] = z \left[\frac{\frac{1}{2}}{z} + \frac{\frac{1}{2}}{z + 2} \right]$$

$$\therefore x(k) = \frac{1}{2} \delta(k) + \frac{1}{2} (-2)^k$$

$$(b) \quad x(0) = 1, \quad x(1) = -1, \quad x(2) = 2, \quad x(3) = -4$$

$$\begin{array}{r}
 (c) \quad \frac{1 - z^{-1} + 2z^{-2} - 4z^{-3} + \dots}{z + 2} \\
 \underline{z + 2} \\
 -1 \\
 \underline{-1 - 2z^{-1}} \\
 2z^{-1} \\
 \underline{2z^{-1} + 4z^{-2}} \\
 -4z^{-2} \\
 \dots
 \end{array}$$

$$(d) \quad x(k+2) = e(k) - 3x(k+1) - 2x(k)$$

$$x(2) = 1 - 3(-1) - 2(1) = 2$$

$$x(3) = 0 - 3(2) - 2(-1) = -4$$

2.6-5. Given the difference equation

$$x(k+3) - 2.2x(k+2) + 1.57x(k+1) - 0.36x(k) = e(k)$$

where $e(k) = 1$ for all $k \geq 0$, and $x(0) = x(1) = x(2) = 0$.

(a) Write a digital computer program that will calculate $x(k)$. Run this program solving for $x(3)$, $x(4)$, \dots , $x(25)$.

(b) Using the sequential technique, check the values of $x(k)$, $0 \leq k \leq 5$.

(c) Use the z -transform and the power-series method to verify the values $x(k)$, $0 \leq k \leq 5$.

Solution:

$$(a) \quad x_0 = 0;$$

$$x_1 = 0;$$

$$x_2 = 0;$$

$$\text{for } k = 0:5;$$

$$x_3 = 2.2*x_2 - 1.57*x_1 + 0.36*x_0 + 1$$

$$x_0 = x_1;$$

$$x_2 = x_3;$$

(b) $x(k+3) = e(k) + 2.2x(k+2) - 1.57x(k+1) + 0.36x(k)$

$$x(4) = 1 + 2.2(1) - 0 + 0 = 3.2$$

(c) $[z^3 - 2.2z^2 + 1.57z - 0.36]X(z) = E(z) = \frac{z}{z-1}$

$$\begin{array}{r} z^{-3} + 3.2z^{-4} + 6.47z^{-5} + \dots \\ z^4 - 3.2z^3 + 3.77z^2 - 1.93z + 0.36 \overline{) } \\ \hline z - 3.2 + 3.77z^{-1} - \dots \\ 3.2 - 3.77z^{-1} \\ \hline 3.2 - 10.24z^{-1} + \dots \\ 6.47z^{-1} + \dots \\ \hline \vdots \end{array}$$

$$x(4) = 3.2$$

$$x(5) = 6.47$$

$$E(z) = \frac{0.1}{z(z - 0.9)}$$

(b) Check the value of $e(0)$ using the initial-value property.

(c) Check the values calculated in part (a) using partial fractions.

(d) Find $e(k)$ for $k = 0, 1, 2, 3$, and 4 if $\mathcal{Z}[e(k)]$ is given by

$$E(z) = \frac{1.98z}{(z^2 - 0.9z + 0.9)(z - 0.8)(z^2 - 1.2z + 0.27)}$$

(e) Find a function $e(t)$ which, when sampled at a rate of 10 Hz ($T = 0.1s$), results in the transform $E(z) = 2z/(z - 0.8)$.

(f) Repeat part (e) for $E(z) = 2z/(z + 0.8)$.

(g) From parts (e) and (f), what is the effect on the inverse z -transform of changing the sign on a real pole?

Solution:

$$(a) e(k) = \sum_{\text{residues}} \frac{0.1z^{k-1}}{z(z-0.9)} = \sum_{\text{residues}} \frac{0.1z^{k-2}}{z-0.9}$$

$$k=0: \text{fcn} = \frac{0.1}{z^2(z-0.9)}, \therefore \text{residue}|_{z=0.9} = \frac{0.1}{(0.9)^2} = 0.1235$$

$$\text{residue}|_{z=0} = \frac{d}{dz} \left[\frac{0.1}{z-0.9} \right]_{z=0} = \frac{-0.1(1)}{(z-0.9)^2} \Big|_{z=0} = \frac{-0.1}{(0.9)^2} = -0.1235$$

$$\therefore e(0) = 0$$

$$k=1: e(1) = \frac{0.1}{z-0.9} \Big|_{z=0} + \frac{0.1}{z} \Big|_{z=0.9} = 0$$

$$k=10: e(10) = 0.1(0.9)^8$$

$$(b) e(0) = \lim_{z \rightarrow \infty} E(z) = \lim_{z \rightarrow \infty} \frac{0.1}{z(z-0.9)} = 0$$

$$(c) \frac{E(z)}{z} = \frac{0.1}{z^2(z-0.9)} = \frac{k_1}{z^2} + \frac{k_2}{z} + \frac{k_3}{z-0.9}$$

$$k_1 = \frac{-0.1}{0.9} = -\frac{1}{9}; \quad k_3 = \frac{0.1}{(0.9)^2} = \frac{1}{8.1}$$

$$k_2 = \frac{d}{dz} \left[\frac{0.1}{z-0.9} \right]_{z=0} = \frac{-1}{8.1}, \text{ from (a)}$$

$$\therefore e(k) = \frac{-1}{8.1} \delta(k) - \frac{1}{9} \delta(k-1) + \frac{1}{8.1} (0.9)^k$$

$$x(0) = -\frac{1}{8.1} + 0 + \frac{1}{8.1} = 0; \quad x(1) = -0 - \frac{1}{9} + \frac{0.9}{8.1} = 0$$

$$x(10) = -0 - 0 + \frac{0.1}{(0.9)^2} (0.9)^{10} = 0.1(0.9)^8$$

$$(d) \quad E(z) = \frac{1.98z}{z^5 + \dots} = 1.98z^{-4} + (\cdot)z^{-5} + (\cdot)z^{-6} + \dots$$

$$\therefore e(0) = e(1) = e(2) = e(3) = 0; \quad e(4) = 1.98$$

$$(e) \quad E(z) = \frac{2z}{z - 0.8} = \frac{2z}{z - \epsilon^{-aT}} \quad \therefore \epsilon^{-aT} = 0.8 \Rightarrow aT = 0.2231$$

$$\therefore a = \frac{0.2231}{0.1} = 2.231, \quad \therefore e(t) = 2\epsilon^{-2.231t} u(t)$$

$$(f) \quad E(z) = \frac{2z}{z - (-0.8)}; \quad \therefore \epsilon^{-aT} \epsilon^{j\pi} = -0.8 \Rightarrow aT = 2.231$$

$$\therefore e(t) = 2e^{-2.231t} \cos 10\pi t \quad \text{where } \frac{\omega_s}{2} = 10\pi$$

$$(g) \quad (e) \quad e(k) = (0.8)^k; \quad (f) \quad e(k) = (-0.8)^k$$

\therefore sign alternates on $e(k)$.

2.7-2. For the number sequence $\{e(k)\}$,

$$E(z) = \frac{z}{(z+1)^2}$$

(a) Apply the final-value theorem to $E(z)$.

(b) Check your result in part (a) by finding the inverse z -transform of $E(z)$.

(c) Repeat parts (a) and (b) with $E(z) = z/(z-1)^2$.

(d) Repeat parts (a) and (b) with $E(z) = z/(z-0.9)^2$.

(e) Repeat parts (a) and (b) with $E(z) = z/(z - 1.1)^2$.

Solution:

$$(a) \quad e(\infty) = \lim_{z \rightarrow 1} (z-1)E(z) = \left. \frac{z(z-1)}{(z+1)^2} \right|_{z=1} = 0$$

$$(b) \quad e(k) = z^{-1} \left[\frac{z}{(z-1)^2} \right] = k(-1)^k, \quad \therefore e(\infty) \text{ unbounded}$$

$$(c) \quad (a) \quad e(\infty) = \lim_{z \rightarrow 1} (z-1) \frac{z}{(z-1)^2}, \quad \therefore \text{unbounded}$$

$$(b) \quad e(k) = k, \quad \therefore \text{unbounded}$$

$$(d) \quad (a) \quad e(\infty) = \lim_{z \rightarrow 1} (z-1) \frac{z}{(z-0.9)^2} = 0$$

$$(b) \quad e(k) = k(0.9)^k; \quad \therefore e(\infty) \rightarrow 0$$

$$(e) \quad (a) \quad e(\infty) = \lim_{z \rightarrow 1} (z-1) \frac{z}{(z-1.1)^2} = 0$$

$$(b) \quad e(k) = k(1.1)^k; \quad \therefore e(\infty) \text{ is unbounded.}$$

2.7-3. Find the inverse z -transform of each $E(z)$ below by the four methods given in the text. Compare the values of $e(z)$, for $k = 0, 1, 2$, and 3 , obtained by the four methods.

$$(a) \quad E(z) = \frac{0.5z}{(z-1)(z-0.6)}$$

$$(b) \quad E(z) = \frac{0.5}{(z-1)(z-0.6)}$$

$$(c) \quad E(z) = \frac{0.5(z+1)}{(z-1)(z-0.6)}$$

$$(d) \quad E(z) = \frac{z(z-0.7)}{(z-1)(z-0.6)}$$

(e) Use MATLAB to verify the partial-fraction expansions.

Solution:

$$(a) \text{ (i) } \begin{array}{r} 0.5z^{-1} + 0.8z^{-2} + 0.98z^{-3} + \dots \\ z^2 - 1.6z + 0.6 \overline{) 0.5z} \\ \underline{0.5z - 0.8 + 0.3z^{-1}} \\ 0.8 - 0.3z^{-1} \\ \underline{0.8 - 1.28z^{-1} + \dots} \\ 0.98z^{-1} + \dots \end{array}$$

$$(ii) \frac{E(z)}{z} = \frac{0.5}{(z-1)(z-0.6)} = \frac{1.25}{z-1} + \frac{-1.25}{z-0.6}; \quad \therefore E(z) = \frac{1.25z}{z-1} - \frac{1.25z}{z-0.6}$$

$$\therefore e(k) = 1.25(1 - 0.6^k)u(k)$$

$$(iii) z^{k-1}E(z) = \frac{0.5z^k}{(z-1)(z-0.6)}$$

$$e(k) = \frac{0.5(1)^k}{1-0.6} + \frac{0.5(0.6)^k}{0.6-1} = 1.25(1 - 0.6^k)u(k)$$

$$(iv) E_1(z) = \frac{0.5z}{z-0.6} \Rightarrow e_1(k) = 0.5(0.6)^k$$

$$E_2(z) = \frac{1}{z-1} \Rightarrow e_2(0) = 0; \quad e_2(k) = 1, \quad k \geq 1$$

$$e(0) = e_1(0)e_2(0) = (0.5)(0) = 0$$

$$e(1) = e_1(0)e_2(1) + e_1(1)e_2(0) = (0.5)(1) + (0.3)(0) = 0.5$$

$$e(2) = e_1(0)e_2(2) + e_1(1)e_2(1) + e_1(2)e_2(0)$$

$$= 0.5 \times 1 + 0.3 \times 1 + 0.18 \times 0 = 0.8$$

$$e(3) = 0.5 \times 1 + 0.3 \times 1 + 0.18 \times 1 + 0.108 \times 0 = 0.98$$

$$(b) \quad e(0) = 0$$

$$e(k) = 1.25 - 2.083(0.6)^k, \quad k \geq 1$$

$$E(z) = 0.5z^{-2} + 0.8z^{-3} + 0.98z^{-4} + 1.088z^{-5} + \dots$$

$$(c) \quad e(0) = 0; \quad e(k) = 2.5 - 3.33(0.6)^k, \quad k \geq 1$$

$$E(z) = 0.5z^{-1} + 1.30z^{-2} + 1.78z^{-3} + 2.068z^{-4} + 2.2408z^{-5} + \dots$$

$$(d) \quad e(k) = 0.75 + 0.25(0.6)^k$$

$$E(z) = 1 + 0.9z^{-1} + 0.84z^{-2} + 0.804z^{-3} + \dots$$

(e) num=[0 0 0.5];

den=[1 -1.6 0.6];

[r, p, k] = residue (num, den)

2.8-1. Given in Fig. P2.8-1 are two digital-filter structures, or realizations, for second-order filters.

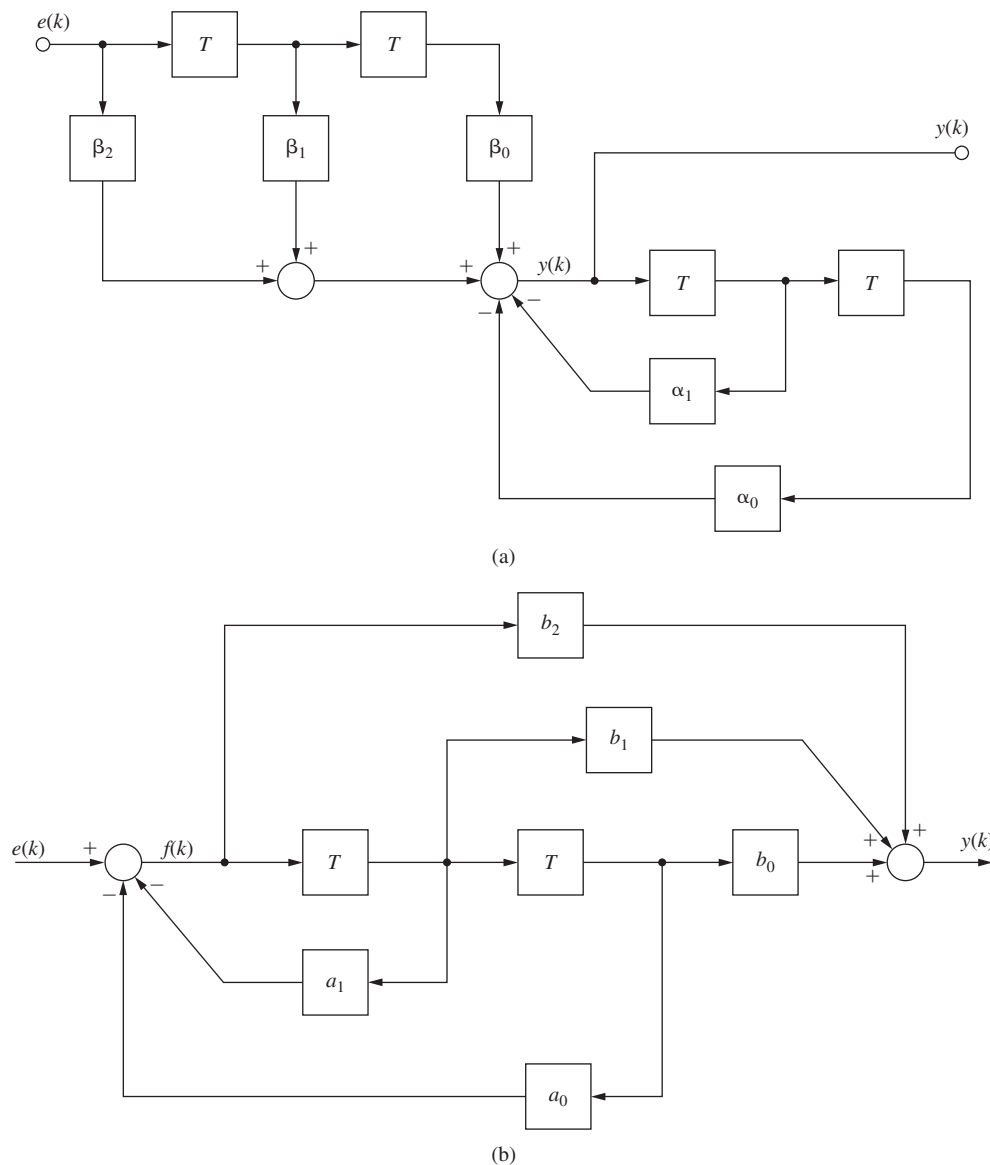


FIGURE P2.8-1 Digital-filter structures: (a) 3D; (b) 1D.

- (a) Write the difference equation for the 3D structure of Fig. P2.8-1(a), expressing $y(k)$ as a function of $y(k-i)$ and $e(k-i)$.
- (b) Derive the filter transfer function $Y(z)/E(z)$ for the 3D structure by taking the z -transform of the equation in part (a).
- (c) Write the difference equation for the 1D structure of Fig. P2.8-1(b). Two equations are required, with one for $f(k)$ and one for $y(k)$.
- (d) Derive the filter transfer function $Y(z)/E(z)$ for the 1D structure by taking the z -transform of the equations in part (c) and eliminating $F(z)$.
- (e) From parts (b) and (d), relate the coefficients α_i, β_i to a_i, b_i such that the two filters realize the same transfer function.
- (f) Write a computer-program segment that realizes the 3D structure. This program should be of the form used in Example 2.10.
- (g) Write a MATLAB-program segment that realizes the 1D structure. This program should be of the form used in Example 2.10.

Solution:

$$(a) \quad y(k) = \beta_2 e(k) + \beta_1 e(k-1) + \beta_0 e(k-2) - \alpha_1 y(k-1) - \alpha_0 y(k-2)$$

$$(b) \quad [1 + \alpha_1 z^{-1} + \alpha_0 z^{-2}] Y(z) = [\beta_2 + \beta_1 z^{-1} + \beta_0 z^{-2}] E(z)$$

$$\frac{Y(z)}{E(z)} = \frac{\beta_2 z^2 + \beta_1 z + \beta_0}{z^2 + \alpha_1 z + \alpha_0}$$

$$(c) \quad f(k) = e(k) - a_1 f(k-1) - a_0 f(k-2)$$

$$y(k) = b_2 f(k) + b_1 f(k-1) + b_0 f(k-2)$$

$$(d) \quad F(z) = E(z) - (a_1 z^{-1} + a_0 z^{-2}) F(z) \Rightarrow F(z) = \frac{E(z)}{1 + a_1 z^{-1} + a_0 z^{-2}}$$

$$Y(z) = (b_2 + b_1 z^{-1} + b_0 z^{-2}) F(z) = \frac{b_2 z^2 + b_1 z + b_0}{z^2 + a_1 z + a_0} E(z)$$

$$(e) \quad \alpha_i = a_i \quad \text{and} \quad \beta_i = b_i, \quad i = 1, 2$$

(f)

```
ykminus2 = 0;
```

```
ykminus1 = 0;
```

```
ekminus2 = 0;
```

```
ekminus1 = 0;
```

```
ek = 1;
```

```
for k = 0:5
```

```
    yk=b2*ek+b1*ekminus1+b0*ekminus2-a1*ykminus1-a0*ykminus2;
```

```
    [k, ek, yk]
```

```
    ekminus2 = ekminus1;
```

```
    ekminus1 = ek;
```

```
    ykminus2 = ykminus1;
```

```
    ykminus1 = yk;
```

```
end
```

(g)

```
fkminus2 = 0;
```

```
fkminus1 = 0;
```

```
ek = 1;
```

```
for k = 0:5
```

```
    fk=ek-a1*fkminus1-a0*fkminus2;
```

```
    yk = b2*fk+b1*fkminus1+b0*fkminus2;
```

```
    [k, ek, yk]
```

```
    fkminus2 = fkminus1;
```

```
    fkminus1 = fk;
```

```
end
```

2.8-2. Shown in Fig. P2.8-2 is the second-order digital-filter structure 1X.

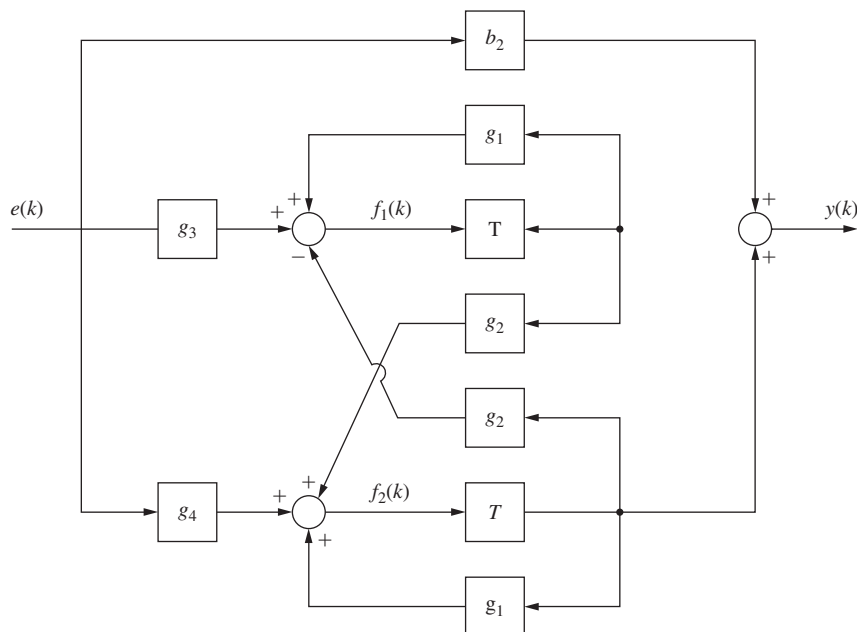


FIGURE P2.8-2 Digital-filter structure 1X.

This structure realizes the filter transfer function

$$D(z) = b_2 + \frac{A}{z-p} + \frac{A^*}{z-p^*}$$

where p and p^* (conjugate of p) are complex. The relationships between the filter coefficients and the coefficients in Fig. P2.8-2 are given by

$$g_1 = \operatorname{Re}(p) \quad g_3 = -2 \operatorname{Im}(A)$$

$$g_2 = \text{Im}(p) \quad g_4 = 2 \text{Re}(A)$$

- To realize this filter, difference equations are required for $f_1(k)$, $f_2(k)$, and $y(k)$. Write these equations.
- Find the filter transfer function $Y(z)/E(z)$ by taking the z -transform of the equations of part (a) and eliminating $F_1(z)$ and $F_2(z)$.
- Verify the results in part (b) using Mason's gain formula.
- Write a MATLAB-program segment that realizes the 1X structure. This program should be of the form of that is used in Example 2.10.

Solution:

$$(a) \quad f_1(k) = g_1 f_1(k-1) - g_2 f_2(k-1) + g_3 e(k)$$

$$f_2(k) = g_1 f_2(k-1) + g_1 f_1(k-1) + g_4 e(k)$$

$$y(k) = b_2 e(k) + f_2(k-1)$$

$$(b) \quad (1) \quad F_1(z) = g_1 z^{-1} F_1(z) - g_2 z^{-1} F_2(z) + g_3 E(z)$$

$$(2) \quad F_2(z) = g_1 z^{-1} F_2(z) + g_2 z^{-1} F_1(z) + g_4 E(z)$$

$$(3) \quad Y(z) = b_0 E(z) + z^{-1} F_2(z)$$

$$\therefore (1) \quad (z - g_1) F_1(z) + g_2 F_2(z) = g_3 z E(z)$$

$$(2) \quad -g_2 F_1(z) + (z - g_1) F_2(z) = g_4 z E(z)$$

$$\therefore F_2(z) = \frac{\begin{vmatrix} z - g_1 & g_3 z E(z) \\ -g_2 & g_4 z E(z) \end{vmatrix}}{\begin{vmatrix} z - g_1 & g_2 \\ -g_2 & z - g_1 \end{vmatrix}} = \frac{(g_4 z^2 - g_1 g_4 z + g_2 g_3 z)}{(z - g_1)^2 + g_2^2} E(z)$$

$$\therefore \frac{Y(z)}{E(z)} = b_2 + \frac{g_4 z + g_2 g_3 - g_1 g_4}{(z - g_1)^2 + g_2^2}$$

$$\text{also, } D(z) = b_2 + \frac{\operatorname{Re}(A) + j \operatorname{Im}(A)}{z - \operatorname{Re}(p) - j \operatorname{Im}(p)} + \frac{\operatorname{Re}(A) - j \operatorname{Im}(A)}{z - \operatorname{Re}(p) + j \operatorname{Im}(p)}$$

$$= b_2 + \frac{\frac{1}{2}(g_4 - j g_3)}{z - g_1 - j g_2} + \frac{\frac{1}{2}(g_4 + j g_3)}{z - g_1 + j g_2}$$

$$= b_2 + \frac{g_4 z - g_1 g_4 + g_2 g_3}{(z - g_1)^2 + g_2^2}$$

$$(c) \quad D(z) = b_0 + \frac{g_2 g_3 z^{-2} + g_4 (1 - g_1 z^{-1})}{1 - g_1 z^{-1} - g_1 z^{-1} + g_1^2 z^{-2} + g_2^2 z^{-2}}$$

$$= b_0 + \frac{g_4 z + g_2 g_3 - g_1 g_4}{z^2 - 2g_1 z + g_1^2 + g_2^2}$$

$$(d) \quad f1_{k \text{ minus } 1} = 0;$$

$$f2_{k \text{ minus } 1} = 0;$$

```

ek = 1;

for k = 0:5

    yk = b0*ek+f2kminus1;

    [k, ek, yk]

    f1k = g1*f1kminus1 - g2*f2kminus1 + g3*ek;

    f2k = g1*f2kminus1 + g2*f1kminus1 + g3*ek;

    f1kminus1 = f1k;

    f2kminus1 = f2k;

end

```

2.8-3. Given the second-order digital-filter transfer function

$$D(z) = \frac{2z^2 - 2.4z + 0.72}{z^2 - 1.4z + 0.98}$$

- (a) Find the coefficients of the 3D structure of Fig. P2.8-1 such that $D(z)$ is realized.
- (b) Find the coefficients of the ID structure of Fig. P2.8-1 such that $D(z)$ is realized.
- (c) Find the coefficients of the IX structure of Fig. P2.8-2 such that $D(z)$ is realized.

The coefficients are identified in Problem 2.8-2.

- (d) Use MATLAB to verify the partial-fraction expansions in part (c).
- (e) Verify the results in part (c) using Mason's gain formula.

Solution:

- (a) $\beta_2 = 2, \beta_1 = -2.4, \beta_0 = 0.72, \alpha_1 = -1.4, \alpha_0 = 0.98$
- (b) $b_2 = 2, b_1 = -2.4, b_0 = 0.72, a_1 = -1.4, a_0 = 0.98$

(c) poles: $z = \frac{1.4 \pm (1.4^2 - 4(0.98))^{1/2}}{2} = 0.7 \pm j0.7 = 0.99 \angle \pm 45^\circ$

$$D(z) = 2 + \frac{A}{z - 0.7 - j0.7} + \frac{A^*}{z - 0.7 + j0.7}$$

$$\therefore A = \left. \frac{2z^2 - 2.4z + 0.72}{z - 0.7 + j0.7} \right|_{z=0.99\angle 45^\circ} = \frac{j1.96 - (1.68 + j1.68) + 0.72}{j1.4}$$

$$= 0.2 + j0.6857$$

$$\therefore g_1 = 0.7 \quad g_3 = 1.371$$

$$g_2 = 0.7 \quad g_4 = 0.4$$

$$(d) \text{ num} = [2 \quad -2.4 \quad .72] ;$$

$$\text{den} = [1 \quad -1.4 \quad 0.98] ;$$

$$[r,p,k]=\text{residue}(\text{num}, \text{den})$$

$$(e) \Delta = 1 - (0.7z^{-1} + 0.7z^{-1} + 0.4z^{-2}) + 0.49z^{-2}$$

$$= 1 - 1.4z^{-1} + 0.98z^{-2}$$

$$D(z) = 2 + \frac{1}{\Delta} [1.371(0.7)z^{-2} + 0.4z^{-1}(1 + 0.7z^{-1})]$$

$$= 2 + \frac{0.4z - 1.24}{z^2 - 1.4z + 0.98} = \frac{2z^2 - 2.4z + 0.72}{z^2 - 1.4z + 0.98}$$

2.9-1. Find two different state-variable formulations that model the system whose difference equation is given by:

$$(a) y(k+2) + 6y(k+1) + 5y(k) = 2e(k)$$

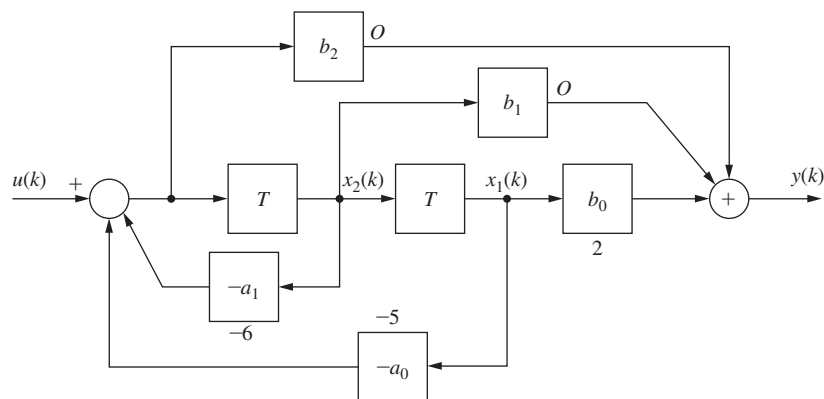
$$(b) y(k+2) + 6y(k+1) + 5y(k) = e(k+1) + 2e(k)$$

$$(c) y(k+2) + 6y(k+1) + 5y(k) = 3e(k+2) + e(k+1) + 2e(k)$$

Solution:

$$(a) \frac{Y(z)}{U(z)} = \frac{2}{z^2 + 6z + 5}$$

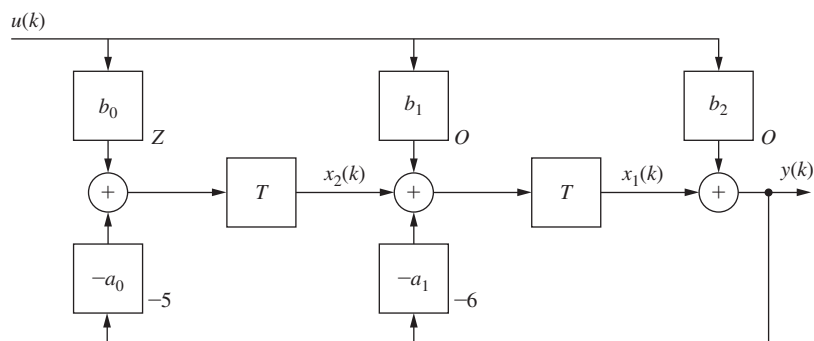
(1) control canonical:



$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = [2 \quad 0] \mathbf{x}(k)$$

(2) observer canonical:



$$\mathbf{x}(k+1) = \begin{bmatrix} -6 & 1 \\ -5 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u(k)$$

$$y(k) = [1 \quad 0] \mathbf{x}(k)$$

(b) $\frac{Y(z)}{U(z)} = \frac{z+2}{z^2+6z+5}$ (1) control canonical: $\mathbf{x}(k+1) = \text{same as (a)}$
 $y(k) = [2 \quad 1] \mathbf{x}(k)$

(2) observer canonical:

$$\mathbf{x}(k+1) = \begin{bmatrix} -6 & 1 \\ -5 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(k)$$

$$y(k) = [1 \quad 0] \mathbf{x}(k)$$

(c) $\frac{Y(z)}{U(z)} = \frac{3z^2+z+2}{z^2+6z+5}$ (1) control canonical: $\mathbf{x}(k+1) = \text{same as (a)}$
 $y(k) = [-13 \quad -17] \mathbf{x}(k) + 3u(k)$

(2) observer canonical:

$$\mathbf{x}(k+1) = \begin{bmatrix} -6 & 1 \\ -5 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(k)$$

$$y(k) = [1 \quad 0] \mathbf{x}(k) + 3u(k)$$

2.9-2. Write the state equations for the observer canonical form of a system, shown in Fig. 2-10, which has the transfer function given in (2-51) and (2-61)

$$G(z) = \frac{b_{n-1}z^{n-1} + \cdots + b_1z + b_0}{z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0}$$

Solution:

$$\mathbf{x}(k+1) = \begin{bmatrix} a_{n-1} & 1 & 0 & \cdots & 0 \\ a_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_0 \end{bmatrix} u(k)$$

$$y(k) = [1 \quad 0 \quad 0 \quad \cdots \quad 0] \mathbf{x}(k)$$

2.10-1. Find a state-variable formulation for the system described by the coupled second-order difference equations given. The system output is $y(k)$, and $e_1(k)$ and $e_2(k)$ are the system inputs. *Hint:*

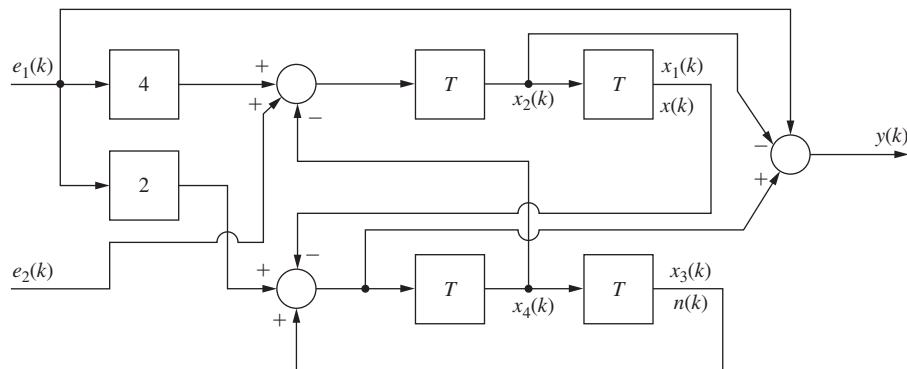
Draw a simulation diagram first.

$$x(k+2) + v(k+1) = 4e_1(k) + e_2(k)$$

$$v(k+2) - v(k) + x(k) = 2e_1(k)$$

$$y(k) = v(k+2) - x(k+1) + e_1(k)$$

Solution:



$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 & 0 \\ 4 & 1 \\ 0 & 0 \\ 2 & 0 \end{bmatrix} e(k)$$

$$y(k) = x_4(k+1) - x_2(k) + e_1(k) = -x_1(k) + x_3(k) - x_2(k) + e_1(k)$$

$$\therefore y(k) = [-1 \quad -1 \quad 1 \quad 0] \mathbf{x}(k) + [1 \quad 0] e(k)$$

2.10-2. Consider the system described by

$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} -2 & 1 \end{bmatrix} \mathbf{x}(k)$$

- Find the transfer function $Y(z)/U(z)$.
- Using any similarity transformation, find a different state model for this system.
- Find the transfer function of the system from the transformed state equations.
- Verify that \mathbf{A} given and \mathbf{A}_w derived in part (b) satisfy the first three properties of similarity transformations. The fourth property was verified in part (c).

Solution:

$$(a) \quad z\mathbf{I} - \mathbf{A} = \begin{bmatrix} z & -1 \\ 0 & z-3 \end{bmatrix}; \quad \Delta = |z\mathbf{I} - \mathbf{A}| = z(z-3) = \Delta$$

$$\frac{Y(z)}{U(z)} = \mathbf{C}[z\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} = \frac{1}{\Delta} \begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} z-3 & 1 \\ 0 & z \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\Delta} \begin{bmatrix} -2z+6 & z-2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{-z+4}{z(z-3)}$$

$$(b) \quad \mathbf{P} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}; \quad \mathbf{P}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\mathbf{A}_w = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$$

$$\mathbf{B}_w = \mathbf{P}^{-1}\mathbf{B} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathbf{C}_w = \mathbf{C}\mathbf{P} = \begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 3 \end{bmatrix}$$

$$\therefore \mathbf{w}(k+1) = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \mathbf{w}(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathbf{u}(k)$$

$$\mathbf{y}(k) = \begin{bmatrix} -1 & 3 \end{bmatrix} \mathbf{w}(k)$$

$$(c) \quad z\mathbf{I} - \mathbf{A}_w = \begin{bmatrix} z-2 & -2 \\ -1 & z-1 \end{bmatrix}; \quad \Delta = |z\mathbf{I} - \mathbf{A}_w| = z^2 - 3z + 2 - 2 = z(z-3)$$

$$\frac{Y(z)}{U(z)} = \mathbf{C}_w[z\mathbf{I} - \mathbf{A}_w]^{-1}\mathbf{B}_w = \frac{1}{\Delta} \begin{bmatrix} -1 & 3 \end{bmatrix} \begin{bmatrix} z-1 & 2 \\ 1 & z-2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\Delta} \begin{bmatrix} -1 & 3 \end{bmatrix} \begin{bmatrix} z-1 \\ 1 \end{bmatrix} = \frac{-z+4}{z(z-3)}$$

$$(d) \quad |z\mathbf{I} - \mathbf{A}| = \begin{vmatrix} z & 1 \\ 0 & z-3 \end{vmatrix} = z^2 - 3z; \quad |z\mathbf{I} - \mathbf{A}_w| = \begin{vmatrix} z-2 & -2 \\ -1 & z-1 \end{vmatrix} = z(z-3)$$

$$\therefore z_1 = 0, z_2 = 3$$

$$|\mathbf{A}| = \begin{vmatrix} 0 & 1 \\ 0 & 3 \end{vmatrix} = 0 = z_1 z_2; \quad |\mathbf{A}_w| = \begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} = 0$$

$$\text{tr } \mathbf{A} = 3 = z_1 + z_2; \quad \text{tr } \mathbf{A}_w = 3$$

2.10-3. Consider the system of Problem 2.10-2. A similarity transformation on these equations yields

$$\mathbf{w}(k+1) = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \mathbf{w}(k) + \mathbf{B}_w u(k)$$

$$y(k) = \mathbf{C}_w \mathbf{x}(k)$$

(a) Find d_1 and d_2 .

(b) Find a similarity transformation that results in the \mathbf{A}_w matrix given. Note that this matrix is diagonal.

(c) Find \mathbf{B}_w and \mathbf{C}_w .

(d) Find the transfer functions of both sets of state equations to verify the results of this problem.

Solution:

(a) Let z_1, z_2 be the characteristic value of \mathbf{A} . $d_1 = z_1$, $d_2 = z_2$

$$z\mathbf{I} - \mathbf{A} = \begin{bmatrix} z & -1 \\ 0 & z-3 \end{bmatrix}, \quad \therefore |z\mathbf{I} - \mathbf{A}| = z(z-3); \quad \therefore z_1 = 0, \quad z_2 = 3$$

$$(b) \quad (z_1\mathbf{I} - \mathbf{A})\mathbf{m}_1 = \begin{bmatrix} 0 & -1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} m_{11} \\ m_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} -m_{21} &= 0 \\ -3m_{21} &= 0 \end{aligned}$$

$$\therefore m_{21} = 0, \text{ let } m_{11} = 1, \quad \therefore \mathbf{m}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(z_2\mathbf{I} - \mathbf{A})\mathbf{m}_2 = \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} m_{12} \\ m_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 3m_{12} - m_{22} = 0$$

$$\therefore \text{let } m_{12} = 1, \quad m_{22} = 3, \quad \therefore \mathbf{m}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\therefore \mathbf{M} = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}, \quad |\mathbf{M}| = 3, \quad \mathbf{M}^{-1} = \begin{bmatrix} 1 & -1/3 \\ 0 & 1/3 \end{bmatrix}$$

$$\mathbf{M}^{-1}\mathbf{A}\mathbf{M} = \begin{bmatrix} 1 & -1/3 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1/3 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$$

$$(c) \quad \mathbf{B}_w = \mathbf{M}^{-1}\mathbf{B} = \begin{bmatrix} 1 & -1/3 \\ 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$$

$$\mathbf{C}_w = \mathbf{C}\mathbf{M} = \begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 1 \end{bmatrix}$$

$$\therefore \mathbf{w}(k+1) = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} \mathbf{w}(k) + \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} \mathbf{u}(k)$$

$$\mathbf{y}(k) = \begin{bmatrix} -2 & 1 \end{bmatrix} \mathbf{w}(k)$$

(d) See Problem 2.10-2(a) for the first transfer function.

$$z\mathbf{I} - \mathbf{A}_w = \begin{bmatrix} z & 0 \\ 0 & z-3 \end{bmatrix}; |z\mathbf{I} - \mathbf{A}_w| = z(z-3) = \Delta$$

$$\frac{Y(z)}{U(z)} = \mathbf{C}_w [z\mathbf{I} - \mathbf{A}_w]^{-1} \mathbf{B}_w = \frac{1}{\Delta} \begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} z-3 & 0 \\ 0 & z \end{bmatrix} \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$$

$$= \frac{1}{\Delta} \begin{bmatrix} -2z+6 & z \end{bmatrix} \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} = \frac{-\frac{4}{3}z+4+\frac{1}{3}z}{\Delta} = \frac{-z+4}{z(z-3)}$$

2.10-4. Repeat Problem 2.10-2 for the system described by

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 2 \end{bmatrix} \mathbf{x}(k)$$

(a) Find the transfer function $Y(z)/U(z)$.

(b) Using any similarity transformation, find a different state model for this system.

(c) Find the transfer function of the system from the transformed state equations.

(d) Verify that \mathbf{A} given and \mathbf{A}_w derived in part (b) satisfy the first three properties of similarity transformations. The fourth property was verified in part (c).

Solution:

(a)

$$\begin{aligned}\frac{Y(z)}{U(z)} &= \mathbf{C}[\mathbf{zI} - \mathbf{A}]^{-1}\mathbf{B} = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{z-1} & 0 \\ 0 & \frac{1}{z-0.5} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{2}{z-1} \\ \frac{1}{z-0.5} \end{bmatrix} = \frac{2}{z-1} + \frac{2}{z-0.5} = \frac{4z-3}{(z-1)(z-0.5)}\end{aligned}$$

$$(b) \quad \mathbf{P} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{P}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\therefore \mathbf{A}_w = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

$$\mathbf{B}_w = \mathbf{P}^{-1}\mathbf{B} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$\mathbf{C}_w = \mathbf{C}\mathbf{P} = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \end{bmatrix}$$

$$\therefore \mathbf{w}(k+1) = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix} \mathbf{w}(k) + \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{bmatrix} \mathbf{u}(k)$$

$$\mathbf{y}(k) = \begin{bmatrix} 3 & 1 \end{bmatrix} \mathbf{x}(k)$$

$$(c) \quad \mathbf{zI} - \mathbf{A}_w = \begin{bmatrix} z - \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & z - \frac{3}{4} \end{bmatrix}, \quad |\mathbf{zI} - \mathbf{A}_w| = z^2 - 1.5z + \frac{9}{16} - \frac{1}{16} = z^2 - 1.5z + 0.5 = \Delta$$

$$\begin{aligned}\frac{Y(z)}{U(z)} &= \mathbf{C}_w[\mathbf{zI} - \mathbf{A}_w]^{-1}\mathbf{B}_w = \begin{bmatrix} 3 & 1 \end{bmatrix} \frac{1}{\Delta} \begin{bmatrix} z - \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & z - \frac{3}{4} \end{bmatrix} \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{bmatrix} \\ &= \frac{1}{\Delta} \begin{bmatrix} 3z & -2.5 & z-1.5 \end{bmatrix} \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{bmatrix} = \frac{4z-3}{(z-1)(z-0.5)}\end{aligned}$$

$$(d) \quad |\mathbf{zI} - \mathbf{A}| = \begin{vmatrix} z-1 & 0 \\ 0 & z-0.5 \end{vmatrix} = z^2 - 1.5z + 0.5; \quad |\mathbf{zI} - \mathbf{A}_w| = z^2 - 1.5z + 0.5$$

$$\therefore z_1 = 1, z_2 = 0.5$$

$$|\mathbf{A}| = \begin{vmatrix} 1 & 0 \\ 0 & 0.5 \end{vmatrix} = 0.5 = z_1 z_2; \quad |\mathbf{A}_w| = \begin{vmatrix} \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} \end{vmatrix} = \frac{9}{16} - \frac{1}{16} = 0.5$$

$$\text{tr } \mathbf{A} = 1.5 = z_1 + z_2; \quad \text{tr } \mathbf{A}_w = 1.5$$

2.11-1. Consider a system with the transfer function

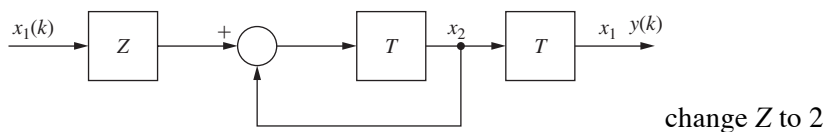
$$G(z) = \frac{Y(z)}{U(z)} = \frac{2}{z(z-1)}$$

- (a) Find three different state-variable models of this system.
(b) Verify the transfer function of each state model in part (a), using (2-84).

Solution:

$$(a) \quad G(z) = G_1(z)G_2(z) = \frac{2}{z^2 - z} = \frac{2z^{-2}}{1 - z^{-1}}$$

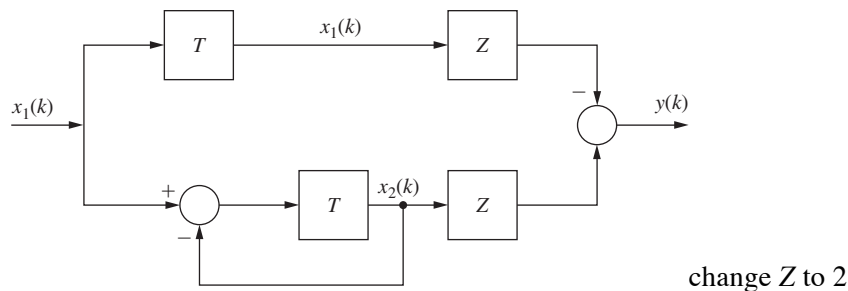
(1)



$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(k)$$

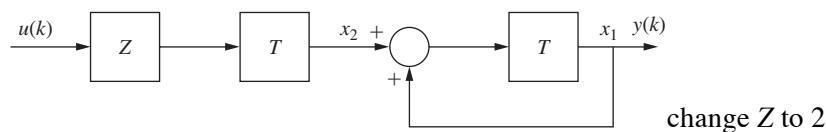
$$(2) \quad G(z) = \frac{2}{z(z-1)} = \frac{-2}{z} + \frac{2}{z-1} = G_1(z) + G_2(z)$$



$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} -2 & 2 \end{bmatrix} \mathbf{x}(k)$$

(3)



$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u(k)$$

$$y(k) = [1 \quad 0] \mathbf{x}(k)$$

(b) (1) $\mathbf{zI} - \mathbf{A} = \begin{bmatrix} z & -1 \\ 0 & z-1 \end{bmatrix}; |\mathbf{zI} - \mathbf{A}| = z^2 - z = \Delta$

$$G(z) = \mathbf{C}[\mathbf{zI} - \mathbf{A}]^{-1} \mathbf{B} = \frac{1}{\Delta} [1 \quad 0] \begin{bmatrix} z-1 & 1 \\ 0 & z \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \frac{1}{\Delta} [z-1 \quad 1] \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \frac{2}{z(z-1)}$$

(2) $\mathbf{zI} - \mathbf{A} = \begin{bmatrix} z & 0 \\ 0 & z-1 \end{bmatrix}; |\mathbf{zI} - \mathbf{A}| = \Delta = z^2 - z$

$$G(z) = \mathbf{C}[\mathbf{zI} - \mathbf{A}]^{-1} \mathbf{B} = \frac{1}{\Delta} [-1 \quad 1] \begin{bmatrix} z-1 & 0 \\ 0 & z \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \frac{1}{\Delta} [-1 \quad 1] \begin{bmatrix} 2z-2 \\ 2z \end{bmatrix} = \frac{2}{z(z-1)}$$

(3) $\mathbf{zI} - \mathbf{A} = \begin{bmatrix} z-1 & -1 \\ 0 & z \end{bmatrix}; |\mathbf{zI} - \mathbf{A}| = z^2 - z = \Delta$

$$G(z) = \mathbf{C}[\mathbf{zI} - \mathbf{A}]^{-1} \mathbf{B} = \frac{1}{\Delta} [1 \quad 0] \begin{bmatrix} z & 1 \\ 0 & z-1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \frac{1}{\Delta} [z \quad 1] \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \frac{2}{z(z-1)}$$

2.11-2. Consider a system described by the coupled difference equation

$$y(k+2) - v(k) = 0$$

$$v(k+1) + y(k+1) = u(k)$$

where $u(k)$ is the system input.

(a) Find a state-variable formulation for this system. Consider the outputs to be $y(k+1)$ and $v(k)$.

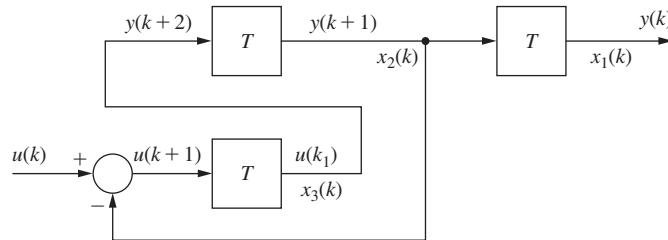
Hint: Draw a simulation diagram first.

(b) Repeat part (a) with $y(k)$ and $v(k)$ as the outputs.

- (c) Repeat part (a) with the single output $v(k)$.
- (d) Use (2-84) to calculate the system transfer function with $v(k)$ as the system output, as in part (c); that is, find $V(z)/U(z)$.
- (e) Verify the transfer function $V(z)/U(z)$ in part (d) by taking the z -transform of the given system difference equations and eliminating $Y(z)$.
- (f) Verify the transfer function $V(z)/U(z)$ in part (d) by using Mason's gain formula on the simulation diagram of part (a).

Solution:

(a)



$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$y_0(k) = \begin{bmatrix} x_2(k) \\ x_3(k) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}(k); \quad y_0(k) = \text{output}$$

(b) $\mathbf{x}(k+1)$ = same as (a)

$$y_0(k) = \begin{bmatrix} x_1(k) \\ x_3(k) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}(k)$$

(c) $\mathbf{x}(k+1)$ = same as (a)

$$y_0(k) = x_3(k) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathbf{x}(k)$$

$$(d) \quad z\mathbf{I} - \mathbf{A} = \begin{bmatrix} z & -1 & 0 \\ 0 & z & -1 \\ 0 & 1 & z \end{bmatrix}; \quad |z\mathbf{I} - \mathbf{A}| = z^3 - (-z) = z^3 + z = \Delta$$

$$\text{Cof}[z\mathbf{I} - \mathbf{A}] = \begin{bmatrix} z^2 + 1 & z^2 & 0 \\ z & z^2 & z \\ 1 & z & z^2 \end{bmatrix}; [z\mathbf{I} - \mathbf{A}]^{-1} = \frac{1}{\Delta} \begin{bmatrix} z^2 + 1 & z & 1 \\ z^2 & z^2 & z \\ 0 & z & z^2 \end{bmatrix}$$

$$\therefore \frac{Y_0(z)}{U(z)} = \mathbf{C}[z\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} = \frac{1}{\Delta} [0 \ 0 \ 1] \begin{bmatrix} z^2 + 1 & z & 1 \\ z^2 & z^2 & z \\ 0 & z & z^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

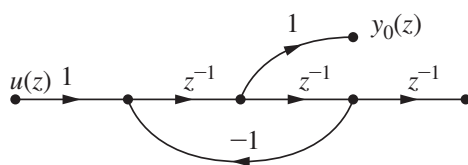
$$= \frac{1}{\Delta} [0 \ z \ z^2] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{z^2}{z^3 - z} = \frac{z}{z^2 + 1}$$

$$(e) \quad z^2 Y(z) - V(z) = 0 \Rightarrow Y(z) = \frac{1}{z^2} V(z)$$

$$zV(z) + zY(z) = zV(z) + \frac{1}{z}V(z) = U(z)$$

$$\therefore \frac{V(z)}{U(z)} = \frac{Y_0(z)}{U(z)} = \frac{1}{z + \frac{1}{z}} = \frac{z}{z^2 + 1}$$

(f) From (a):



make u and y capital letters

$$\therefore \frac{Y_0(z)}{U(z)} = \frac{z^{-1}}{1 + z^{-2}} = \frac{z}{z^2 + 1}$$

2.11-3. Given the system described by the state equations

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathbf{x}(k)$$

- (a) Calculate the transfer function $Y(z)/U(z)$, using (2-84).
- (b) Draw a simulation diagram for this system, from the state equations given.
- (c) Use Mason's gain formula and the simulation diagram to verify the transfer function found in part (a).

Solution:

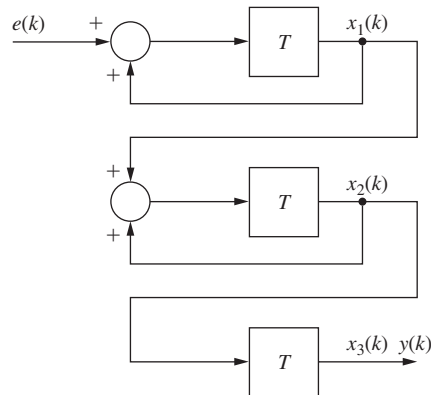
$$(a) \ z\mathbf{I} - \mathbf{A} = \begin{bmatrix} z-1 & 0 & 0 \\ -1 & z-1 & 0 \\ 0 & -1 & z \end{bmatrix}; \Delta = z^3 - 2z^2 + z = z(z-1)^2$$

$$\text{Cof } (z\mathbf{I} - \mathbf{A}) = \begin{bmatrix} z(z-1) & z & 1 \\ 0 & z(z-1) & z-1 \\ 0 & 0 & (z-1)^2 \end{bmatrix}, (z\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} \frac{1}{z-1} & 0 & 0 \\ \frac{1}{(z-1)^2} & \frac{1}{z-1} & 0 \\ \frac{1}{z(z-1)^2} & \frac{1}{z(z-1)} & \frac{1}{z} \end{bmatrix}$$

$$G(z) = \mathbf{C}[z\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} [z\mathbf{I} - \mathbf{A}]^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{z(z-1)^2} & \frac{1}{z(z-1)} & \frac{1}{z} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{z(z-1)^2} = \frac{1}{z^3 - 2z^2 + z}$$

(b)



$$(c) \ \Delta = 1 - z^{-1} - z^{-1} + z^{-2} = 1 - 2z^{-1} + z^{-2}$$

$$\therefore G(z) = \frac{z^{-3}}{\Delta} = \frac{1}{z^3 - 2z^2 + z}$$

2.11-4.Section 2.9 gives some standard forms for state equations (simulation diagrams for the control canonical and observer canonical forms). The MATLAB statement

$$[A,B,C,D] = \text{tf2ss}(\text{num},\text{den})$$

generates a standard set of state equations for the transfer function whose numerator coefficients are given in the vector *num* and denominator coefficients in the vector *den*.

(a) Use the MATLAB statement given to generate a set of state equations for the transfer function

$$G(z) = \frac{3z + 4}{z^2 + 5z + 6}$$

(b) Draw a simulation diagram for the state equations in part (a).

(c) Determine if the simulation diagram in part (b) is one of the standard forms in Section 2.9.

Solution:

$$(a) \mathbf{n} = [0 \ 3 \ 4];$$

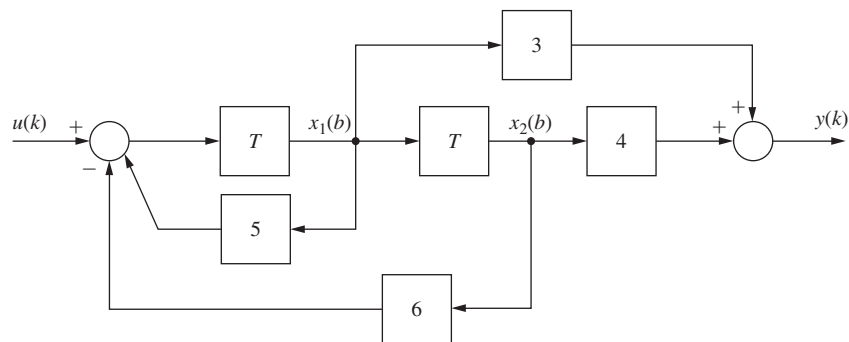
$$\mathbf{d} = [1 \ 5 \ 6];$$

$$[A,B,C,D] = \text{tf2ss}(\mathbf{n}, \mathbf{d})$$

$$\mathbf{x}(k+1) = \begin{bmatrix} -5 & -6 \\ 1 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)$$

$$y(k) = [3 \ 4] \mathbf{x}(k)$$

(b)



(c) Yes, it is the control canonical form with the states renumbered.

2.12-1. Consider the system described in Problem 2.10-2.

$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} -2 & 1 \end{bmatrix} \mathbf{x}(k)$$

(a) Find the transfer function of this system.

(b) Let $u(k) = 1, k \geq 0$ (a unit step function) and $\mathbf{x}(0) = 0$. Use the transfer function of part (a) to find the system response.

(c) Find the state transition matrix $\Phi(k)$ for this system.

(d) Use (2-90) to verify the step response calculated in part (b). This calculation results in the response expressed as a summation. Then check the values $y(0)$, $y(1)$, and $y(2)$.

(e) Verify the results of part (d) by the iterative solution of the state equations.

Solution:

$$(a) \quad z\mathbf{I} - \mathbf{A} = \begin{bmatrix} z & -1 \\ 0 & z-3 \end{bmatrix}; \quad \Delta = |z\mathbf{I} - \mathbf{A}| = z(z-3) = \Delta$$

$$\begin{aligned} \frac{Y(z)}{U(z)} &= \mathbf{C}[z\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} = \frac{1}{\Delta} \begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} z-3 & 1 \\ 0 & z \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{\Delta} \begin{bmatrix} -2z+6 & z-2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{-z+4}{z(z-3)} \end{aligned}$$

$$(b) \quad Y(z) = \frac{(-z+4)z}{z(z-3)(z-1)}$$

$$\frac{Y(z)}{z} = \frac{-z+4}{z(z-1)(z-3)} = \frac{4/3}{z} + \frac{-3/2}{z-1} + \frac{1/6}{z-3}$$

$$\therefore y(k) = \begin{cases} \frac{4}{3} - \frac{3}{2} + \frac{1}{6} = 0, & k = 0 \\ -\frac{3}{2} + \frac{1}{6}(3)^k & k \geq 1 \end{cases}$$

$$\therefore y(0) = 0$$

$$y(1) = -\frac{3}{2} + \frac{1}{2} = -1$$

$$y(2) = -\frac{3}{2} + \frac{3}{2} = 0$$

(c)

$$\Phi(z) = z(z\mathbf{I} - \mathbf{A})^{-1} = z \begin{bmatrix} \frac{z-3}{z(z-3)} & \frac{1}{z(z-3)} \\ 0 & \frac{z}{z(z-3)} \end{bmatrix} = z \begin{bmatrix} \frac{1}{z} & \frac{-\frac{1}{3}}{z} + \frac{\frac{1}{3}}{z-3} \\ 0 & \frac{1}{z-3} \end{bmatrix}$$

$$\therefore \Phi(k) = \begin{bmatrix} \delta(k) & -\frac{1}{3}\delta(k) + \frac{1}{3}(3)^k \\ 0 & (3)^k \end{bmatrix}$$

$$(d) \quad \mathbf{y}(k) = \sum_{j=0}^{k-1} \mathbf{C}\Phi(k-1-j)\mathbf{B}u(j) = \sum_{j=0}^{k-1} \begin{bmatrix} -2 & 1 \end{bmatrix} \Phi(k-1-j) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \sum_{j=0}^{k-1} \begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{3}\delta(k-1-j) + \frac{1}{3}(3)^{k-j-1} \\ (3)^{k-j-1} \end{bmatrix} = \sum_{j=0}^{k-1} \left[\frac{-4}{3}\delta(k-j-1) + \frac{1}{3}(3)^{k-j-1} \right]$$

$$= \sum_{j=0}^{k-1} \left[\frac{-4}{3}\delta(k-1-j) + \frac{1}{3}(3)^{k-1-j} \right]$$

$$y(0) = 0; \quad y(1) = -\frac{4}{3}\delta(0) + \frac{1}{3}(3)^0 = -\frac{4}{3} + \frac{1}{3} = -1$$

$$y(2) = -\frac{4}{3}\delta(1) + \frac{1}{3}(3)^1 - \frac{4}{3}\delta(0) + \frac{1}{3}(3)^0 = 1 - \frac{4}{3} + \frac{1}{3} = 0$$

$$(e) \quad \mathbf{x}(1) = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad y(1) = \begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -1$$

$$\mathbf{x}(2) = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}; \quad y(2) = \begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 0$$

2.12-2. The system described by the equations

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 2 \end{bmatrix} \mathbf{x}(k)$$

is excited by the initial conditions $\mathbf{x}(0) = \begin{bmatrix} -1 & 2 \end{bmatrix}^T$ with $u(k) = 0$ for all k .

(a) Use (2-89) to solve for $\mathbf{x}(k)$, $k \geq 0$.

(b) Find the output $y(z)$.

(c) Show that $\Phi(k)$ in (a) satisfies the property $\Phi(0) = \mathbf{I}$.

(d) Show that the solution in part (a) satisfies the given initial conditions.

(e) Use an iterative solution of the state equations to show that the values $y(k)$, for $k = 0, 1, 2$, and 3, in part (b) are correct.

(f) Verify the results in part (e) using MATLAB.

Solution:

$$(a) \quad z\mathbf{I} - \mathbf{A} = \begin{bmatrix} z-1 & 0 \\ 0 & z-0.5 \end{bmatrix}; \quad |z\mathbf{I} - \mathbf{A}| = \Delta = (z-1)(z-0.5)$$

$$(z\mathbf{I} - \mathbf{A}^{-1}) = \frac{1}{\Delta} \begin{bmatrix} z-0.5 & 0 \\ 0 & z-1 \end{bmatrix} = \begin{bmatrix} \frac{1}{z-1} & 0 \\ 0 & \frac{1}{z-0.5} \end{bmatrix}$$

$$\therefore \Phi(k) = f^{-1} \begin{bmatrix} \frac{z}{z-1} & 0 \\ 0 & \frac{z}{z-0.5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5^k \end{bmatrix}$$

$$\therefore \mathbf{x}(k) = \Phi(k)\mathbf{x}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 0.5^k \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2(0.5)^k \end{bmatrix}$$

$$(b) \quad y(k) = \mathbf{C}\mathbf{x}(k) = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2(0.5)^k \end{bmatrix} = 1 + 4(0.5)^k$$

$$(c) \quad \Phi(0) = \begin{bmatrix} 1 & 0 \\ 0 & 0.5^0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

$$(d) \quad x(k)|_{k=0} = \begin{bmatrix} 1 \\ 2(0.5)^k \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$(e) \quad \text{From (b),} \quad \begin{array}{ll} y(0) = 5 & y(2) = 2 \\ y(1) = 3 & y(3) = 1.5 \end{array}$$

$$y(0) = \mathbf{C}\mathbf{x}(0) = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 5$$

$$x(1) = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad y(1) = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3$$

$$x(2) = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, \quad y(2) = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} = 2$$

$$x(3) = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.25 \end{bmatrix}, \quad y(3) = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = 1.5$$

$$(f) \quad \mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} 1 & 2 \end{bmatrix};$$

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix};$$

$$u = 0;$$

$$\text{for } k = 0:3$$

$$\mathbf{x}_1 = \mathbf{A} * \mathbf{x} + \mathbf{B} * u;$$

$$y = \mathbf{C} * \mathbf{x};$$

$$[k, y]$$

$$\mathbf{x} = \mathbf{x}_1;$$

end

2.12-3. The system described by the equations

$$\mathbf{x}(k+1) = \begin{bmatrix} 1.1 & 1 \\ -0.3 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & -1 \end{bmatrix} \mathbf{x}(k)$$

is excited by the initial conditions $\mathbf{x}(0) = \begin{bmatrix} -1 & 2 \end{bmatrix}^T$ with $u(k) = 0$ for all k .

(a) Use (2-89) to solve for $\mathbf{x}(k)$, $k \geq 0$.

(b) Find the output $y(k)$.

(c) Show that $\Phi(k)$ in part (a) satisfies the property $\Phi(0) = \mathbf{I}$.

(d) Show that the solution in part (a) satisfies the given initial conditions.

(e) Use an iterative solution of the state equations to show that the values $y(k)$, for $k = 0, 1, 2$, and 3, in part (b) are correct.

Solution:

$$(a) \quad z\mathbf{I} - \mathbf{A} = \begin{bmatrix} z-1.1 & -1 \\ 0.3 & z \end{bmatrix}; |z\mathbf{I} - \mathbf{A}| = \Delta = z^2 - 1.1z + 0.3 = (z-0.5)(z-0.6)$$

$$(z\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{\Delta} \begin{bmatrix} z & 1 \\ -0.3 & z-1.1 \end{bmatrix}$$

$$\Phi(k) = \mathcal{Z}^{-1}[z(z\mathbf{I} - \mathbf{A})^{-1}] = \mathcal{Z}^{-1} \left(z \begin{bmatrix} \frac{z}{(z-0.5)(z-0.6)} & \frac{1}{(z-0.5)(z-0.6)} \\ \frac{-0.3}{(z-0.5)(z-0.6)} & \frac{z-1.1}{(z-0.5)(z-0.6)} \end{bmatrix} \right)$$

$$= \mathcal{Z}^{-1} \left(z \begin{bmatrix} \frac{-5}{z-0.5} + \frac{6}{z-0.6} & \frac{-10}{z-0.5} + \frac{10}{z-0.6} \\ \frac{3}{z-0.5} + \frac{-3}{z-0.6} & \frac{6}{z-0.5} + \frac{-5}{z-0.6} \end{bmatrix} \right)$$

$$= \begin{bmatrix} -5(0.5)^k + 6(0.6)^k & -10(0.5)^k + 10(0.6)^k \\ 3(0.5)^k - 3(0.6)^k & 6(0.5)^k - 5(0.6)^k \end{bmatrix}$$

$$\therefore \mathbf{x}(k) = \Phi(k)\mathbf{x}(0) = \begin{bmatrix} -5(0.5)^k + 6(0.6)^k & -10(0.5)^k + 10(0.6)^k \\ 3(0.5)^k - 3(0.6)^k & 6(0.5)^k - 5(0.6)^k \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -15(0.5)^k + 14(0.6)^k \\ 9(0.5)^k - 7(0.6)^k \end{bmatrix}$$

$$(b) \quad y(k) = \mathbf{C}\mathbf{x}(k) = [1 \quad -1] \begin{bmatrix} -15(0.5)^k + 14(0.6)^k \\ 9(0.5)^k - 7(0.6)^k \end{bmatrix} = -24(0.5)^k + 21(0.6)^k$$

$$(c) \quad \Phi(0) = \begin{bmatrix} -5+6 & -10+10 \\ 3-3 & 6-5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

$$(d) \quad \mathbf{x}(k)|_{k=0} = \begin{bmatrix} -15+14 \\ 9-7 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$(e) \quad \text{From (b),} \quad \begin{array}{ll} y(0) = -3 & y(2) = 1.56 \\ y(1) = 0.6 & y(3) = 1.536 \end{array}$$

$$y(0) = \mathbf{C}\mathbf{x}(0) = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -3$$

$$\mathbf{x}(1) = \begin{bmatrix} 1.1 & 1 \\ -0.3 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.9 \\ 0.3 \end{bmatrix}; \quad y(1) = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 0.9 \\ 0.3 \end{bmatrix} = 0.6$$

$$\mathbf{x}(2) = \begin{bmatrix} 1.1 & 1 \\ -0.3 & 0 \end{bmatrix} \begin{bmatrix} 0.9 \\ 0.3 \end{bmatrix} = \begin{bmatrix} 1.29 \\ -0.27 \end{bmatrix}; \quad y(2) = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1.29 \\ -0.27 \end{bmatrix} = 1.56$$

$$\mathbf{x}(3) = \begin{bmatrix} 1.1 & 1 \\ -0.3 & 0 \end{bmatrix} \begin{bmatrix} 1.29 \\ -0.27 \end{bmatrix} = \begin{bmatrix} 1.149 \\ -0.387 \end{bmatrix}; \quad y(3) = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1.149 \\ -0.389 \end{bmatrix} = 1.536$$

MATLAB:

$$\mathbf{A} = \begin{bmatrix} 1.1 & 1 \\ -0.3 & 0 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} 1 & -1 \end{bmatrix};$$

$$\mathbf{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix};$$

$$\mathbf{u} = 0;$$

$$\text{for } k = 0:3$$

$$\mathbf{x}1 = \mathbf{A}*\mathbf{x} + \mathbf{B}*\mathbf{u};$$

$$\mathbf{y} = \mathbf{C}*\mathbf{x};$$

$$[k, \mathbf{y}]$$

$$\mathbf{x} = \mathbf{x}1;$$

end

2.12-4. Let $\Phi(k)$ be the state transition matrix for the equations

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k)$$

Show that $\Phi(k)$ satisfies the difference equation

$$\Phi(k+1) = \mathbf{A}\Phi(k)$$

Solution:

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k); \mathbf{x}(k) = \mathbf{\Phi}(k)\mathbf{x}(0)$$

$$\therefore \mathbf{\Phi}(k+1)\mathbf{x}(0) = \mathbf{A}\mathbf{\Phi}(k)\mathbf{x}(0)$$

Since this is true for any $\mathbf{x}(0)$, $\therefore \mathbf{\Phi}(k+1) = \mathbf{A}\mathbf{\Phi}(k)$