CHAPTER 1

FIRST-ORDER DIFFERENTIAL EQUATIONS

SECTION 1.1

DIFFERENTIAL EQUATIONS AND MATHEMATICAL MODELS

The main purpose of Section 1.1 is simply to introduce the basic notation and terminology of differential equations, and to show the student what is meant by a solution of a differential equation. Also, the use of differential equations in the mathematical modeling of real-world phenomena is outlined.

Problems 1-12 are routine verifications by direct substitution of the suggested solutions into the given differential equations. We include here just some typical examples of such verifications.

3. If $y_1 = \cos 2x$ and $y_2 = \sin 2x$, then $y_1' = -2\sin 2x$ $y_2' = 2\cos 2x$, so $y_1'' = -4\cos 2x = -4y_1$ and $y_2'' = -4\sin 2x = -4y_2$. Thus $y_1'' + 4y_1 = 0$ and $y_2'' + 4y_2 = 0$.

4. If $y_1 = e^{3x}$ and $y_2 = e^{-3x}$, then $y_1 = 3e^{3x}$ and $y_2 = -3e^{-3x}$, so $y_1'' = 9e^{3x} = 9y_1$ and $y_2'' = 9e^{-3x} = 9y_2$.

5. If $y = e^x - e^{-x}$, then $y' = e^x + e^{-x}$, so $y' - y = \left(e^x + e^{-x}\right) - \left(e^x - e^{-x}\right) = 2e^{-x}$. Thus $y' = y + 2e^{-x}$.

6. If $y_1 = e^{-2x}$ and $y_2 = xe^{-2x}$, then $y_1' = -2e^{-2x}$, $y_1'' = 4e^{-2x}$, $y_2' = e^{-2x} - 2xe^{-2x}$, and $y_2'' = -4e^{-2x} + 4xe^{-2x}$. Hence

$$y_1'' + 4y_1' + 4y_1 = \left(4e^{-2x}\right) + 4\left(-2e^{-2x}\right) + 4\left(e^{-2x}\right) = 0$$

and

$$y_2'' + 4y_2' + 4y_2 = \left(-4e^{-2x} + 4xe^{-2x}\right) + 4\left(e^{-2x} - 2xe^{-2x}\right) + 4\left(xe^{-2x}\right) = 0.$$

8. If $y_1 = \cos x - \cos 2x$ and $y_2 = \sin x - \cos 2x$, then $y_1' = -\sin x + 2\sin 2x$, $y_1'' = -\cos x + 4\cos 2x$, $y_2' = \cos x + 2\sin 2x$, and $y_2'' = -\sin x + 4\cos 2x$. Hence

$$y_1'' + y_1 = \left(-\cos x + 4\cos 2x\right) + \left(\cos x - \cos 2x\right) = 3\cos 2x$$

and

$$y_2'' + y_2 = \left(-\sin x + 4\cos 2x\right) + \left(\sin x - \cos 2x\right) = 3\cos 2x.$$

11. If $y = y_1 = x^{-2}$, then $y' = -2x^{-3}$ and $y'' = 6x^{-4}$, so

$$x^2 y'' + 5x\,y' + 4y \;=\; x^2\left(6x^{-4}\right) + 5x\left(-2x^{-3}\right) + 4\left(x^{-2}\right) \;=\; 0.$$

If $y = y_2 = x^{-2}\ln x$, then $y' = x^{-3} - 2x^{-3}\ln x$ and $y'' = -5x^{-4} + 6x^{-4}\ln x$, so

$$\begin{aligned} x^2 y'' + 5x\,y' + 4y &= x^2\left(-5x^{-4} + 6x^{-4}\ln x\right) + 5x\left(x^{-3} - 2x^{-3}\ln x\right) + 4\left(x^{-2}\ln x\right) \\ &= \left(-5x^{-2} + 5x^{-2}\right) + \left(6x^{-2} - 10x^{-2} + 4x^{-2}\right)\ln x \;=\; 0. \end{aligned}$$

13. Substitution of $y = e^{rx}$ into $3y' = 2y$ gives the equation $3r\,e^{rx} \;=\; 2e^{rx}$, which simplifies to $3r = 2$. Thus $r = 2/3$.

14. Substitution of $y = e^{rx}$ into $4y'' = y$ gives the equation $4r^2 e^{rx} \;=\; e^{rx}$, which simplifies to $4r^2 = 1$. Thus $r = \pm 1/2$.

15. Substitution of $y = e^{rx}$ into $y'' + y' - 2y \;=\; 0$ gives the equation $r^2 e^{rx} + r\,e^{rx} - 2e^{rx} = 0$, which simplifies to $r^2 + r - 2 \;=\; (r+2)(r-1) \;=\; 0$. Thus $r = -2$ or $r = 1$.

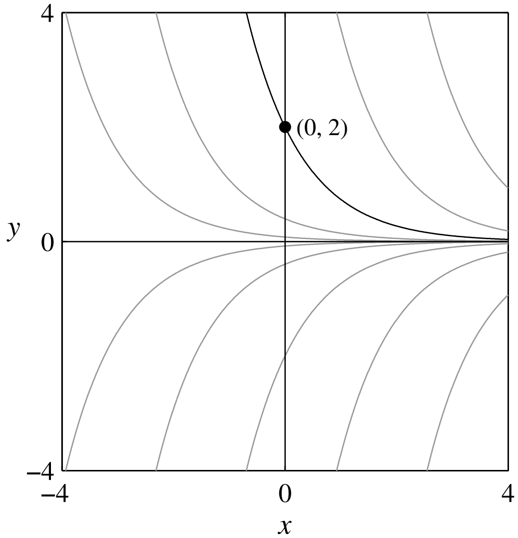
16. Substitution of $y = e^{rx}$ into $3y'' + 3y' - 4y = 0$ gives the equation $3r^2 e^{rx} + 3r\,e^{rx} - 4e^{rx} = 0$, which simplifies to $3r^2 + 3r - 4 = 0$. The quadratic formula then gives the solutions $r = \left(-3 \pm \sqrt{57}\right)\big/6$.

The verifications of the suggested solutions in Problems 17-26 are similar to those in Problems 1-12. We illustrate the determination of the value of *C* only in some typical cases. However, we illustrate typical solution curves for each of these problems.

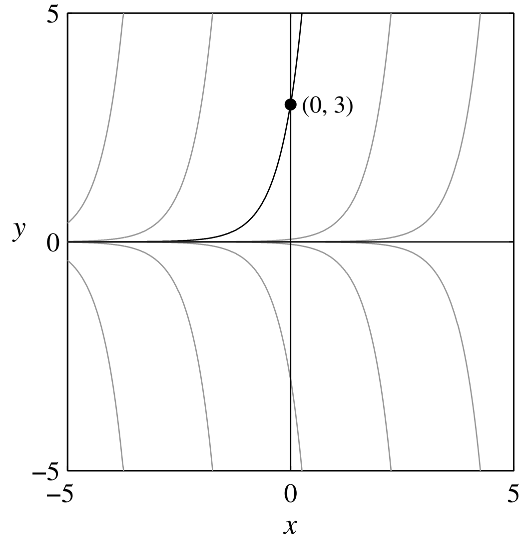
17. $C = 2$

18. $C = 3$

Problem 17



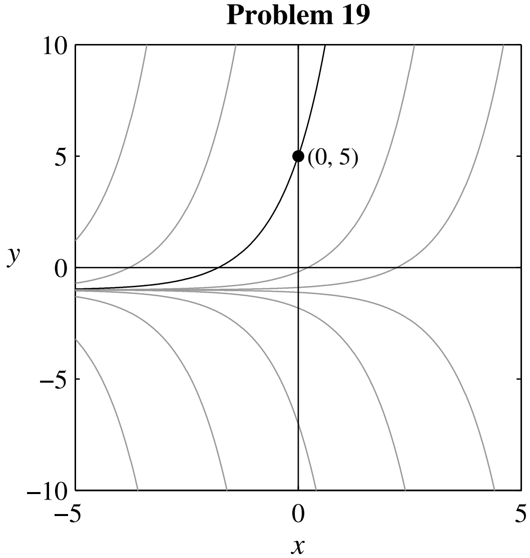
Problem 18



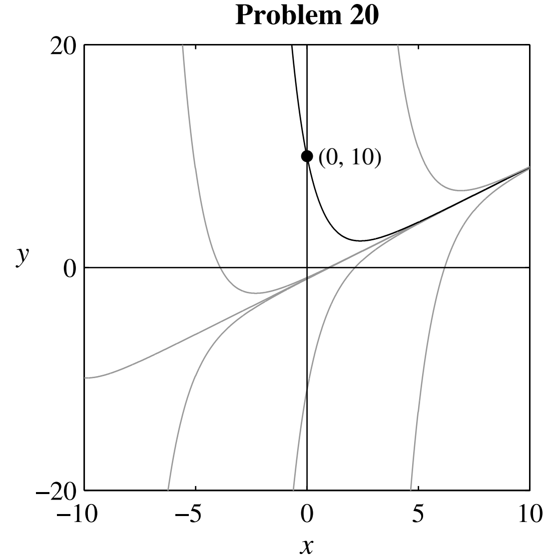
19. If $y(x) = Ce^{x} - 1$, then $y(0) = 5$ gives $C - 1 = 5$, so $C = 6$.

20. If $y(x) = C\,e^{-x} + x - 1$, then $y(0) = 10$ gives $C - 1 = 10$, or $C = 11$.

Problem 19

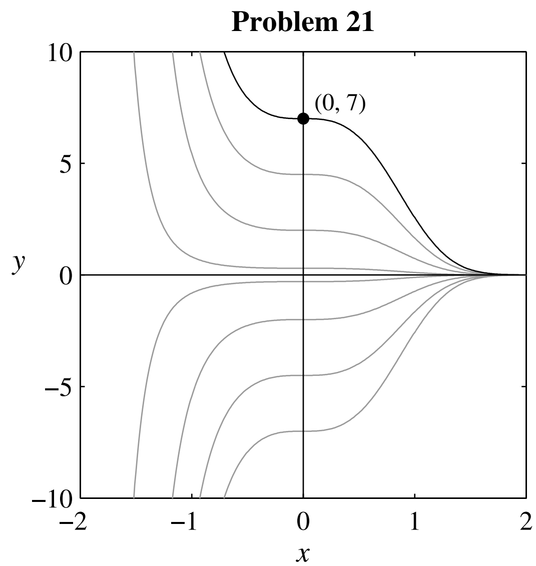


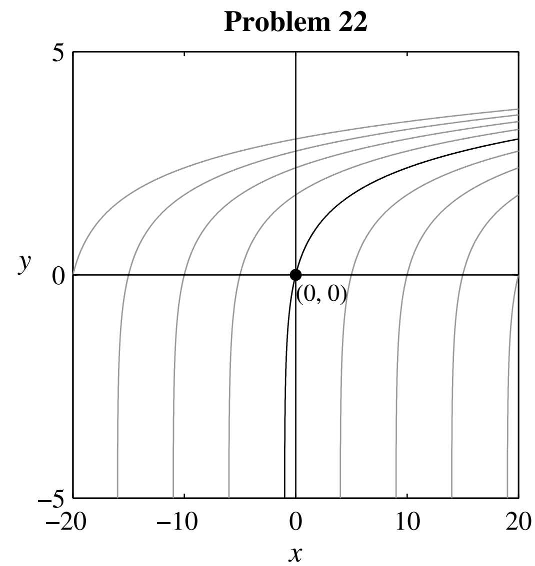
Problem 20



21. $C = 7$.

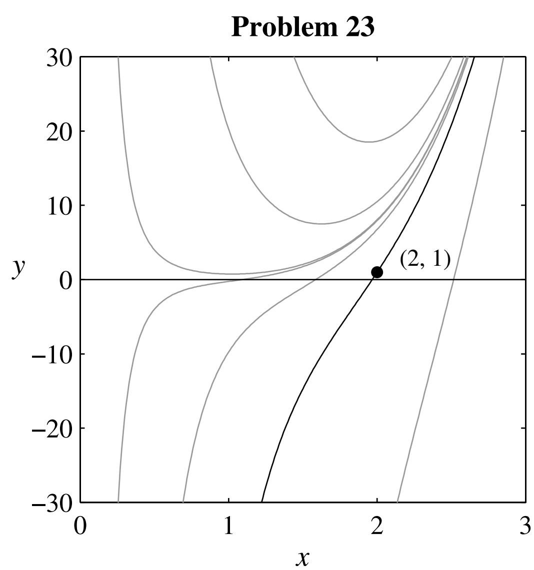
22. If $y(x) = \ln(x + C)$, then $y(0) = 0$ gives $\ln C = 0$, so $C = 1$.

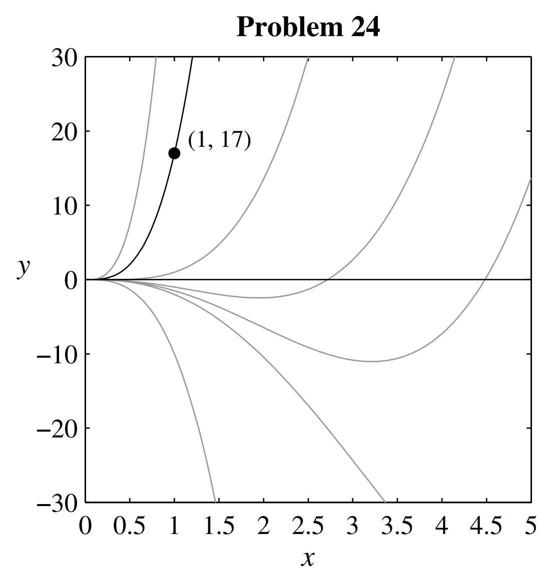




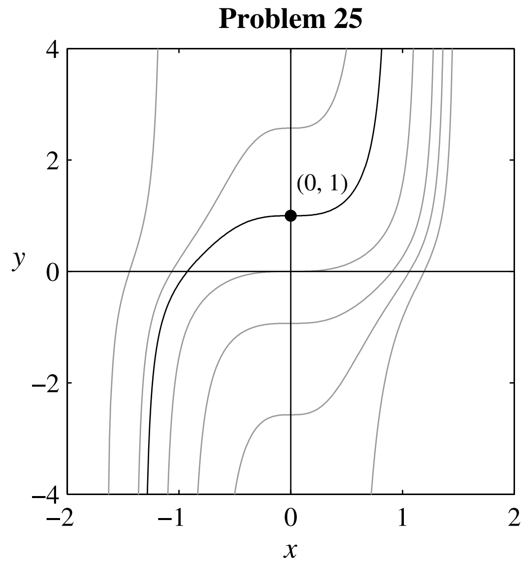
23. If $y(x)=\frac{1}{4}x^5+Cx^{-2}$, then $y(2)=1$ gives $\frac{1}{4}\cdot 32+C\cdot\frac{1}{8}=1$, or $C=-56$.

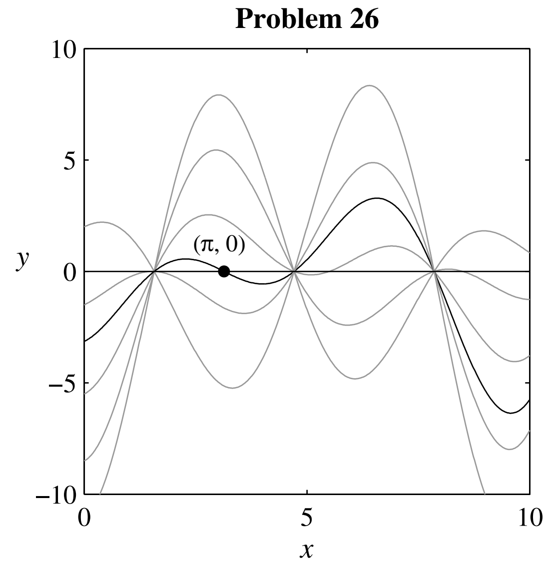
24. $C=17$.





25. If $y=\tan\left(x^3+C\right)$, then $y(0)=1$ gives the equation $\tan C=1$. Hence one value of C is $C=\pi/4$, as is this value plus any integral multiple of π.





26. Substitution of $x=\pi$ and $y=0$ into $y=(x+C)\cos x$ yields $0=(\pi+C)(-1)$, so $C=-\pi$.

27. $y'=x+y$

28. The slope of the line through (x,y) and $(x/2,0)$ is $y'=\dfrac{y-0}{x-x/2}=2y/x$, so the differential equation is $xy'=2y$.

29. If $m=y'$ is the slope of the tangent line and m' is the slope of the normal line at (x,y), then the relation $m\,m'=-1$ yields $m'=-1/y'=(y-1)/(x-0)$. Solving for y' then gives the differential equation $(1-y)\,y'=x$.

30. Here $m=y'$ and $m'=D_x(x^2+k) = 2x$, so the orthogonality relation $m\,m'=-1$ gives the differential equation $2xy' = -1$.

31. The slope of the line through (x,y) and $(-y,x)$ is $y'=(x-y)/(-y-x)$, so the differential equation is $(x+y)y'=y-x$.

In Problems 32-36 we get the desired differential equation when we replace the "time rate of change" of the dependent variable with its derivative with respect to time *t*, the word "is" with the = sign, the phrase "proportional to" with *k*, and finally translate the remainder of the given sentence into symbols.

32. $dP/dt=k\sqrt{P}$

33. $dv/dt=kv^2$

34. $dv/dt = k(250 - v)$

35. $dN/dt = k(P - N)$

36. $dN/dt = kN(P - N)$

37. The second derivative of any linear function is zero, so we spot the two solutions $y(x) \equiv 1$ and $y(x) = x$ of the differential equation $y'' = 0$.

38. A function whose derivative equals itself, and is hence a solution of the differential equation $y' = y$, is $y(x) = e^x$.

39. We reason that if $y = kx^2$, then each term in the differential equation is a multiple of x^2. The choice $k = 1$ balances the equation and provides the solution $y(x) = x^2$.

40. If y is a constant, then $y' \equiv 0$, so the differential equation reduces to $y^2 = 1$. This gives the two constant-valued solutions $y(x) \equiv 1$ and $y(x) \equiv -1$.

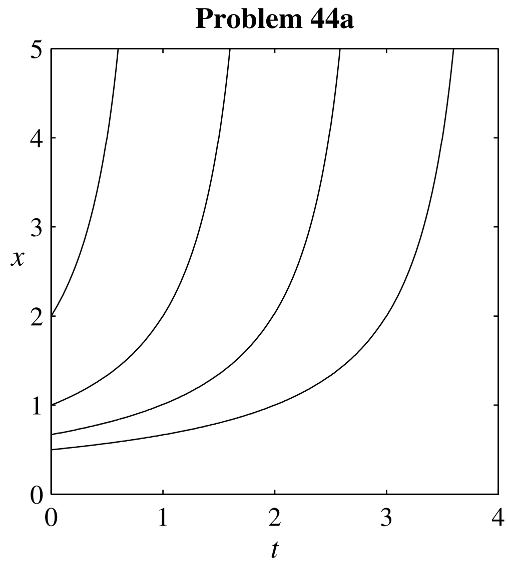
41. We reason that if $y = ke^x$, then each term in the differential equation is a multiple of e^x. The choice $k = \frac{1}{2}$ balances the equation and provides the solution $y(x) = \frac{1}{2}e^x$.

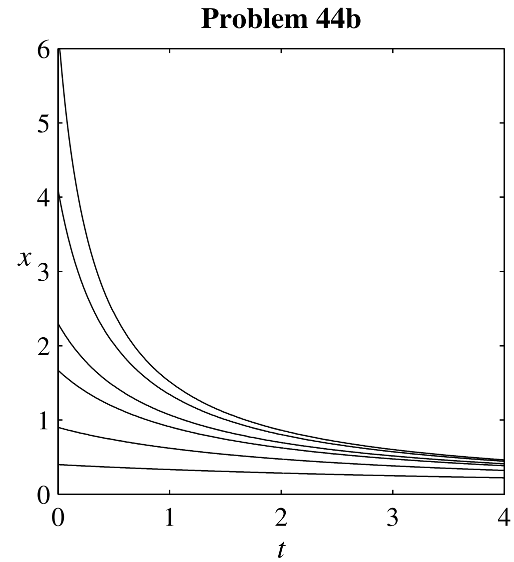
42. Two functions, each equaling the negative of its own second derivative, are the two solutions $y(x) = \cos x$ and $y(x) = \sin x$ of the differential equation $y'' = -y$.

43. **(a)** We need only substitute $x(t) = 1/(C - kt)$ in both sides of the differential equation $x' = kx^2$ for a routine verification.
(b) The zero-valued function $x(t) \equiv 0$ obviously satisfies the initial value problem $x' = kx^2$, $x(0) = 0$.

44. **(a)** The figure shows typical graphs of solutions of the differential equation $x' = \frac{1}{2}x^2$.

(b) The figure shows typical graphs of solutions of the differential equation $x' = -\frac{1}{2}x^2$. We see that—whereas the graphs with $k = \frac{1}{2}$ appear to "diverge to infinity"—each solution with $k = -\frac{1}{2}$ appears to approach 0 as $t \to \infty$. Indeed, we see from the Problem 43(a) solution $x(t) = 1/(C - \frac{1}{2}t)$ that $x(t) \to \infty$ as $t \to 2C$. However, with $k = -\frac{1}{2}$ it is clear from the resulting solution $x(t) = 1/(C + \frac{1}{2}t)$ that $x(t)$ remains bounded on any bounded interval, but $x(t) \to 0$ as $t \to +\infty$.





45. Substitution of $P' = 1$ and $P = 10$ into the differential equation $P' = kP^2$ gives $k = \frac{1}{100}$, so Problem 43(a) yields a solution of the form $P(t) = 1\big/\left(C - \frac{1}{100}t\right)$. The initial condition $P(0) = 2$ now yields $C = \frac{1}{2}$, so we get the solution

$$P(t) = \frac{1}{\frac{1}{2} - \frac{t}{100}} = \frac{100}{50 - t}.$$

We now find readily that $P = 100$ when $t = 49$ and that $P = 1000$ when $t = 49.9$. It appears that P grows without bound (and thus "explodes") as t approaches 50.

46. Substitution of $v' = -1$ and $v = 5$ into the differential equation $v' = kv^2$ gives $k = -\frac{1}{25}$, so Problem 43(a) yields a solution of the form $v(t) = 1\big/\left(C + t/25\right)$. The initial condition $v(0) = 10$ now yields $C = \frac{1}{10}$, so we get the solution

$$v(t) = \frac{1}{\frac{1}{10} + \frac{t}{25}} = \frac{50}{5 + 2t}.$$

We now find readily that $v = 1$ when $t = 22.5$ and that $v = 0.1$ when $t = 247.5$. It appears that v approaches 0 as t increases without bound. Thus the boat gradually slows, but never comes to a "full stop" in a finite period of time.

47. **(a)** $y(10) = 10$ yields $10 = 1\big/\left(C - 10\right)$, so $C = 101/10$.

(b) There is no such value of *C*, but the constant function $y(x) \equiv 0$ satisfies the conditions $y' = y^2$ and $y(0) = 0$.

(c) It is obvious visually (in Fig. 1.1.8 of the text) that one and only one solution curve passes through each point (a,b) of the *xy*-plane, so it follows that there exists a unique solution to the initial value problem $y' = y^2$, $y(a) = b$.

48. **(b)** Obviously the functions $u(x) = -x^4$ and $v(x) = +x^4$ both satisfy the differential equation $xy' = 4y$. But their derivatives $u'(x) = -4x^3$ and $v'(x) = +4x^3$ match at $x = 0$, where both are zero. Hence the given piecewise-defined function $y(x)$ is differentiable, and therefore satisfies the differential equation because $u(x)$ and $v(x)$ do so (for $x \le 0$ and $x \ge 0$, respectively).

(c) If $a \ge 0$ (for instance), then choose C_+ fixed so that $C_+ a^4 = b$. Then the function

$$y(x) = \begin{cases} C_- x^4 & \text{if } x \le 0 \\ C_+ x^4 & \text{if } x \ge 0 \end{cases}$$

satisfies the given differential equation for every real number value of C_-.

SECTION 1.2

INTEGRALS AS GENERAL AND PARTICULAR SOLUTIONS

This section introduces **general solutions** and **particular solutions** in the very simplest situation — a differential equation of the form $y' = f(x)$ — where only direct integration and evaluation of the constant of integration are involved. Students should review carefully the elementary concepts of velocity and acceleration, as well as the fps and mks unit systems.

1. Integration of $y' = 2x + 1$ yields $y(x) = \int (2x+1)\,dx = x^2 + x + C$. Then substitution of $x = 0$, $y = 3$ gives $3 = 0 + 0 + C = C$, so $y(x) = x^2 + x + 3$.

2. Integration of $y' = (x-2)^2$ yields $y(x) = \int (x-2)^2\,dx = \tfrac{1}{3}(x-2)^3 + C$. Then substitution of $x = 2$, $y = 1$ gives $1 = 0 + C = C$, so $y(x) = \tfrac{1}{3}(x-2)^3 + 1$.

3. Integration of $y' = \sqrt{x}$ yields $y(x) = \int \sqrt{x}\,dx = \tfrac{2}{3}x^{3/2} + C$. Then substitution of $x = 4$, $y = 0$ gives $0 = \tfrac{16}{3} + C$, so $y(x) = \tfrac{2}{3}\left(x^{3/2} - 8\right)$.

4. Integration of $y' = x^{-2}$ yields $y(x) = \int x^{-2}dx = -1/x + C$. Then substitution of $x = 1$, $y = 5$ gives $5 = -1 + C$, so $y(x) = -1/x + 6$.

5. Integration of $y' = (x+2)^{-1/2}$ yields $y(x) = \int (x+2)^{-1/2}\,dx = 2\sqrt{x+2} + C$. Then substitution of $x = 2$, $y = -1$ gives $-1 = 2\cdot 2 + C$, so $y(x) = 2\sqrt{x+2} - 5$.

6. Integration of $y' = x(x^2+9)^{1/2}$ yields $y(x) = \int x(x^2+9)^{1/2}\,dx = \frac{1}{3}(x^2+9)^{3/2} + C$. Then substitution of $x = -4$, $y = 0$ gives $0 = \frac{1}{3}(5)^3 + C$, so $y(x) = \frac{1}{3}\left[(x^2+9)^{3/2} - 125\right]$.

7. Integration of $y' = \dfrac{10}{x^2+1}$ yields $y(x) = \int \dfrac{10}{x^2+1}\,dx = 10\tan^{-1} x + C$. Then substitution of $x = 0$, $y = 0$ gives $0 = 10\cdot 0 + C$, so $y(x) = 10\tan^{-1} x$.

8. Integration of $y' = \cos 2x$ yields $y(x) = \int \cos 2x\,dx = \frac{1}{2}\sin 2x + C$. Then substitution of $x = 0$, $y = 1$ gives $1 = 0 + C$, so $y(x) = \frac{1}{2}\sin 2x + 1$.

9. Integration of $y' = \dfrac{1}{\sqrt{1-x^2}}$ yields $y(x) = \int \dfrac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$. Then substitution of $x = 0$, $y = 0$ gives $0 = 0 + C$, so $y(x) = \sin^{-1} x$.

10. Integration of $y' = xe^{-x}$ yields

$$y(x) = \int xe^{-x}\,dx = \int ue^u\,du = (u-1)e^u = -(x+1)e^{-x} + C,$$

using the substitution $u = -x$ together with Formula #46 inside the back cover of the textbook. Then substituting $x = 0$, $y = 1$ gives $1 = -1 + C$, so $y(x) = -(x+1)e^{-x} + 2$.

11. If $a(t) = 50$, then $v(t) = \int 50dt = 50t + v_0 = 50t + 10$. Hence

$$x(t) = \int (50t + 10)\,dt = 25t^2 + 10t + x_0 = 25t^2 + 10t + 20.$$

12. If $a(t) = -20$, then $v(t) = \int (-20)\,dt = -20t + v_0 = -20t - 15$. Hence

$$x(t) = \int (-20t - 15)\,dt = -10t^2 - 15t + x_0 = -10t^2 - 15t + 5.$$

13. If $a(t) = 3t$, then $v(t) = \int 3t\,dt = \frac{3}{2}t^2 + v_0 = \frac{3}{2}t^2 + 5$. Hence

$$x(t) = \int \left(\tfrac{3}{2}t^2 + 5\right)dt = \tfrac{1}{2}t^3 + 5t + x_0 = \tfrac{1}{2}t^3 + 5t.$$

14. If $a(t) = 2t + 1$, then $v(t) = \int (2t+1)\,dt = t^2 + t + v_0 = t^2 + t - 7$. Hence

$$x(t) = \int (t^2 + t - 7)\,dt = \tfrac{1}{3}t^3 + \tfrac{1}{2}t - 7t + x_0 = \tfrac{1}{3}t^3 + \tfrac{1}{2}t - 7t + 4.$$

15. If $a(t)=4(t+3)^2$, then $v(t)=\int 4(t+3)^2\,dt=\frac{4}{3}(t+3)^3+C=\frac{4}{3}(t+3)^3-37$ (taking $C=-37$ so that $v(0)=-1$). Hence

$$x(t)=\int \tfrac{4}{3}(t+3)^3-37\,dt=\tfrac{1}{3}(t+3)^4-37t+C=\tfrac{1}{3}(t+3)^4-37t-26.$$

16. If $a(t)=\dfrac{1}{\sqrt{t+4}}$, then $v(t)=\int \dfrac{1}{\sqrt{t+4}}\,dt=2\sqrt{t+4}+C=2\sqrt{t+4}-5$ (taking $C=-5$ so that $v(0)=-1$). Hence

$$x(t)=\int \left(2\sqrt{t+4}-5\right)dt=\tfrac{4}{3}(t+4)^{3/2}-5t+C=\tfrac{4}{3}(t+4)^{3/2}-5t-\tfrac{29}{3}$$

(taking $C=-29/3$ so that $x(0)=1$).

17. If $a(t)=(t+1)^{-3}$, then $v(t)=\int (t+1)^{-3}\,dt=-\frac{1}{2}(t+1)^{-2}+C=-\frac{1}{2}(t+1)^{-2}+\frac{1}{2}$ (taking $C=\frac{1}{2}$ so that $v(0)=0$). Hence

$$x(t)=\int -\tfrac{1}{2}(t+1)^{-2}+\tfrac{1}{2}\,dt=\tfrac{1}{2}(t+1)^{-1}+\tfrac{1}{2}t+C=\tfrac{1}{2}\left[(t+1)^{-1}+t-1\right]$$

(taking $C=-\frac{1}{2}$ so that $x(0)=0$).

18. If $a(t)=50\sin 5t$, then $v(t)=\int 50\sin 5t\,dt=-10\cos 5t+C=-10\cos 5t$ (taking $C=0$ so that $v(0)=-10$). Hence

$$x(t)=\int -10\cos 5t\,dt=-2\sin 5t+C=-2\sin 5t+10$$

(taking $C=-10$ so that $x(0)=8$).

Students should understand that Problems 19-22, though different at first glance, are solved in the same way as the preceding ones, that is, by means of the fundamental theorem of calculus in the form $x(t)=x(t_0)+\int_{t_0}^{t} v(s)\,ds$ cited in the text. Actually in these problems $x(t)=\int_0^t v(s)\,ds$, since t_0 and $x(t_0)$ are each given to be zero.

19. The graph of $v(t)$ shows that $v(t)=\begin{cases}5 & \text{if } 0\le t\le 5\\ 10-t & \text{if } 5\le t\le 10\end{cases}$, so that

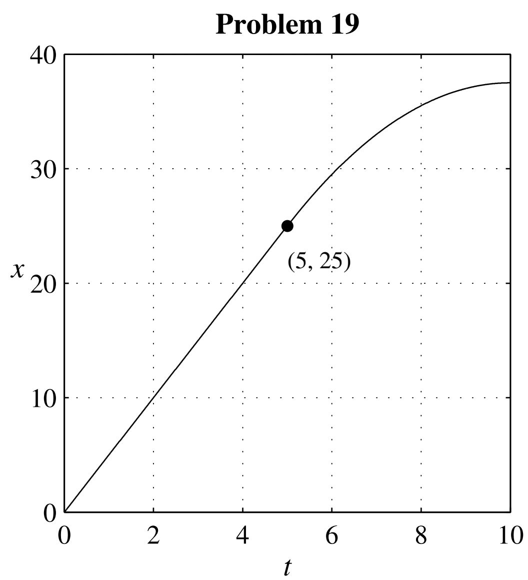
$x(t)=\begin{cases}5t+C_1 & \text{if } 0\le t\le 5\\ 10t-\frac{1}{2}t^2+C_2 & \text{if } 5\le t\le 10\end{cases}$. Now $C_1=0$ because $x(0)=0$, and continuity of $x(t)$ requires that $x(t)=5t$ and $x(t)=10t-\frac{1}{2}t^2+C_2$ agree when $t=5$. This implies that $C_2=-\frac{25}{2}$, leading to the graph of $x(t)$ shown.

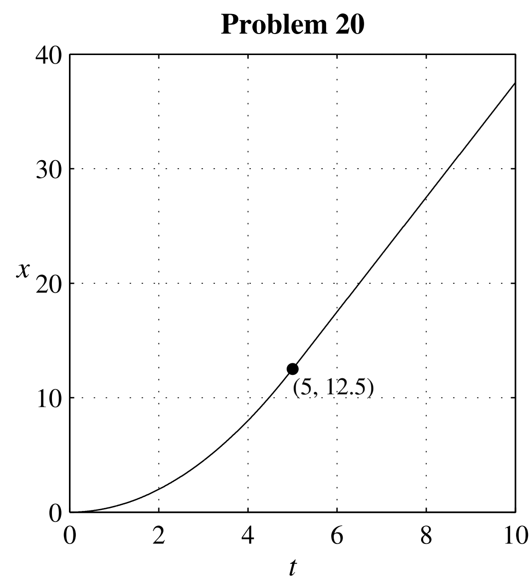
Alternate solution for Problem 19 (and similar for 20-22): The graph of $v(t)$ shows that $v(t)=\begin{cases}5 & \text{if } 0\le t\le 5\\ 10-t & \text{if } 5\le t\le 10\end{cases}$. Thus for $0\le t\le 5$, $x(t)=\int_0^t v(s)\,ds$ is given by $\int_0^t 5ds=5t$, whereas for $5\le t\le 10$ we have

$$x(t)=\int_0^t v(s)\,ds=\int_0^5 5\,ds+\int_5^t 10-s\,ds$$
$$=25+\left(10s-\frac{s^2}{2}\Big|_{s=5}^{s=t}\right)=25+10t-\frac{t^2}{2}-\frac{75}{2}=10t-\frac{t^2}{2}-\frac{25}{2}.$$

The graph of $x(t)$ is shown.

20. The graph of $v(t)$ shows that $v(t)=\begin{cases}t & \text{if } 0\le t\le 5\\ 5 & \text{if } 5\le t\le 10\end{cases}$, so that $x(t)=\begin{cases}\frac{1}{2}t^2+C_1 & \text{if } 0\le t\le 5\\ 5t+C_2 & \text{if } 5\le t\le 10\end{cases}$. Now $C_1=0$ because $x(0)=0$, and continuity of $x(t)$ requires that $x(t)=\frac{1}{2}t^2$ and $x(t)=5t+C_2$ agree when $t=5$. This implies that $C_2=-\frac{25}{2}$, leading to the graph of $x(t)$ shown.





21. The graph of $v(t)$ shows that $v(t)=\begin{cases}t & \text{if } 0\le t\le 5\\ 10-t & \text{if } 5\le t\le 10\end{cases}$, so that

$x(t)=\begin{cases}\frac{1}{2}t^2+C_1 & \text{if } 0\le t\le 5\\ 10t-\frac{1}{2}t^2+C_2 & \text{if } 5\le t\le 10\end{cases}$. Now $C_1=0$ because $x(0)=0$, and continuity of

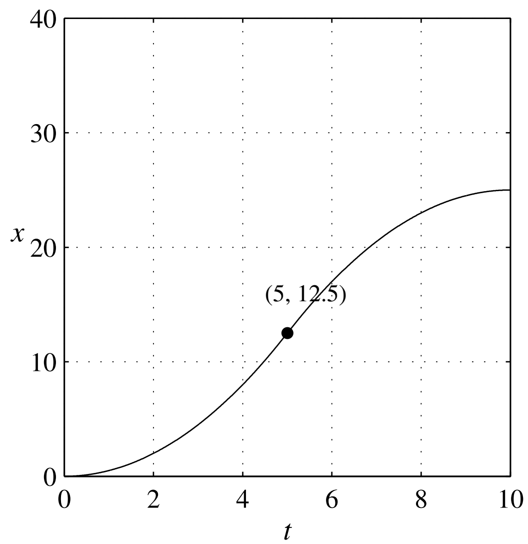
$x(t)$ requires that $x(t) = \frac{1}{2}t^2$ and $x(t) = 10t - \frac{1}{2}t^2 + C_2$ agree when $t = 5$. This implies that $C_2 = -25$, leading to the graph of $x(t)$ shown.

22. For $0 \le t \le 3$, $v(t) = \frac{5}{3}t$, so $x(t) = \frac{5}{6}t^2 + C_1$. Now $C_1 = 0$ because $x(0) = 0$, so $x(t) = \frac{5}{6}t^2$ on this first interval, and its right-endpoint value is $x(3) = \frac{15}{2}$.

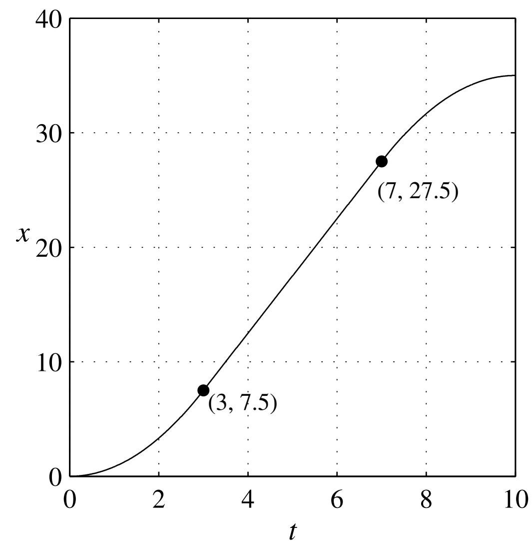
For $3 \le t \le 7$, $v(t) = 5$, so $x(t) = 5t + C_2$ Now $x(3) = \frac{15}{2}$ implies that $C_2 = -\frac{15}{2}$, so $x(t) = 5t - \frac{15}{2}$ on this second interval, and its right-endpoint value is $x(7) = \frac{55}{2}$.

For $7 \le t \le 10$, $v - 5 = -\frac{5}{3}(t-7)$, so $v(t) = -\frac{5}{3}t + \frac{50}{3}$. Hence $x(t) = -\frac{5}{6}t^2 + \frac{50}{3}t + C_3$, and $x(7) = \frac{55}{2}$ implies that $C_3 = -\frac{290}{6}$. Finally, $x(t) = \frac{1}{6}(-5t^2 + 100t - 290)$ on this third interval, leading to the graph of $x(t)$ shown.

Problem 21



Problem 22



23. $v(t) = -9.8t + 49$, so the ball reaches its maximum height ($v = 0$) after $t = 5$ seconds. Its maximum height then is $y(5) = -4.9(5)^2 + 49(5) = 122.5$ meters.

24. $v = -32t$ and $y = -16t^2 + 400$, so the ball hits the ground ($y = 0$) when $t = 5$ sec, and then $v = -32(5) = -160$ ft/sec.

25. $a = -10\ \text{m/s}^2$ and $v_0 = 100$ km/h ≈ 27.78 m/s, so $v = -10t + 27.78$, and hence $x(t) = -5t^2 + 27.78t$. The car stops when $v = 0$, that is $t \approx 2.78$ s, and thus the distance traveled before stopping is $x(2.78) \approx 38.59$ meters.

26. $v = -9.8t + 100$ and $y = -4.9t^2 + 100t + 20$.

(a) $v = 0$ when $t = 100/9.8$ s, so the projectile's maximum height is $y(100/9.8) = -4.9(100/9.8)^2 + 100(100/9.8) + 20 \approx 530$ meters.

(b) It passes the top of the building when $y(t) = -4.9t^2 + 100t + 20 = 20$, and hence after $t = 100/4.9 \approx 20.41$ seconds.

(c) The roots of the quadratic equation $y(t) = -4.9t^2 + 100t + 20 = 0$ are $t = -0.20, 20.61$. Hence the projectile is in the air 20.61 seconds.

27. $a = -9.8 \text{ m/s}^2$, so $v = -9.8t - 10$ and $y = -4.9t^2 - 10t + y_0$. The ball hits the ground when $y = 0$ and $v = -9.8t - 10 = -60$ m/s, so $t \approx 5.10$ s. Hence the height of the building is

$$y_0 = 4.9(5.10)^2 + 10(5.10) \approx 178.57 \text{ m}.$$

28. $v = -32t - 40$ and $y = -16t^2 - 40t + 555$. The ball hits the ground ($y = 0$) when $t \approx 4.77$ s, with velocity $v = v(4.77) \approx -192.64$ ft/s, an impact speed of about 131 mph.

29. Integration of $dv/dt = 0.12t^2 + 0.6t$ with $v(0) = 0$ gives $v(t) = 0.04t^3 + 0.3t^2$. Hence $v(10) = 70$ ft/s. Then integration of $dx/dt = 0.04t^3 + 0.3t^2$ with $x(0) = 0$ gives $x(t) = 0.01t^4 + 0.1t^3$, so $x(10) = 200$ ft. Thus after 10 seconds the car has gone 200 ft and is traveling at 70 ft/s.

30. Taking $x_0 = 0$ and $v_0 = 60 \text{ mph} = 88 \text{ ft/s}$, we get $v = -at + 88$, and $v = 0$ yields $t = 88/a$. Substituting this value of *t*, as well as $x = 176$ ft, into $x = -at^2/2 + 88t$ leads to $a = 22 \text{ ft/s}^2$. Hence the car skids for $t = 88/22 = 4$ s.

31. If $a = -20 \text{ m/s}^2$ and $x_0 = 0$, then the car's velocity and position at time *t* are given by $v = -20t + v_0$ and $x = -10t^2 + v_0 t$. It stops when $v = 0$ (so $v_0 = 20t$), and hence when $x = 75 = -10t^2 + (20t)t = 10t^2$. Thus $t = \sqrt{7.5}$ s, so

$$v_0 = 20\sqrt{7.5} \approx 54.77 \text{ m/s} \approx 197 \text{ km/hr}.$$

32. Starting with $x_0 = 0$ and $v_0 = 50 \text{ km/h} = 5 \times 10^4 \text{ m/h}$, we find by the method of Problem 30 that the car's deceleration is $a = (25/3) \times 10^7 \text{ m/h}^2$. Then, starting with $x_0 = 0$ and $v_0 = 100 \text{ km/h} = 10^5 \text{ m/h}$, we substitute $t = v_0/a$ into $x = -\frac{1}{2}at^2 + v_0 t$ and find that $x = 60$ m when $v = 0$. Thus doubling the initial velocity quadruples the distance the car skids.

33. If $v_0 = 0$ and $y_0 = 20$, then $v = -at$ and $y = -\frac{1}{2}at^2 + 20$. Substitution of $t = 2$, $y = 0$ yields $a = 10\text{ ft/s}^2$. If $v_0 = 0$ and $y_0 = 200$, then $v = -10t$ and $y = -5t^2 + 200$. Hence $y = 0$ when $t = \sqrt{40} = 2\sqrt{10}\text{ s}$ and $v = -20\sqrt{10} \approx -63.25\text{ ft/s}$.

34. **On Earth:** $v = -32t + v_0$, so $t = v_0/32$ at maximum height (when $v = 0$). Substituting this value of *t* and $y = 144$ in $y = -16t^2 + v_0 t$, we solve for $v_0 = 96\text{ ft/s}$ as the initial speed with which the person can throw a ball straight upward.
On Planet Gzyx: From Problem 33, the surface gravitational acceleration on planet Gzyx is $a = 10\text{ ft/s}^2$, so $v = -10t + 96$ and $y = -5t^2 + 96t$. Therefore $v = 0$ yields $t = 9.6\text{ s}$ and so $y_{\max} = y(9.6) = 460.8\text{ ft}$ is the height a ball will reach if its initial velocity is 96 ft/s.

35. If $v_0 = 0$ and $y_0 = h$, then the stone's velocity and height are given by $v = -gt$ and $y = -0.5gt^2 + h$, respectively. Hence $y = 0$ when $t = \sqrt{2h/g}$, so $v = -g\sqrt{2h/g} = -\sqrt{2gh}$.

36. The method of solution is precisely the same as that in Problem 30. We find first that, on Earth, the woman must jump straight upward with initial velocity $v_0 = 12\text{ ft/s}$ to reach a maximum height of 2.25 ft. Then we find that, on the Moon, this initial velocity yields a maximum height of about 13.58 ft.

37. We use units of miles and hours. If $x_0 = v_0 = 0$, then the car's velocity and position after *t* hours are given by $v = at$ and $x = \frac{1}{2}at^2$, respectively. Since $v = 60$ when $t = 5/6$, the velocity equation yields . Hence the distance traveled by 12:50 pm is $x = \frac{1}{2} \cdot 72 \cdot (5/6)^2 = 25\text{ miles}$.

38. Again we have $v = at$ and $x = \frac{1}{2}at^2$. But now $v = 60$ when $x = 35$. Substitution of $a = 60/t$ (from the velocity equation) into the position equation yields $35 = \frac{1}{2}(60/t)t^2 = 30t$, whence $t = 7/6\text{ h}$, that is, 1:10 pm.

39. Integration of $y' = (9/v_S)(1 - 4x^2)$ yields $y = (3/v_S)(3x - 4x^3) + C$, and the initial condition $y(-1/2) = 0$ gives $C = 3/v_S$. Hence the swimmer's trajectory is $y(x) = (3/v_S)(3x - 4x^3 + 1)$. Substitution of $y(1/2) = 1$ now gives $v_S = 6\text{ mph}$.

40. Integration of $y' = 3(1 - 16x^4)$ yields $y = 3x - (48/5)x^5 + C$, and the initial condition $y(-1/2) = 0$ gives $C = 6/5$. Hence the swimmer's trajectory is

$$y(x) = (1/5)\left(15x - 48x^5 + 6\right),$$

and so his downstream drift is $y(1/2) = 2.4\,\text{miles}$.

41. The bomb equations are $a = -32$, $v = -32t$, and $s_B = s = -16t^2 + 800$ with $t = 0$ at the instant the bomb is dropped. The projectile is fired at time $t = 2$, so its corresponding equations are $a = -32$, $v = -32(t-2) + v_0$, and $s_P = s = -16(t-2)^2 + v_0(t-2)$ for $t \geq 2$ (the arbitrary constant vanishing because $s_P(2) = 0$). Now the condition $s_B(t) = -16t^2 + 800 = 400$ gives $t = 5$, and then the further requirement that $s_P(5) = 400$ yields $v_0 = 544/3 \approx 181.33$ ft/s for the projectile's needed initial velocity.

42. Let $x(t)$ be the (positive) altitude (in miles) of the spacecraft at time *t* (hours), with $t = 0$ corresponding to the time at which its retrorockets are fired; let $v(t) = x'(t)$ be the velocity of the spacecraft at time *t*. Then $v_0 = -1000$ and $x_0 = x(0)$ is unknown. But the (constant) acceleration is $a = +20000$, so $v(t) = 20000t - 1000$ and $x(t) = 10000t^2 - 1000t + x_0$. Now $v(t) = 20000t - 1000 = 0$ (soft touchdown) when $t = \frac{1}{20}\,\text{h}$ (that is, after exactly 3 minutes of descent). Finally, the condition $0 = x\left(\frac{1}{20}\right) = 10000\left(\frac{1}{20}\right)^2 - 1000\left(\frac{1}{20}\right) + x_0$ yields $x_0 = 25\,\text{miles}$ for the altitude at which the retrorockets should be fired.

43. The velocity and position functions for the spacecraft are $v_S(t) = 0.0098t$ and $x_S(t) = 0.0049t^2$, and the corresponding functions for the projectile are $v_P(t) = \frac{1}{10}c = 3\times 10^7$ and $x_P(t) = 3\times 10^7 t$. The condition that $x_S = x_P$ when the spacecraft overtakes the projectile gives $0.0049t^2 = 3\times 10^7 t$, whence

$$t = \frac{3\times 10^7}{0.0049} \approx 6.12245\times 10^9\ \text{s} \approx \frac{6.12245\times 10^9}{(3600)(24)(365.25)} \approx 194\ \text{years}.$$

Since the projectile is traveling at $\frac{1}{10}$ the speed of light, it has then traveled a distance of about 19.4 light years, which is about 1.8367×10^{17} meters.

44. Let $a > 0$ denote the constant deceleration of the car when braking, and take $x_0 = 0$ for the car's position at time $t = 0$ when the brakes are applied. In the police experiment with $v_0 = 25$ ft/s, the distance the car travels in *t* seconds is given by

$$x(t) = -\frac{1}{2}at^2 + \frac{88}{60}\cdot 25t,$$

with the factor $\frac{88}{60}$ used to convert the velocity units from mi/h to ft/s. When we solve simultaneously the equations $x(t) = 45$ and $x'(t) = 0$ we find that $a = \frac{1210}{81} \approx 14.94\ \text{ft/s}^2$.

With this value of the deceleration and the (as yet) unknown velocity v_0 of the car involved in the accident, its position function is

$$x(t) = -\frac{1}{2}\cdot\frac{1210}{81}t^2 + v_0 t.$$

The simultaneous equations $x(t) = 210$ and $x'(t) = 0$ finally yield $v_0 = \frac{110}{9}\sqrt{42} \approx 79.21\,\text{ft/s}$, that is, almost exactly 54 miles per hour.

SECTION 1.3

SLOPE FIELDS AND SOLUTION CURVES

The instructor may choose to delay covering Section 1.3 until later in Chapter 1. However, before proceeding to Chapter 2, it is important that students come to grips at some point with the question of the existence of a unique solution of a differential equation — and realize that it makes no sense to look for the solution without knowing in advance that it exists. It may help some students to simplify the statement of the existence-uniqueness theorem as follows:

> Suppose that the function $f(x, y)$ and the partial derivative $\partial f/\partial y$ are both continuous in some neighborhood of the point (a,b). Then the initial value problem
>
> $$\frac{dy}{dx} = f(x, y),\ \ y(a) = b$$
>
> has a unique solution in some neighborhood of the point *a*.

Slope fields and geometrical solution curves are introduced in this section as a concrete aid in visualizing solutions and existence-uniqueness questions. Instead, we provide some details of the construction of the figure for the Problem 1 answer, and then include without further comment the similarly constructed figures for Problems 2 through 9.

1. The following sequence of *Mathematica* 7 commands generates the slope field and the solution curves through the given points. Begin with the differential equation $dy/dx = f(x, y)$, where

```
f[x_, y_] := -y - Sin[x]
```

Then set up the viewing window

```
a = -3; b = 3; c = -3; d = 3;
```

The slope field is then constructed by the command

```
dfield = VectorPlot[{1, f[x, y]}, {x, a, b}, {y, c, d},
   PlotRange -> {{a, b}, {c, d}}, Axes -> True, Frame -> True,
   FrameLabel -> {TraditionalForm[x], TraditionalForm[y]},
   AspectRatio -> 1, VectorStyle -> {Gray, "Segment"},
   VectorScale -> {0.02, Small, None},
```

```
   FrameStyle -> (FontSize -> 12), VectorPoints -> 21,
   RotateLabel -> False]
```

The original curve shown in Fig. 1.3.15 of the text (and its initial point not shown there) are plotted by the commands

```
x0 = -1.9; y0 = 0;
point0 = Graphics[{PointSize[0.025], Point[{x0, y0}]}];
soln = NDSolve[{y'[x] == f[x, y[x]], y[x0] == y0}, y[x],
   {x, a, b}];
curve0 = Plot[soln[[1, 1, 2]], {x, a, b}, PlotStyle ->
   {Thickness[0.0065], Blue}];
Show[curve0, point0]
```

(The *Mathematica* **NDSolve** command carries out an approximate numerical solution of the given differential equation. Numerical solution techniques are discussed in Sections 2.4–2.6 of the textbook.)

The coordinates of the 12 points are marked in Fig. 1.3.15 in the textbook. For instance the 7th point is $(-2.5,1)$. It and the corresponding solution curve are plotted by the commands

```
x0 = -2.5; y0 = 1;
point7 = Graphics[{PointSize[0.025], Point[{x0, y0}]}];
soln = NDSolve[{y'[x] == f[x, y[x]], y[x0] == y0}, y[x],
   {x, a, b}];
curve7 = Plot[soln[[1, 1, 2]], {x, a, b},
   PlotStyle -> {Thickness[0.0065], Blue}];
Show[curve7, point7]
```

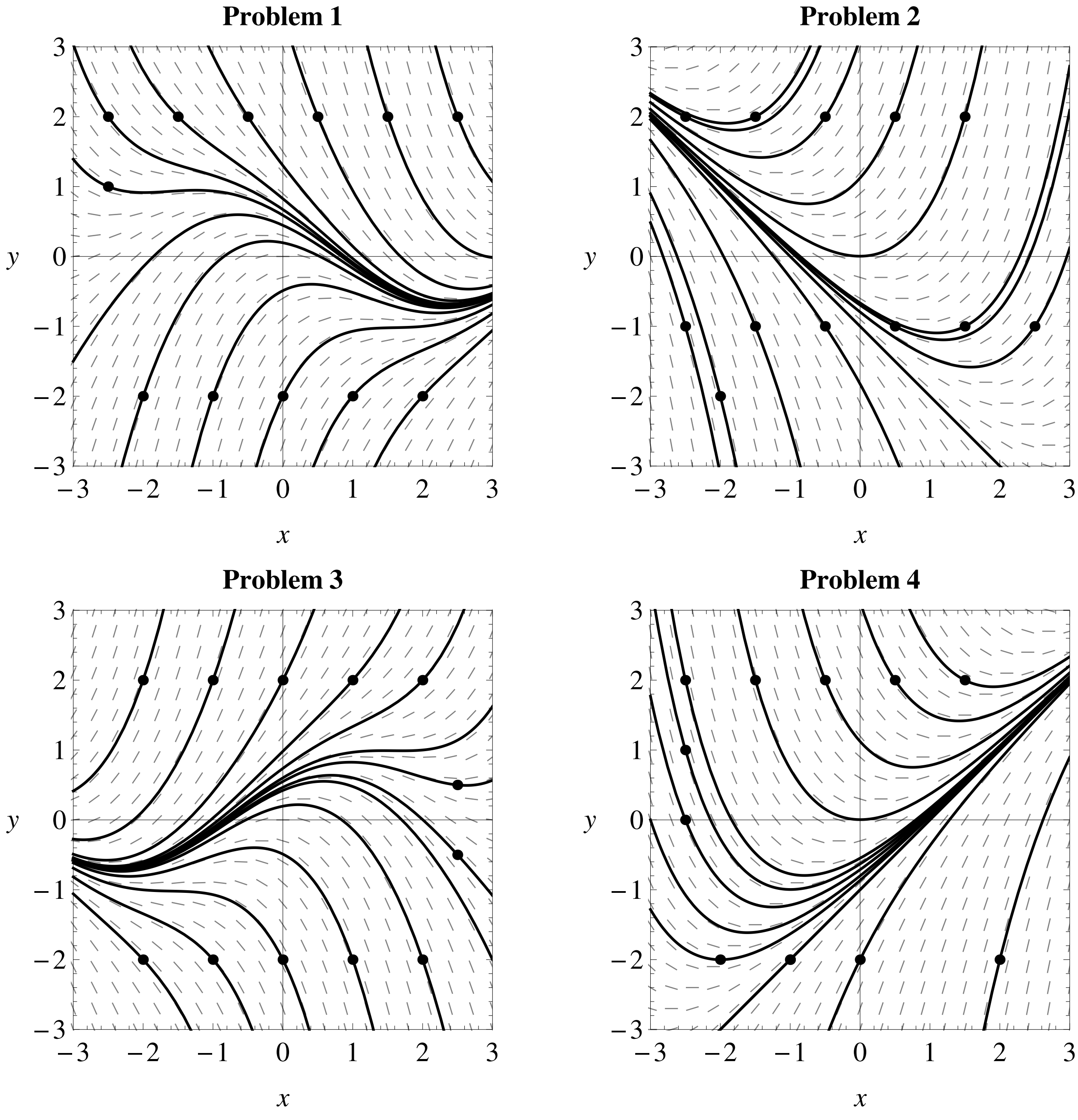
The following command superimposes the two solution curves and starting points found so far upon the slope field:

```
Show[dfield, point0, curve0, point7, curve7]
```

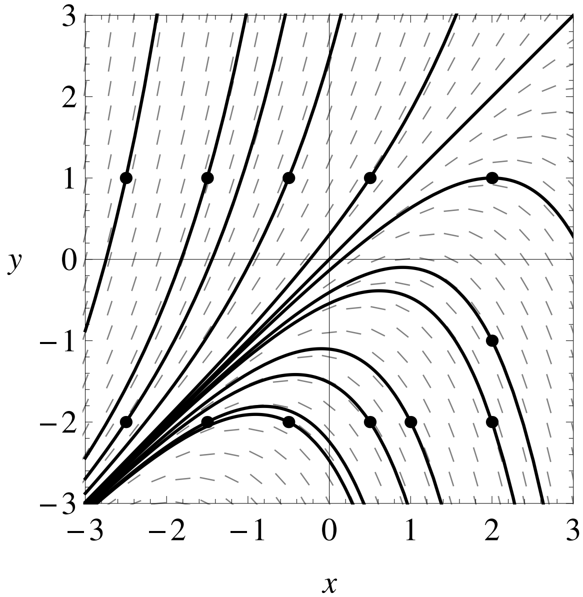
We could continue in this way to build up the entire graphic called for in the problem. Here is an alternative looping approach, variations of which were used to generate the graphics below for Problems 1-10:

```
points = {{-2.5,2}, {-1.5,2}, {-0.5,2}, {0.5,2}, {1.5,2},
   {2.5,2}, {-2,-2}, {-1,-2}, {0,-2}, {1,-2}, {2,-2}, {-2.5,1}};
curves = {}; (* start with null lists *)
dots = {};
Do [
   x0 = points[[i, 1]];
   y0 = points[[i, 2]];
   newdot = Graphics[{PointSize[0.025],Point[{x0, y0}]}];
   dots = AppendTo[dots, newdot];
   soln = NDSolve[{y'[x] == f[x, y[x]],y[x0] == y0}, y[x],
      {x, a, b}];
   newcurve = Plot[soln[[1, 1, 2]], {x, a, b},
      PlotStyle -> {Thickness[0.0065], Black}];
```

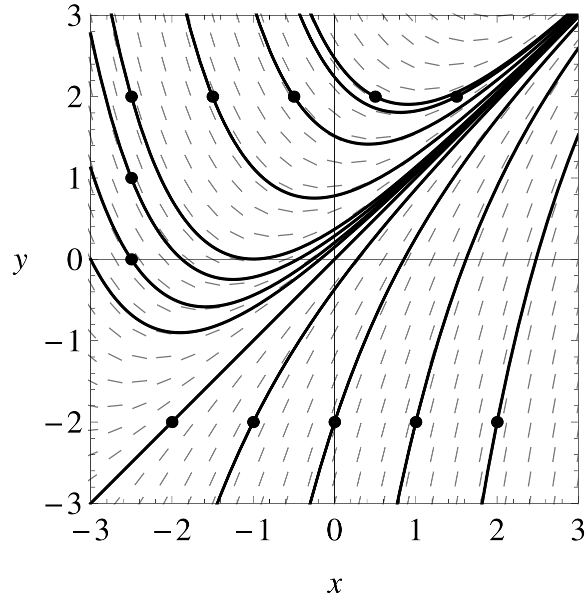
```
    AppendTo[curves, newcurve],
    {i, 1, Length[points]}];
  Show[dfield, curves, dots, PlotLabel -> Style["Problem 1", Bold,
    11]]
```



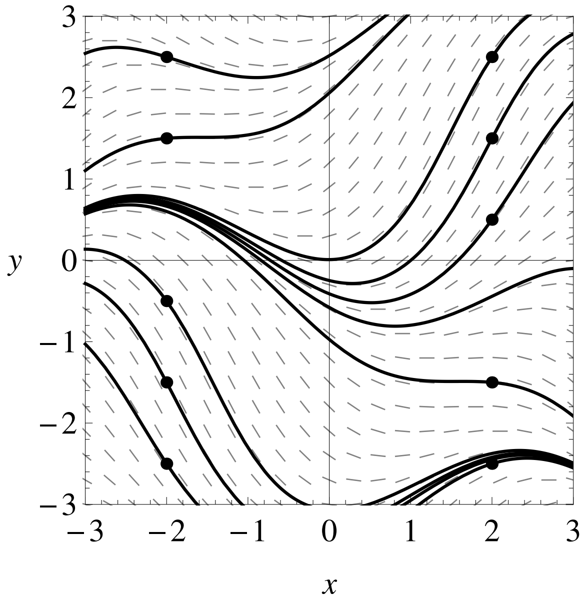
Problem 5



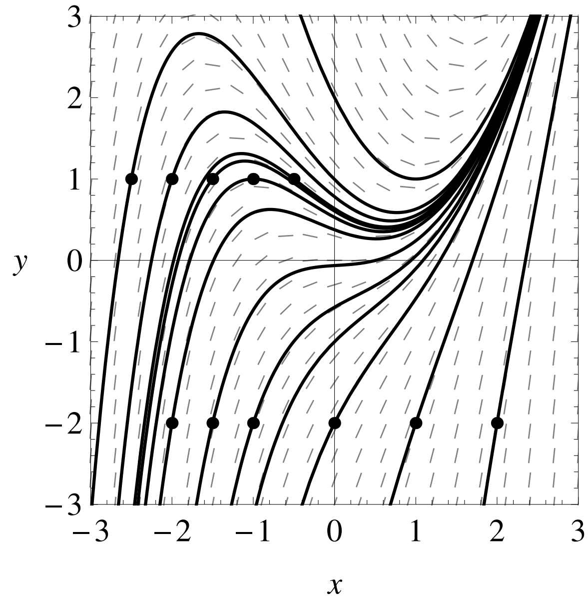
Problem 6



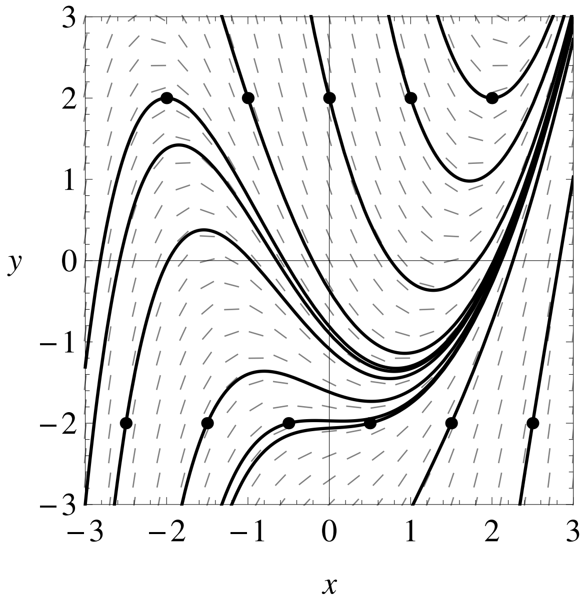
Problem 7



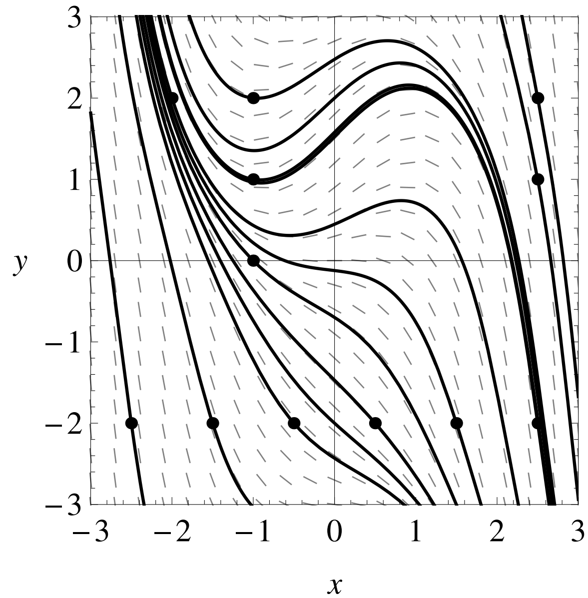
Problem 8



Problem 9



Problem 10



11. Because both $f(x,y) = 2x^2y^2$ and $D_y f(x,y) = 4x^2y$ are continuous everywhere, the existence-uniqueness theorem of Section 1.3 in the textbook guarantees the existence of a unique solution in some neighborhood of $x = 1$.

12. Both $f(x,y) = x\ln y$ and $\partial f/\partial y = x/y$ are continuous in a neighborhood of $(1,1)$, so the theorem guarantees the existence of a unique solution in some neighborhood of $x = 1$.

13. Both $f(x,y) = y^{1/3}$ and $\partial f/\partial y = \frac{1}{3}y^{-2/3}$ are continuous near $(0,1)$, so the theorem guarantees the existence of a unique solution in some neighborhood of $x = 0$.

14. The function $f(x,y) = y^{1/3}$ is continuous in a neighborhood of $(0,0)$, but $\partial f/\partial y = \frac{1}{3}y^{-2/3}$ is not, so the theorem guarantees existence but not uniqueness in some neighborhood of $x = 0$. (See Remark 2 following the theorem.)

15. The function $f(x,y) = (x-y)^{1/2}$ is not continuous at $(2,2)$ because it is not even defined if $y > x$. Hence the theorem guarantees neither existence nor uniqueness in any neighborhood of the point $x = 2$.

16. The function $f(x,y) = (x-y)^{1/2}$ and $\partial f/\partial y = -\frac{1}{2}(x-y)^{-1/2}$ are continuous in a neighborhood of $(2,1)$, so the theorem guarantees both existence and uniqueness of a solution in some neighborhood of $x = 2$.

17. Both $f(x,y)=(x-1)/y$ and $\partial f/\partial y=-(x-1)/y^2$ are continuous near $(0,1)$, so the theorem guarantees both existence and uniqueness of a solution in some neighborhood of $x=0$.

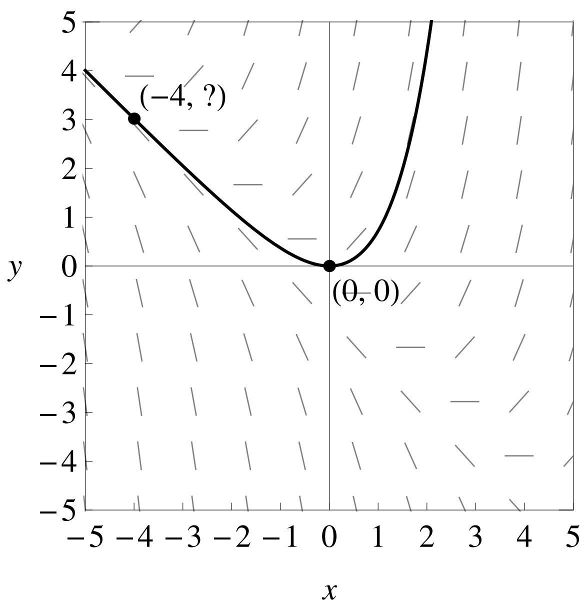
18. Neither $f(x,y)=(x-1)/y$ nor $\partial f/\partial y=-(x-1)/y^2$ is continuous near $(1,0)$, so the existence-uniqueness theorem guarantees nothing.

19. Both $f(x,y)=\ln(1+y^2)$ and $\partial f/\partial y=2y/(1+y^2)$ are continuous near $(0,0)$, so the theorem guarantees the existence of a unique solution near $x=0$.

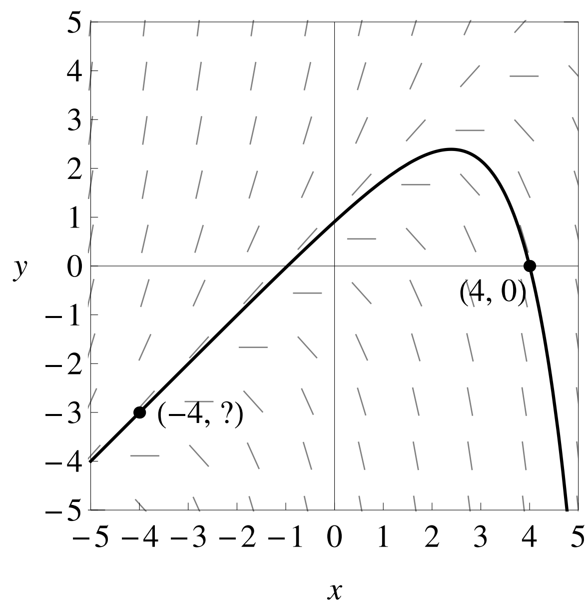
20. Both $f(x,y)=x^2-y^2$ and $\partial f/\partial y=-2y$ are continuous near $(0,1)$, so the theorem guarantees both existence and uniqueness of a solution in some neighborhood of $x=0$.

21. The figure shown can be constructed using commands similar to those in Problem 1, above. Tracing this solution curve, we see that $y(-4)\approx 3$. (An exact solution of the differential equation yields the more accurate approximation $y(-4)=3+e^{-4}\approx 3.0183$.)

Problem 21

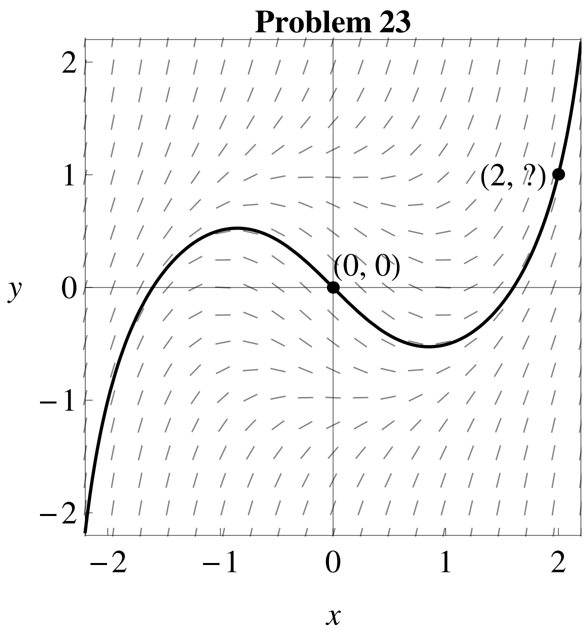


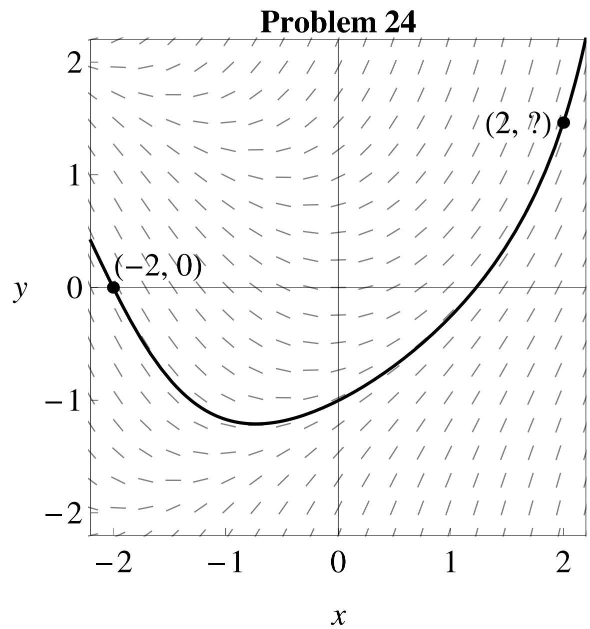
Problem 22



22. Tracing the curve in the figure shown, we see that $y(-4)\approx -3$. An exact solution of the differential equation yields the more accurate approximation $y(-4)\approx -3.0017$.

23. Tracing the curve in the figure shown, we see that $y(2)\approx 1$. A more accurate approximation is $y(2)\approx 1.0044$.

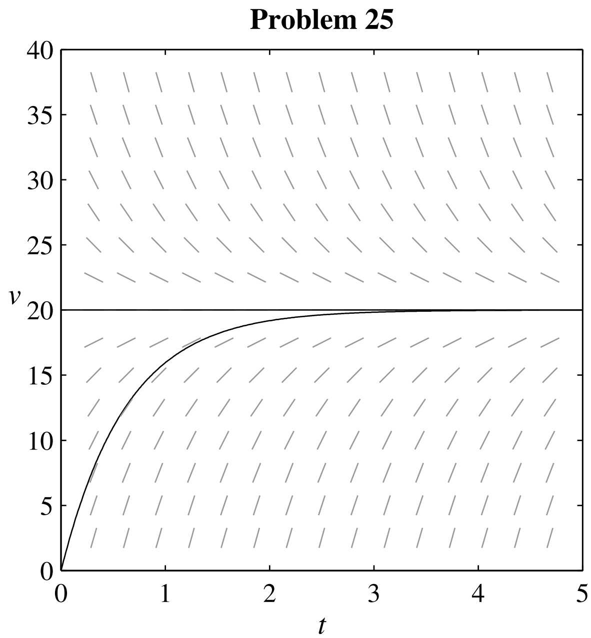


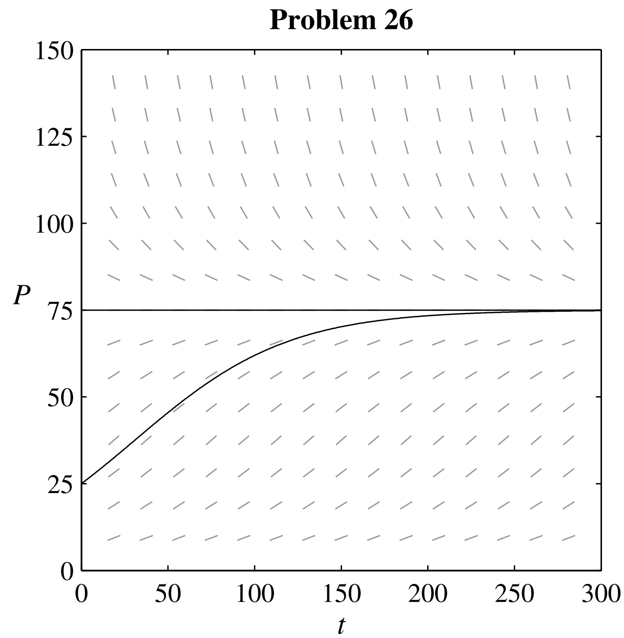


24. Tracing the curve in the figure shown, we see that $y(2) \approx 1.5$. A more accurate approximation is $y(2) \approx 1.4633$.

25. The figure indicates a limiting velocity of 20 ft/sec — about the same as jumping off a $6\frac{1}{4}$-foot wall, and hence quite survivable. Tracing the curve suggests that $v(t) = 19$ ft/sec when t is a bit less than 2 seconds. An exact solution gives $t \approx 1.8723$ then.

26. The figure suggests that there are 40 deer after about 60 months; a more accurate value is $t \approx 61.61$. And it's pretty clear that the limiting population is 75 deer.

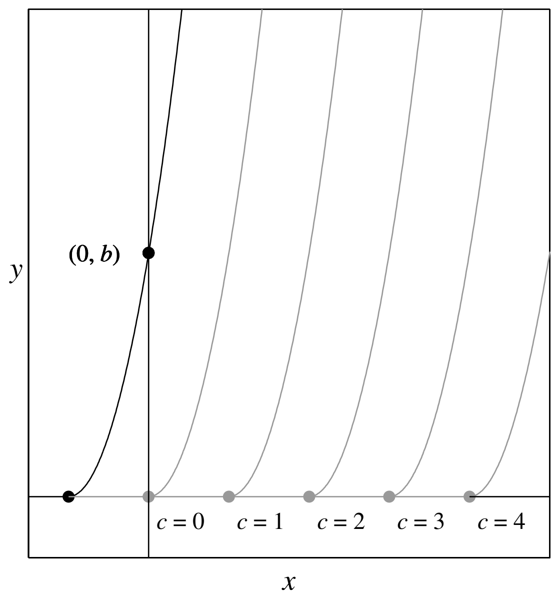




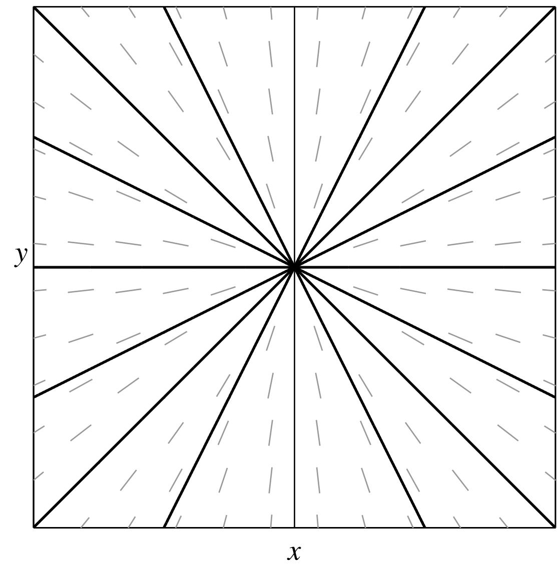
27. **a)** It is clear that $y(x)$ satisfies the differential equation at each *x* with $x < c$ or $x > c$,and by examining left- and right-hand derivatives we see that the same is true at $x = c$. Thus $y(x)$ not only satisfies the differential equation for all *x*, it also satisfies the given initial value problem whenever $c \geq 0$. The infinitely many solutions of the initial value problem are illustrated in the figure. Note that $f(x, y) = 2\sqrt{y}$ is not continuous in any neighborhood of the origin, and so Theorem 1 guarantees neither existence nor uniqueness of solution to the given initial value problem. As it happens, existence occurs, but not uniqueness.

b) If $b < 0$, then the initial value problem $y' = 2\sqrt{y}$, $y(0) = b$ has <u>no</u> solution, because the square root of a negative number would be involved. If $b > 0$, then we get a unique solution curve through $(0, b)$ defined for all *x* by following a parabola (as in the figure, in black) — down (and leftward) to the *x*-axis and then following the *x*-axis to the left. Finally if $b = 0$, then starting at $(0,0)$ we can follow the positive *x*-axis to the point $(c, 0)$ and then branch off on the parabola $y = (x - c)^2$, as shown in gray. Thus there are infinitely many solutions in this case.

Problem 27a



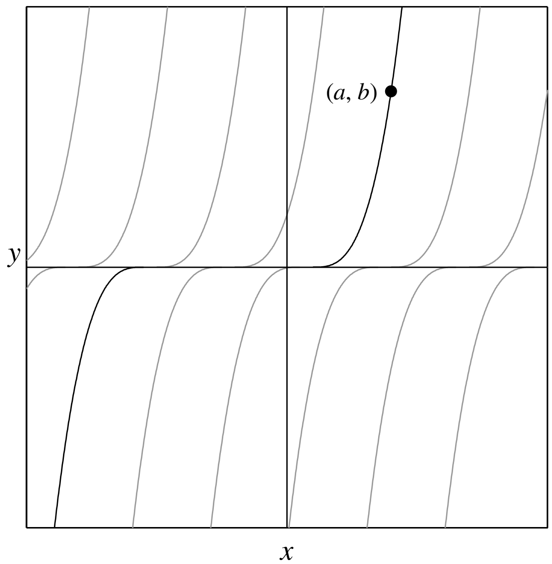
Problem 28



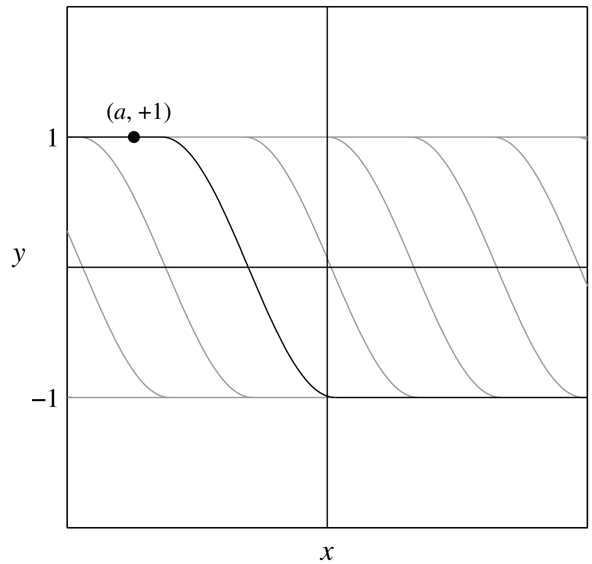
28. The figure makes it clear that the initial value problem $xy' = y$, $y(a) = b$ has a unique solution if $a \neq 0$, infinitely many solutions if $a = b = 0$, and no solution if $a = 0$ but $b \neq 0$ (so that the point (a, b) lies on the positive or negative y-axis). Each of these conclusions is consistent with Theorem 1.

29. As with Problem 27, it is clear that $y(x)$ satisfies the differential equation at each x with $x < c$ or $x > c$, and by examining left- and right-hand derivatives we see that the same is true at $x = c$. Looking at the figure on the left below, we see that if, for instance, $b > 0$, then we can start at the point (a, b) and follow a branch of a cubic down to the x-axis, then follow the x-axis an arbitrary distance before branching down on another cubic. This gives infinitely many solutions of the initial value problem $y' = 3y^{2/3}$, $y(a) = b$ that are defined for all x. However, if $b \neq 0$, then there is only a single cubic $y = (x - c)^3$ passing through (a, b), so the solution is unique near $x = a$ (as Theorem 1 would predict).

Problem 29



Problem 30



30. The function $y(x)$ satisfies the given differential equation on the interval $c < x < c + \pi$, since $y'(x) = -\sin(x - c) < 0$ there and thus

$$-\sqrt{1 - y^2} = -\sqrt{1 - \cos^2(x - c)} = -\sqrt{\sin^2(x - c)} = -\sin(x - c) = y'.$$

Moreover, the same is true for $x < c$ and $x > c + \pi$ (since $y^2 \equiv 1$ and $y' \equiv 0$ there), and at $x = c, c + \pi$ by examining one-sided derivatives. Thus $y(x)$ satisfies the given differential equation for all *x*.

If $|b| > 1$, then the initial value problem $y' = -\sqrt{1 - y^2}$, $y(a) = b$ has no solution, because the square root of a negative number would be involved. If $|b| < 1$, then there is only one curve of the form $y = \cos(x - c)$ through the point (a, b), giving a unique solution. But if $b = \pm 1$, then we can combine a left ray of the line $y = +1$, a cosine curve from the line $y = +1$ to the line $y = -1$, and then a right ray of the line $y = -1$. Looking at the figure, we see that this gives infinitely many solutions (defined for all *x*) through any point of the form $(a, \pm 1)$.

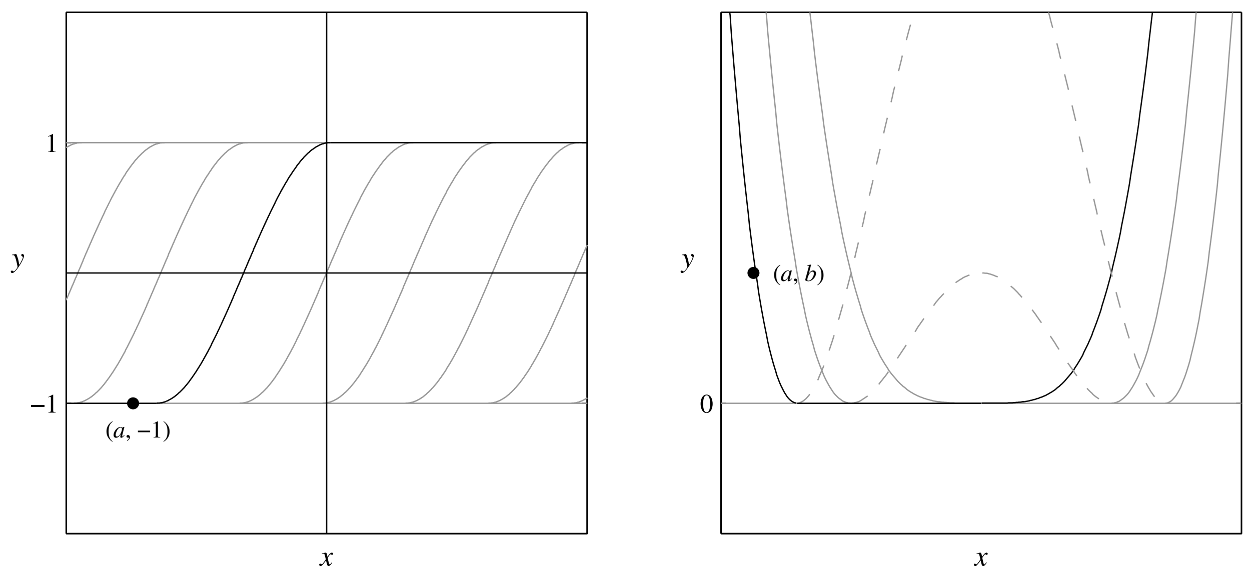
31. The function $y(x) = \begin{cases} -1 & \text{if } x < c - \pi/2 \\ \sin(x - c) & \text{if } c - \pi/2 < x < c + \pi/2 \\ +1 & \text{if } x < c + \pi/2 \end{cases}$ satisfies the given differential equation on the interval $c - \dfrac{\pi}{2} < x < c + \dfrac{\pi}{2}$, since $y'(x) = \cos(x - c) > 0$ there and thus

$$\sqrt{1 - y^2} = \sqrt{1 - \sin^2(x - c)} = \sqrt{\cos^2(x - c)} = \cos(x - c) = y'.$$

Moreover, the same is true for $x < \frac{\pi}{2}$ and $x > c + \frac{\pi}{2}$ (since $y^2 \equiv 1$ and $y' \equiv 0$ there), and at $x = \frac{\pi}{2}, c + \frac{\pi}{2}$ by examining one-sided derivatives. Thus $y(x)$ satisfies the given differential equation for all *x*.

If $|b| > 1$, then the initial value problem $y' = \sqrt{1 - y^2}$, $y(a) = b$ has no solution because the square root of a negative number would be involved. If $|b| < 1$, then there is only one curve of the form $y = \sin(x - c)$ through the point (a, b); this gives a unique solution. But if $b = \pm 1$, then we can combine a left ray of the line $y = -1$, a sine curve from the line $y = -1$ to the line $y = +1$, and then a right ray of the line $y = +1$. Looking at the figure, we see that this gives infinitely many solutions (defined for all *x*) through any point of the form $(a, \pm 1)$.

Problem 31 **Problem 32**



32. The function $y(x)$ satisfies the given differential equation for $x^2 > c$, since $y'(x) = 4x(x^2 - c) = 4x\sqrt{y}$ there. Moreover, the same is true for $x^2 < c$ (since $y = y' \equiv 0$ there), and at $x = \pm\sqrt{c}$ by examining one-sided derivatives. Thus $y(x)$ satisfies the given differential equation for all *x*.

Looking at the figure, we see that we can piece together a "left half" of a quartic for *x* negative, an interval along the *x*-axis, and a "right half" of a quartic curve for *x* positive. This makes it clear that the initial value problem $y' = 4x\sqrt{y}$, $y(a) = b$ has infinitely many solutions (defined for all *x*) if $b \geq 0$. There is no solution if $b < 0$ because this would involve the square root of a negative number.

33. Looking at the figure provided in the answers section of the textbook, it suffices to observe that, among the pictured curves $y = x/(cx-1)$ for all possible values of c,

- there is a unique one of these curves through any point not on either coordinate axis;
- there is no such curve through any point on the y-axis other than the origin; and
- there are infinitely many such curves through the origin (0,0).

But in addition we have the constant-valued solution $y(x) \equiv 0$ that "covers" the x-axis. It follows that the given differential equation has near (a,b)

- a unique solution if $a \neq 0$;
- no solution if $a = 0$ but $b \neq 0$;
- infinitely many different solutions if $a = b = 0$.

Once again these findings are consistent with Theorem 1.

34. **(a)** With a computer algebra system we find that the solution of the initial value problem $y' = y - x + 1$, $y(-1) = -1.2$ is $y(x) = x - 0.2e^{x+1}$, whence $y(1) \approx -0.4778$. With the same differential equation but with initial condition $y(-1) = -0.8$ the solution is $y(x) = x + 0.2e^{x+1}$, whence $y(1) \approx 2.4778$

(b) Similarly, the solution of the initial value problem $y' = y - x + 1$, $y(-3) = -3.01$ is $y(x) = x - 0.01e^{x+3}$, whence $y(3) \approx -1.0343$. With the same differential equation but with initial condition $y(-3) = -2.99$ the solution is $y(x) = x + 0.01e^{x+3}$, whence $y(3) \approx 7.0343$. Thus close initial values $y(-3) = -3 \pm 0.01$ yield $y(3)$ values that are far apart.

35. **(a)** With a computer algebra system we find that the solution of the initial value problem $y' = x - y + 1$, $y(-3) = -0.2$ is $y(x) = x + 2.8e^{-x-3}$, whence $y(2) \approx 2.0189$. With the same differential equation but with initial condition $y(-3) = +0.2$ the solution is $y(x) = x + 3.2e^{-x-3}$, whence $y(2) \approx 2.0216$.

(b) Similarly, the solution of the initial value problem $y' = x - y + 1$, $y(-3) = -0.5$ is $y(x) = x + 2.5e^{-x-3}$, whence $y(2) \approx 2.0168$. With the same differential equation but with initial condition $y(-3) = +0.5$ the solution is $y(x) = x + 3.5e^{-x-3}$, whence $y(2) \approx 2.0236$. Thus the initial values $y(-3) = \pm 0.5$ that are not close both yield $y(2) \approx 2.02$.

SECTION 1.4

SEPARABLE EQUATIONS AND APPLICATIONS

Of course it should be emphasized to students that the possibility of separating the variables is the first one you look for. The general concept of natural growth and decay is important for all differential equations students, but the particular applications in this section are optional. Torricelli's law in the form of Equation (24) in the text leads to some nice concrete examples and problems.

Also, in the solutions below, we make free use of the fact that if C is an arbitrary constant, then so is $5-3C$, for example, which we can (and usually do) replace simply with C itself. In the same way we typically replace e^C by C, with the understanding that C is then an arbitrary nonzero constant.

1. For $y \neq 0$ separating variables gives $\int \frac{dy}{y} = -\int 2x\,dx$, so that $\ln|y| = -x^2 + C$, or $y(x) = \pm e^{-x^2+C} = Ce^{-x^2}$, where C is an arbitrary nonzero constant. (The equation also has the singular solution $y \equiv 0$.)

2. For $y \neq 0$ separating variables gives $\int \frac{dy}{y^2} = -\int 2x\,dx$, so that $-\frac{1}{y} = -x^2 + C$, or $y(x) = \frac{1}{x^2 + C}$. (The equation also has the singular solution $y \equiv 0$.)

3. For $y \neq 0$ separating variables gives $\int \frac{dy}{y} = \int \sin x\,dx$, so that $\ln|y| = -\cos x + C$, or $y(x) = \pm e^{-\cos x + C} = Ce^{-\cos x}$, where C is an arbitrary nonzero constant. (The equation also has the singular solution $y \equiv 0$.)

4. For $y \neq 0$ separating variables gives $\int \frac{dy}{y} = \int \frac{4}{1+x}dx$, so that $\ln|y| = 4\ln(1+x) + C$, or $y(x) = C(1+x)^4$, where C is an arbitrary nonzero constant. (The equation also has the singular solution $y \equiv 0$.)

5. For $-1 < y < 1$ and $x > 0$ separating variables gives $\int \frac{dy}{\sqrt{1-y^2}} = \int \frac{1}{2\sqrt{x}}dx$, so that $\sin^{-1} y = \sqrt{x} + C$, or $y(x) = \sin\left(\sqrt{x} + C\right)$. (The equation also has the singular solutions $y \equiv 1$ and $y \equiv -1$.)

6. For $x, y > 0$ separating variables gives $\int \frac{dy}{\sqrt{y}} = \int 3\sqrt{x}\,dx$, so that $2\sqrt{y} = 2x^{3/2} + C$, or $y(x) = \left(x^{3/2} + C\right)^2$. For $x, y < 0$ we write $\frac{dy}{dx} = 3\sqrt{(-x)(-y)}$, leading to $\int \frac{dy}{\sqrt{-y}} = \int 3\sqrt{-x}\,dx$, or $-2\sqrt{-y} = -2(-x)^{3/2} + C$, or $y(x) = -\left[(-x)^{3/2} + C\right]^2$.

7. For $y \neq 0$ separating variables gives $\int \frac{dy}{y^{1/3}} = \int 4x^{1/3}\,dx$, so that $\frac{3}{2}y^{2/3} = 3x^{4/3} + C$, or $y(x) = \left(2x^{4/3} + C\right)^{3/2}$. (The equation also has the singular solution $y \equiv 0$.)

8. For $y \neq \frac{\pi}{2} + k\pi$, k integer, separating variables gives $\int \cos y\,dy = \int 2x\,dx$, so that $\sin y = x^2 + C$, or $y(x) = \sin^{-1}\left(x^2 + C\right)$.

9. For $y \neq 0$ separating variables and decomposing into partial fractions give $\int \frac{dy}{y} = \int \frac{2}{1-x^2}dx = \int \frac{1}{1+x} + \frac{1}{1-x}dx$, so that $\ln|y| = \ln|1+x| - \ln|1-x| + C$, or $|y| = C\left|\frac{1+x}{1-x}\right|$, where C is an arbitrary positive constant, or $y(x) = C\frac{1+x}{1-x}$, where C is an arbitrary nonzero constant. (The equation also has the singular solution $y \equiv 0$.)

10. For $y \neq -1$ and $x \neq -1$ separating variables gives $\int \frac{1}{(1+y)^2}dy = \int \frac{1}{(1+x)^2}dx$, so that $\frac{-1}{1+y} = \frac{-1}{1+x} + C$, or $1 + y = \frac{1}{\frac{1}{1+x} + C} = \frac{1+x}{1+C(1+x)}$, or finally

$$y(x) = \frac{1+x}{1+C(1+x)} - 1 = \frac{1+x-\left[1+C(1+x)\right]}{1+C(1+x)} = \frac{x-C(1+x)}{1+C(1+x)},$$

where C is an arbitrary constant. (The equation also has the singular solution $y \equiv -1$.)

11. For $y > 0$ separating variables gives $\int \frac{dy}{y^3} = \int x\,dx$, so that $-\frac{1}{2y^2} = \frac{x^2}{2} + C$, or $y(x) = \left(C - x^2\right)^{-1/2}$, where C is an arbitrary constant. Likewise $y(x) = -\left(C - x^2\right)^{-1/2}$ for $y < 0$. (The equation also has the singular solution $y \equiv 0$.)

12. Separating variables gives $\int \frac{y}{y^2+1}\,dy = \int x\,dx$, so that $\frac{1}{2}\ln\left(y^2+1\right) = \frac{1}{2}x^2 + C$, or $y^2+1 = Ce^{x^2}$, or $y = \pm\sqrt{Ce^{x^2}-1}$, where C is an arbitrary nonzero constant.

13. Separating variables gives $\int \frac{y^3}{y^4+1}\,dy = \int \cos x\,dx$, so that $\frac{1}{4}\ln\left(y^4+1\right) = \sin x + C$, where C is an arbitrary constant.

14. For $x, y > 0$ separating variables gives $\int 1+\sqrt{y}\,dy = \int 1+\sqrt{x}\,dx$, so that $y + \frac{2}{3}y^{3/2} = x + \frac{2}{3}x^{3/2} + C$, where C is an arbitrary constant.

15. For $x \neq 0$ and $y \neq 0, \frac{\sqrt{2}}{2}$ separating variables gives $\int \frac{2}{y^2} - \frac{1}{y^4}\,dy = \int \frac{1}{x} - \frac{1}{x^2}\,dx$, so that $-\frac{2}{y} + \frac{1}{3y^3} = \ln|x| + \frac{1}{x} + C$, where C is an arbitrary constant.

16. Separating variables gives $\int \tan y\,dy = \int \frac{x}{1+x^2}\,dx$, so that $-\ln\cos y = \frac{1}{2}\ln\left(1+x^2\right) + C$, or $\sec y = C\sqrt{1+x^2}$, or $y(x) = \sec^{-1}\left(C\sqrt{1+x^2}\right)$, where C is an arbitrary positive constant.

17. Factoring gives $y' = 1 + x + y + xy = (1+x)(1+y)$, and then for $y \neq -1$ separating variables gives $\int \frac{1}{1+y}\,dy = \int 1 + x\,dx$, so that $\ln|1+y| = x + \frac{1}{2}x^2 + C$, where C is an arbitrary constant. (The equation also has the singular solution $y \equiv -1$.)

18. Factoring gives $x^2y' = 1 - x^2 + y^2 - x^2y^2 = \left(1-x^2\right)\left(1+y^2\right)$, and then for $x \neq 0$ separating variables gives $\int \frac{1}{1+y^2}\,dy = \int \frac{1}{x^2} - 1\,dx$, so that $\tan^{-1} y = -\frac{1}{x} - x + C$, or $y(x) = \tan\left(C - \frac{1}{x} - x\right)$, where C is an arbitrary nonzero constant.

19. For $y \neq 0$ separating variables gives $\int \frac{1}{y}\,dy = \int e^x\,dx$, so that $\ln|y| = e^x + C$, or $|y| = C\exp\left(e^x\right)$, where C is an arbitrary positive constant, or finally $y = C\exp\left(e^x\right)$, where C is an arbitrary nonzero constant. The initial condition $y(0) = 2e$ implies that $C \cdot \exp\left(e^0\right) = 2e$, or $C = 2$, leading to the particular solution $y(x) = 2\exp\left(e^x\right)$.

20. Separating variables gives $\int \frac{1}{1+y^2}dy = \int 3x^2 dx$, or $\tan^{-1} y = x^3 + C$. The initial condition $y(0)=1$ implies that $C = \frac{\pi}{4}$, leading to the particular solution $y(x) = \tan\left(x^3 + \frac{\pi}{4}\right)$.

21. For $|x| > 4$ separating variables gives $\int 2y\,dy = \int \frac{x}{\sqrt{x^2-16}}dx$, so that $y^2 = \sqrt{x^2-16} + C$. The initial condition $y(5)=2$ implies that $C=1$, leading to the particular solution $y(x) = \sqrt{1+\sqrt{x^2-16}}$.

22. For $y \neq 0$ separating variables gives $\int \frac{1}{y}dy = \int 4x^3 - 1\,dx$, so that $\ln|y| = x^4 - x + C$, or $|y| = Ce^{x^4-x}$, where *C* is an arbitrary positive constant, or $y = Ce^{x^4-x}$, where *C* is an arbitrary nonzero constant. The initial condition $y(1) = -3$ implies that $C = -3$, leading to the particular solution $y(x) = -3e^{x^4-x}$.

23. Rewriting the differential equation as $\frac{dy}{dx} = 2y - 1$, we see that for $y \neq \frac{1}{2}$ separating variables gives $\int \frac{1}{2y-1}dy = \int dx$, so that $\frac{1}{2}\ln|2y-1| = x + C$, or $|2y-1| = Ce^{2x}$, where *C* is an arbitrary positive constant, or finally $2y - 1 = Ce^{2x}$, which is to say $y = \frac{1}{2}\left(Ce^{2x}+1\right)$, where *C* is an arbitrary nonzero constant. The initial condition $y(1)=1$ implies that $C = \frac{1}{e^2}$, leading to the particular solution $y(x) = \frac{1}{2}\left(\frac{1}{e^2}e^{2x}+1\right) = \frac{1}{2}\left(e^{2x-2}+1\right)$.

24. For $y > 0$ and $0 < x < \pi$, separating variables gives $\int \frac{1}{y}dy = \int \cot x\,dx$, so that $\ln y = \ln(\sin x) + C$, or $y = C\sin x$, where *C* is an arbitrary positive constant. The initial condition $y\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$ implies that $C = \frac{\pi}{2}$, leading to the particular solution $y = \frac{\pi}{2}\sin x$.

25. Rewriting the differential equation as $x\frac{dy}{dx} = 2x^2y + y$, we see that for $x, y \neq 0$ separating variables gives $\int \frac{1}{y}dy = \int 2x + \frac{1}{x}dx$, so that $\ln|y| = x^2 + \ln|x| + C$, or $|y| = C|x|e^{x^2}$, where *C* is an arbitrary positive constant, or $y = Cxe^{x^2}$, where *C* is an arbitrary nonzero constant. The initial condition $y(1)=1$ implies that $C=\dfrac{1}{e}$, leading to the particular solution $y(x)=xe^{x^2-1}$.

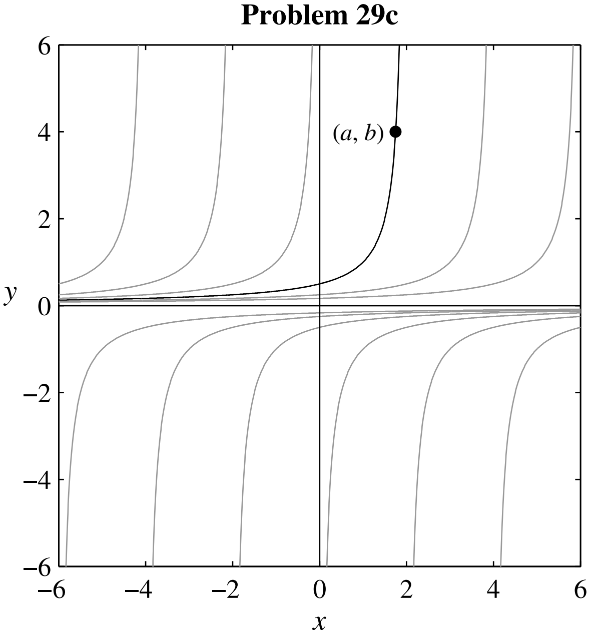
26. For $y\neq 0$ separating variables gives $\int \frac{1}{y^2}dy=\int 2x+3x^2dx$, so that $-\frac{1}{y}=x^2+x^3+C$, or $y=\dfrac{-1}{x^2+x^3+C}$. The initial condition $y(1)=-1$ implies that $C=-1$, leading to the particular solution $y(x)=\dfrac{1}{1-x^2-x^3}$.

27. Separating variables gives $\int e^y dy=\int 6e^{2x}dx$, so that $e^y=3e^{2x}+C$, or $y=\ln\left(3e^{2x}+C\right)$. The initial condition $y(0)=0$ implies that $C=-2$, leading to the particular solution $y(x)=\ln\left(3e^{2x}-2\right)$.

28. For $x\neq 0$ and $y\neq\frac{\pi}{2}+k\pi$, k integer, separating variables gives $\int \sec^2 y\,dy=\int\frac{1}{2\sqrt{x}}dx$, so that $\tan y=\sqrt{x}+C$, or $y(x)=\tan^{-1}\left(\sqrt{x}+C\right)$. The initial condition $y(4)=\frac{\pi}{4}$ implies that $C=-1$, leading to the particular solution $y(x)=\tan^{-1}\left(\sqrt{x}-1\right)$.

29. **(a)** For $y\neq 0$ separation of variables gives the general solution $\int\frac{1}{y^2}dy=\int dx$, so that $-\frac{1}{y}=x+C$, or $y(x)=\dfrac{1}{C-x}$.
(b) Inspection yields the singular solution $y(x)\equiv 0$ that corresponds to *no* value of the constant *C*.
(c) The figure illustrates that there is a unique solution curve through every point in the *xy*-plane.

Problem 29c



30. The set of solutions of $(y')^2=4y$ is the union of the solutions of the two differential equations $y'=\pm 2\sqrt{y}$, where $y\geq 0$. For $y\neq 0$ separation of variables applied to $y'=2\sqrt{y}$ gives $\int\frac{1}{\sqrt{y}}dy=\int 2\,dx$, so that

$\sqrt{y} = x + C$, or $y(x) = (x+C)^2$; replacing C with $-C$ gives the solution family indicated in the text. The same procedure applied to $y' = -2\sqrt{y}$ leads to $y(x) = (-x+C)^2 = (x-C)^2$, again the same solution family (although see Problem 31 and its solution). In both cases the equation also has the singular solution $y(x) \equiv 0$, which corresponds to *no* value of the constant *C*.

(a) The given differential equation $(y')^2 = 4y$ has no solution curve through the point (a,b) if $b < 0$, simply because $(y')^2 \geq 0$.

(b) If $b \geq 0$, then we can combine branches of parabolas with segments along the *x*-axis (in the manner of Problems 27-32, Section 1.3) to form infinitely many solution curves through (a,b) that are defined for all *x*.

(c) Finally, if $b > 0$, then near (a,b) there are exactly *two* solution curves through this point, corresponding to the two indicated parabolas through (a,b), one ascending, and one descending, with increasing *x*. (Again, see Problem 31.)

31. As noted in Problem 30, the solutions of the differential equation $(dy/dx)^2 = 4y$ consist of the solutions of $dy/dx = 2\sqrt{y}$ together with those of $dy/dx = -2\sqrt{y}$, and again we must have $y \geq 0$. Imposing the initial condition $y(a) = b$, where $b > 0$, upon the general solution $y(x) = (x-C)^2$ found in Problem 30 gives $b = (a-C)^2$, which leads to the two values $C = a \pm \sqrt{b}$, and thus to the two particular solutions $y(x) = \left(x - a \pm \sqrt{b}\right)^2$. For these two particular solutions we have $y'(a) = \pm 2\sqrt{b}$, where $(+)$ corresponds to $dy/dx = 2\sqrt{y}$ and $(-)$ corresponds to $dy/dx = -2\sqrt{y}$. It follows that whereas the solutions of $(dy/dx)^2 = 4y$ through (a,b) contain two parabolic segments, one ascending and one descending from left to right, the solutions of $dy/dx = 2\sqrt{y}$ through (a,b) (the black curves in the figure) contain only ascending parabolic segments, whereas for $dy/dx = -2\sqrt{y}$ the (gray) parabolic segments are strictly descending. Thus the answer to the question is "no", because the descending parabolic segments represent solutions of $(dy/dx)^2 = 4y$ but not of $dy/dx = 2\sqrt{y}$. From all this we arrive at the following answers to parts (a)-(c):

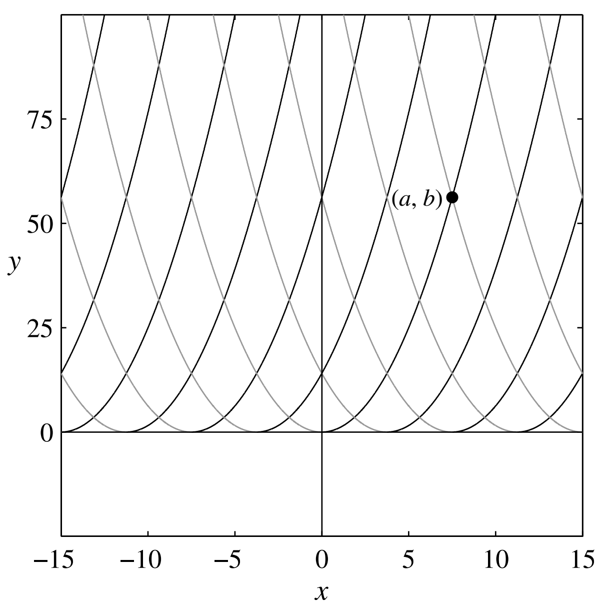
(a) No solution curve if $b < 0$;

(b) A unique solution curve if $b > 0$;

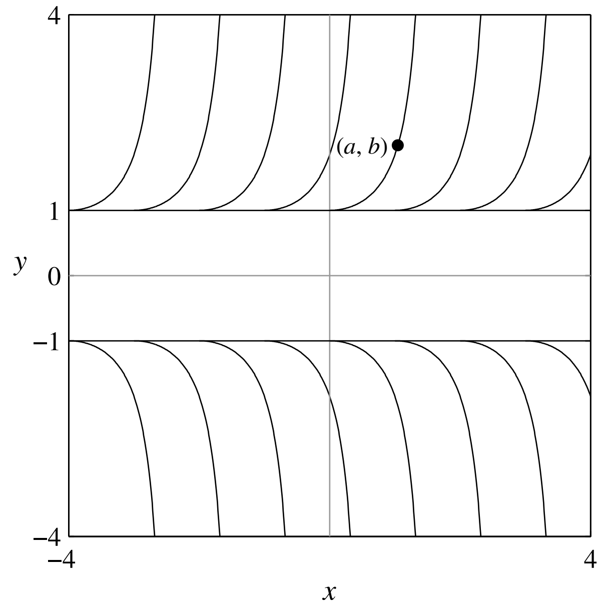
(c) Infinitely many solution curves if $b = 0$, because in this case (as noted in the solution for Problem 30) we can pick any $c > a$ and define the solution

$$y(x)=\begin{cases}0 & \text{if } x<c\\ (x-c)^2 & \text{if } x\ge c\end{cases}.$$

Problem 31



Problem 32



32. For $|y|>1$ separation of variables gives $\int \frac{1}{y\sqrt{y^2-1}}\,dy=\int dx$. We take the inverse secant function to have range $\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$, so that $\frac{d}{dy}\sec^{-1}y=\frac{1}{|y|\sqrt{y^2-1}}$, $|y|>1$. Thus if $y>1$, then the solutions of the differential equation are given by $x=\sec^{-1}y+C$, or $y(x)=\sec(x-C)$, where $C\le x<C+\frac{\pi}{2}$. If instead $y<-1$, then the solutions are given by $x=-\sec^{-1}y+C$, or $y(x)=\sec(C-x)$, where $C-\pi\le x<C-\frac{\pi}{2}$. Finally, the equation also has the singular solutions $y(x)\equiv-1$ and $y(x)\equiv+1$. This leads to the following answers for (a)-(c):

(a) If $-1<b<1$, then the initial value problem has no solution, because the square root of a negative number would be involved.

(b) As the figure illustrates, the initial value problem has a unique solution if $|b|>1$.

(c) If $b=1$ (and similarly if $b=-1$), then we can pick any $c>a$ and define the solution

$$y(x)=\begin{cases}1 & \text{if } x<c\\ \sec(x-c) & \text{if } c\le x<c+\frac{\pi}{2}\end{cases}.$$

So we see that if $b=\pm1$, then the initial value problem has infinitely many solutions.

33. The population growth rate is $k = \ln(30000/25000)/10 \approx 0.01823$, so the population of the city *t* years after 1960 is given by $P(t) = 25000e^{0.01823t}$. The expected year 2000 population is then $P(40) = 25000e^{0.01823\times 40} \approx 51840$.

34. The population growth rate is $k = \ln(6)/10 \approx 0.17918$, so the population after *t* hours is given by $P(t) = P_0 e^{0.17918t}$. To find how long it takes for the population to double, we therefore need only solve the equation $2P_0 = P_0 e^{0.17918t}$ for *t*, finding $t = \ln(2)/0.17198 \approx 3.87$ hours.

35. As in the textbook discussion of radioactive decay, the number of ${}^{14}C$ atoms after *t* years is given by $N(t) = N_0 e^{-0.0001216t}$. Hence we need only solve the equation $\frac{1}{6}N_0 = N_0 e^{-0.0001216t}$ for the age *t* of the skull, finding $t = \dfrac{\ln 6}{0.0001216} \approx 14735$ years.

36. As in Problem 35, the number of ${}^{14}\mathrm{C}$ atoms after *t* years is given by $N(t) = 5.0\times 10^{10} e^{-0.0001216t}$. Hence we need only solve the equation $4.6\times 10^{10} = 5.0\times 10^{10} e^{-0.0001216t}$ for the age *t* of the relic, finding $t = \left[\ln(5.0/4.6)\right]/0.0001216 \approx 686$ years. Thus it appears not to be a genuine relic of the time of Christ 2000 years ago.

37. The amount in the account after *t* years is given by $A(t) = 5000e^{0.08t}$. Hence the amount in the account after 18 years is given by $A(18) = 5000e^{0.08\times 18} \approx 21{,}103.48$ dollars.

38. When the book has been overdue for *t* years, the fine owed is given in dollars by $A(t) = 0.30e^{0.05t}$. Hence the amount owed after 100 years is given by $A(100) = 0.30e^{0.05\times 100} \approx 44.52$ dollars.

39. To find the decay rate of this drug in the dog's blood stream, we solve the equation $\frac{1}{2} = e^{-5k}$ (half-life 5 hours) for *k*, finding $k = (\ln 2)/5 \approx 0.13863$. Thus the amount in the dog's bloodstream after *t* hours is given by $A(t) = A_0 e^{-0.13863t}$. We therefore solve the equation $A(1) = A_0 e^{-0.13863} = 50\times 45 = 2250$ for A_0, finding $A_0 \approx 2585$ mg, the amount to anesthetize the dog properly.

40. To find the decay rate of radioactive cobalt, we solve the equation $\frac{1}{2} = e^{-5.27k}$ (half-life 5.27 years) for $k = (\ln 2)/5.27 \approx 0.13153$. Thus the amount of radioactive cobalt left after *t* years is given by $A(t) = A_0 e^{-0.13153t}$. We therefore solve the equation

$A(t) = A_0 e^{-0.13153t} = 0.01A_0$ for *t*, finding $t = (\ln 100)/0.13153 \approx 35.01$ years. Thus it will be about 35 years until the region is again inhabitable.

41. Taking $t = 0$ when the body was formed and $t = T$ now, we see that the amount $Q(t)$ of ^{238}U in the body at time *t* (in years) is given by $Q(t) = Q_0 e^{-kt}$, where $k = (\ln 2)/(4.51 \times 10^9)$. The given information implies that $\dfrac{Q(T)}{Q_0 - Q(T)} = 0.9$. Upon substituting $Q(t) = Q_0 e^{-kt}$ we solve readily for $e^{kT} = 19/9$, so that $T = (1/k)\ln(19/9) \approx 4.86 \times 10^9$. Thus the body was formed approximately 4.86 billion years ago.

42. Taking $t = 0$ when the rock contained only potassium and $t = T$ now, we see that the amount $Q(t)$ of potassium in the rock at time *t* (in years) is given by $Q(t) = Q_0 e^{-kt}$, where $k = (\ln 2)/(1.28 \times 10^9)$. The given information implies that the amount $A(t)$ of argon at time *t* is $A(t) = \frac{1}{9}[Q_0 - Q(t)]$ and also that $A(T) = Q(T)$. Thus $Q_0 - Q(T) = 9Q(T)$. After substituting $Q(T) = Q_0 e^{-kT}$ we readily solve for $T = (\ln 10/\ln 2)\cdot(1.28 \times 10^9) \approx 4.25 \times 10^9$. Thus the age of the rock is about 1.25 billion years.

43. Because $A = 0$ in Newton's law of cooling, the differential equation reduces to $T' = -kT$, and the given initial temperature then leads to $T(t) = 25e^{-kt}$. The fact that $T(20) = 15$ yields $k = (1/20)\ln(5/3)$, and finally we solve the equation $5 = 25e^{-kt}$ for *t* to find $t = \ln 5/k \approx 63$ min.

44. The amount of sugar remaining undissolved after *t* minutes is given by $A(t) = A_0 e^{-kt}$; we find the value of *k* by solving the equation $A(1) = A_0 e^{-k} = 0.75A_0$ for *k*, finding $k = -\ln 0.75 \approx 0.28768$. To find how long it takes for half the sugar to dissolve, we solve the equation $A(t) = A_0 e^{-kt} = \frac{1}{2}A_0$ for *t*, finding $t = (\ln 2)/0.28768 \approx 2.41$ min.

45. **(a)** The light intensity at a depth of *x* meters is given by $I(x) = I_0 e^{-1.4x}$. We solve the equation $I(x) = I_0 e^{-1.4x} = \frac{1}{2}I_0$ for *x*, finding $x = (\ln 2)/1.4 \approx 0.495$ meters.

(b) At depth 10 meters the intensity is $I(10) = I_0 e^{-1.4\times 10} \approx (8.32 \times 10^{-7})I_0$, that is, 0.832 of one millionth of the light intensity I_0 at the surface.

(c) We solve the equation $I(x) = I_0 e^{-1.4x} = 0.01I_0$ for *x*, finding $x = (\ln 100)/1.4 \approx 3.29$ meters.

46. Solving the initial value problem shows that the pressure at an altitude of *x* miles is given by $p(x) = 29.92e^{-0.2x}$ inches of mercury.

(a) Hence the pressure at altitude 10000 ft is $p(10000/5280) \approx 20.49$ inches of mercury, and likewise the pressure at altitude 30000 ft is $p(30000/5280) \approx 9.60$ inches of mercury.

(b) To find the altitude where $p = 15$ inches of mercury we solve the $29.92e^{-0.2x} = 15$ for x, finding $x = (\ln 29.92/15)/0.2 \approx 3.452 \text{ miles} \approx 18,200 \text{ ft}$.

47. If $N(t)$ denotes the number of people (in thousands) who have heard the rumor after *t* days, then the initial value problem is $N' = k(100 - N)$, $N(0) = 0$. Separating variables leads to $\ln(100 - N) = -kt + C$, and the initial condition $N(0) = 0$ gives $C = \ln 100$. Then $100 - N = 100e^{-kt}$, so $N(t) = 100(1 - e^{-kt})$. Substituting $N(7) = 10$ and solving for *k* gives $k = \ln(100/90)/7 \approx 0.01505$. Finally, 50,000 people have heard the rumor after $t = (\ln 2)/k \approx 46.05$ days, by solving the equation $100(1 - e^{-kt}) = 50$ for *t*.

48. Let $N_8(t)$ and $N_5(t)$ be the numbers of ^{238}U and ^{235}U atoms, respectively, at time *t* (in billions of years after the creation of the universe). Then $N_8(t) = N_0 e^{-kt}$ and $N_5(t) = N_0 e^{-ct}$, where N_0 is the initial number of atoms of each isotope. Also, $k = (\ln 2)/4.51$ and $c = (\ln 2)/0.71$ from the given half-lives. Since $\dfrac{N_8(t)}{N_5(t)} = 137.7$ at present, dividing the equations for $N_8(t)$ and $N_5(t)$ shows that $e^{(c-k)t} = 137.7$ at present, and solving for *t* gives $t = (\ln 137.7)/(c - k) \approx 5.99$. Thus we get an estimate of about 6 billion years for the age of the universe.

49. Newton's law of cooling gives $\dfrac{dT}{dt} = k(70 - T)$, and separating variables and integrating lead to $\ln(T - 70) = -kt + C$. The initial condition $T(0) = 210$ gives $C = \ln 140$, and then $T(30) = 140$ gives $\ln 70 = -30k + \ln 140$, or $k = (\ln 2)/30$, so that $T(t) = e^{-kt+C} + 70 = 140e^{-kt} + 70$. Finally, setting $T(t) = 100$ gives $140e^{-kt} + 70 = 100$, or $t = [\ln(14/3)]/k \approx 66.67$ minutes, or 66 minutes and 40 seconds.

50. **(a)** The initial condition implies that $A(t)=10e^{kt}$. The fact that $A(t)$ triples every 7.5 years implies that $30=A(\tfrac{15}{2})=10e^{15k/2}$, which gives $e^{15k/2}=3$, or $k=\dfrac{2\ln 3}{15}=\ln 3^{2/15}$. Thus $A(t)=10\left(e^{k}\right)^{t}=10\cdot 3^{2t/15}$.

(b) After 5 months we have $A(5)=10\cdot 3^{2/3}\approx 20.80\text{ pu}$.

(c) $A(t)=100$ gives $10\cdot 3^{2t/15}=100$, or $t=\dfrac{15}{2}\cdot\dfrac{\ln 10}{\ln 3}\approx 15.72$ years.

51. **(a)** The initial condition gives $A(t)=15e^{-kt}$, and then $A(5)=10$ implies that $15e^{-kt}=10$, or $e^{kt}=\dfrac{3}{2}$, or $k=\dfrac{1}{5}\ln\dfrac{3}{2}$. Thus

$$A(t)=15\exp\left(-\frac{t}{5}\ln\frac{3}{2}\right)=15\left(\frac{3}{2}\right)^{-t/5}=15\left(\frac{2}{3}\right)^{t/5}.$$

(b) After 8 months we have $A(8)=15\left(\dfrac{2}{3}\right)^{8/5}\approx 7.84\text{ su}$.

(c) $A(t)=1$ when $A(t)=15\left(\dfrac{2}{3}\right)^{t/5}=1$, that is $t=5\dfrac{\ln(\frac{1}{15})}{\ln(\frac{2}{3})}\approx 33.3944$. Thus it will be safe to return after about 33.4 months.

52. If $L(t)$ denotes the number of human language families at time *t* (in years), then $L(t)=e^{kt}$ for some constant *k*. The condition that $L(6000)=e^{6000k}=1.5$ gives $k=\dfrac{1}{6000}\ln\dfrac{3}{2}$. If "now" corresponds to time $t=T$, then we are given that $L(T)=e^{kT}=3300$, so $T=\dfrac{1}{k}\ln 3300=6000\dfrac{\ln 3300}{\ln(3/2)}\approx 119887.18$. This result suggests that the original human language was spoken about 120 thousand years ago.

53. As in Problem 52, if $L(t)$ denotes the number of Native American language families at time *t* (in years), then $L(t)=e^{kt}$ for some constant *k*, and the condition that $L(6000)=e^{6000k}=1.5$ gives $k=\dfrac{1}{6000}\ln\dfrac{3}{2}$. If "now" corresponds to time $t=T$, then we are given that $L(T)=e^{kT}=150$, so $T=\dfrac{1}{k}\ln 150=6000\dfrac{\ln 150}{\ln(3/2)}\approx 74146.48$. This result suggests that the ancestors of today's Native Americans first arrived in the western hemisphere about 74 thousand years ago.

54. With $A(y)$ constant, Equation (30) in the text takes the form $\frac{dy}{dt} = k\sqrt{y}$, which we readily solve to find $2\sqrt{y} = kt + C$. The initial condition $y(0) = 9$ yields $C = 6$, and then $y(1) = 4$ yields $k = 2$. Thus the depth at time *t* (in hours) is $y(t) = (3-t)^2$, and hence it takes 3 hours for the tank to empty.

55. With $A = \pi \cdot 3^2$ and $a = \pi(1/12)^2$, and taking $g = 32 \text{ ft/sec}^2$, Equation (30) reduces to $162y' = -\sqrt{y}$, which we solve to find $324\sqrt{y} = -t + C$. The initial condition $y(0) = 9$ leads to $C = 972$, and so $y = 0$ when $t = 972 \text{ sec}$, that is 16 min 12 sec.

56. The radius of the cross-section of the cone at height *y* is proportional to *y*, so $A(y)$ is proportional to y^2. Therefore Equation (30) takes the form $y^2 y' = -k\sqrt{y}$, and a general solution is given by $2y^{5/2} = -5kt + C$. The initial condition $y(0) = 16$ yields $C = 2048$, and then $y(1) = 9$ gives $5k = 1562$. Hence $y = 0$ when $t = \frac{C}{5k} = \frac{2048}{1562} \approx 1.31 \text{ hr}$.

57. The solution of $y' = -k\sqrt{y}$ is given by $2\sqrt{y} = -kt + C$. The initial condition $y(0) = h$ (the height of the cylinder) yields $C = 2\sqrt{h}$. Then substituting $t = T$ and $y = 0$ gives $k = 2\sqrt{h}/T$. It follows that $y = h\left(1 - \frac{t}{T}\right)^2$. If *r* denotes the radius of the cylinder, then

$$V(y) = \pi r^2 y = \pi r^2 h\left(1 - \frac{t}{T}\right)^2 = V_0\left(1 - \frac{t}{T}\right)^2.$$

58. Since $x = y^{3/4}$, the cross-sectional area is $A(y) = \pi x^2 = \pi y^{3/2}$. Hence the general equation $A(y)y' = -a\sqrt{2gy}$ reduces to the differential equation $yy' = -k$ with general solution $\frac{1}{2}y^2 = -kt + C$. The initial condition $y(0) = 12$ gives $C = 72$, and then $y(1) = 6$ yields $k = 54$. Upon solving for *y* we find that the depth at time *t* is $y(t) = \sqrt{144 - 108t}$. Hence the tank is empty after $t = 144/108 \text{ hr}$, that is, at 1:20 p.m.

59. **(a)** Since $x^2 = by$, the cross-sectional area is $A(y) = \pi x^2 = \pi by$. Hence equation (30) becomes $y^{1/2}y' = -k = -(a/\pi b)\sqrt{2g}$, with general solution $\frac{2}{3}y^{3/2} = -kt + C$. The initial condition $y(0) = 4$ gives $C = 16/3$, and then $y(1) = 1$ yields $k = 14/3$. It follows that the depth at time *t* is $y(t) = (8 - 7t)^{2/3}$.

(b) The tank is empty after $t = 8/7$ hr, that is, at 1:08:34 p.m.

(c) We see above that $k = \dfrac{a}{\pi b}\sqrt{2g} = \dfrac{14}{3}$. Substitution of $a = \pi r^2$ and $b = 1$ and $g = 32 \cdot 3600^2$ ft/hr^2 yields $r = \dfrac{1}{60}\sqrt{\dfrac{7}{12}}$ ft $\approx$ 0.15 in for the radius of the bottom hole.

60. With $g = 32$ ft/sec^2 and $a = \pi(\frac{1}{12})^2$, Equation (30) simplifies to $A(y)\dfrac{dy}{dt} = -\dfrac{\pi}{18}\sqrt{y}$. If z denotes the distance from the center of the cylinder down to the fluid surface, then $y = 3 - z$ and $A(y) = 10\left(9 - z^2\right)^{1/2}$, with $\dfrac{dz}{dt} = -\dfrac{dy}{dt}$. Upon substituting, then, the equation above becomes $10\left(9 - z^2\right)^{1/2}\dfrac{dz}{dt} = \dfrac{\pi}{18}(3 - z)^{1/2}$, or $\int 180(3 + z)^{1/2}\,dz = \int \pi\,dt$, or $120(3 + z)^{1/2} = \pi t + C$. Now $z = 0$ when $t = 0$, so $120 \cdot 3^{3/2}$. The tank is empty when $z = 3$ (that is, when $y = 0$), and thus after $t = \dfrac{120}{\pi}\left(6^{3/2} - 3^{3/2}\right) \approx 362.90$ sec. It therefore takes about 6 min 3 sec for the fluid to drain completely.

61. $A(y) = \pi\left(8y - y^2\right)$ as in Example 6 in the text, but now $a = \dfrac{\pi}{144}$ in Equation (30), so that the initial value problem is $18\left(8y - y^2\right)y' = -\sqrt{y}$, $y(0) = 8$. Separating variables gives $\int 18\left(y^{3/2} - 8y^{1/2}\right)dy = \int dt$, or $18\left(\dfrac{2}{5}y^{5/2} - \dfrac{16}{3}y^{3/2}\right) = t + C$, and the initial condition gives $C = 18\left(\dfrac{2}{5}8^{5/2} - \dfrac{16}{3}8^{3/2}\right)$. We seek the value of t when $y = 0$, which is given by $-C \approx 869 \text{ sec} = 14 \text{ min } 29 \text{ sec}$.

62. Here $A(y) = \pi\left(1 - y^2\right)$ and the area of the bottom hole is $a = 10^{-4}\pi$, so Equation (30) leads to the initial value problem $\pi\left(1 - y^2\right)\dfrac{dy}{dt} = -10^{-4}\pi\sqrt{2 \times 9.8y}$, $y(0) = 1$, or $\left(y^{-1/2} - y^{3/2}\right)\dfrac{dy}{dt} = -1.4 \times 10^{-4}\sqrt{10}$. Separating variables yields

$$2y^{1/2} - \frac{2}{5}y^{5/2} = -1.4 \times 10^{-4}\sqrt{10}\,t + C.$$

The initial condition $y(0) = 1$ implies that $C = 2 - \dfrac{2}{5} = \dfrac{8}{5}$, so $y = 0$ after $t = \dfrac{8/5}{1.4 \times 10^{-4}\sqrt{10}} \approx 3614 \text{ sec} = 1 \text{ hr } 14 \text{ sec}$. Thus the tank is empty at about 14 seconds after 2 pm.

63. **(a)** As in Example 6, the initial value problem is $\pi\left(8y-y^2\right)\dfrac{dy}{dt}=-\pi k\sqrt{y}$, $y(0)=4$, where $k=0.6r^2\sqrt{2g}=4.8r^2$. Separating variables and applying the initial condition just as in the Example 6 solution, we find that $\dfrac{16}{3}y^{3/2}-\dfrac{2}{5}y^{5/2}=-kt+\dfrac{448}{15}$. When we substitute $y=2$ (ft) and $t=1800$ sec (that is, 30 min) we find that $k\approx 0.009469$. Finally, $y=0$ when $t=\dfrac{448}{15k}\approx 3154$ sec $=53$ min 34 sec. Thus the tank is empty at 1:53:34 p.m.

(b) The radius of the bottom hole is $r=\sqrt{\dfrac{k}{4.8}}\approx 0.04442$ ft ≈ 0.53 in, thus about half an inch.

64. The given rate of fall of the water level is $\dfrac{dy}{dt}=-4$ in/hr $=-\dfrac{1}{10800}$ ft/sec. With $A(y)=\pi x^2$ (where $y=f(x)$) and $a=\pi r^2$, Equation (30) becomes

$\dfrac{\pi x^2}{10800}=\pi r^2\sqrt{2gy}=8\pi r^2\sqrt{y}$, or $\sqrt{y}=\dfrac{x^2}{10800\cdot 8r^2}$. Hence the curve is of the form $y=kx^4$, and the diagram shows that $y=4$ when $x=1$, which means that $k=4$. Finally, rewriting $y=4x^4$ as $\sqrt{y}=2x^2$ shows that $\dfrac{1}{10800\cdot 8r^2}=2$, and so the radius r of the bottom hole is given by $r=\dfrac{1}{4\sqrt{10800}}=\dfrac{1}{240\sqrt{3}}$ ft $=\dfrac{1}{20\sqrt{3}}$ in $\approx$ 0.02888 in, that is, about $1/35$ in.

65. The temperature $T(t)$ of the body satisfies the differential equation $\dfrac{dT}{dt}=k(70-T)$. Separating variables gives $\int\dfrac{1}{70-T}\,dT=\int k\,dt$, or (since $T(t)>70$ for all t) $\ln(T-70)=-kt+C$. If we take $t=0$ at the (unknown) time of death, then applying the initial condition $T(0)=98.6$ gives $C=\ln 28.6$, and so $T(t)=70+28.6e^{-kt}$. Now suppose that 12 noon corresponds to $t=a$. This gives the two equations

$$\begin{aligned} T(a) &= 70+28.6e^{-ka}=80 \\ T(a+1) &= 70+28.6e^{-k(a+1)}=75 \end{aligned},$$

which simplify to

$$\begin{aligned} 28.6e^{-ka} &= 10 \\ 28.6e^{-ka}e^{-k} &= 5 \end{aligned}.$$

These latter equations imply that $e^{-k} = 5/10 = 1/2$, so that $k = \ln 2$. Finally, we can substitute this value of k into the first of the previous two equations to find that $a = \dfrac{\ln 2.86}{\ln 2} \approx 1.516 \text{ hr} \approx 1 \text{ hr } 31 \text{ min}$, so the death occurred at 10:29 a.m.

66. **(a)** Let $t = 0$ when it began to snow, and let $t = t_0$ at 7:00 a.m. Also let $x = 0$ where the snowplow begins at 7:00 a.m. If the constant rate of snowfall is given by c, then the snow depth at time t is given by $y = ct$. If $v = dx/dt$ denotes the plow's velocity (and if we assume that the road is of constant width), then "clearing snow at a constant rate" means that the product yv is constant. Hence the snowplow must satisfy the differential equation

$$k\frac{dx}{dt} = \frac{1}{t},$$

where k is a constant.

(b) Separating variables gives $\int k\,dx = \int \frac{1}{t}\,dt$, or $kx = \ln t + C$, and then solving for t gives $t = Ce^{kx}$. The initial condition $x(t_0) = 0$ gives $C = t_0$. We are further given that $x = 2$ when $t = t_0 + 1$ and $x = 4$ when $t = t_0 + 3$, which lead to the equations

$$\begin{aligned} t_0 + 1 &= t_0 e^{2k} \\ t_0 + 3 &= t_0 e^{4k} \end{aligned}.$$

Solving each these for t_0 shows that $t_0 = \dfrac{1}{e^{2k} - 1} = \dfrac{3}{e^{4k} - 1}$, and so $e^{4k} - 1 = 3\left(e^{2k} - 1\right)$, or $e^{4k} - 3e^{2k} + 2 = 0$, or $\left(e^{2k} - 1\right)\left(e^{2k} - 2\right) = 0$, or $e^{2k} = 2$, since $k > 0$. Hence the first of the two equations above gives $t_0 + 1 = 2t_0$, so $t_0 = 1$. Thus it began to snow at 6 a.m.

67. We still have $t = t_0 e^{kx}$, but now the given information yields the conditions

$$\begin{aligned} t_0 + 1 &= t_0 e^{4k} \\ t_0 + 2 &= t_0 e^{7k} \end{aligned}$$

at 8 a.m. and 9 a.m., respectively. Elimination of t_0 gives the equation $2e^{4k} - e^{7k} - 1 = 0$, which cannot be easily factored, unlike the corresponding equation in Problem 66. Letting $u = e^k$ gives $2u^4 - u^7 - 1 = 0$, and solving this equation using MATLAB or other technology leads to three real and four complex roots. Of the three real roots, only $u \approx 1.086286$ satisfies $u > 1$, and thus represents the desired solution. This means that $k \approx \ln 1.086286 \approx 0.08276$. Using this value, we finally solve either of the preceding pair of equations for $t_0 \approx 2.5483 \text{ hr} \approx 2 \text{ hr } 33 \text{ min}$. Thus it began to snow at 4:27 a.m.

68. **(a)** Note first that if β denotes the angle between the tangent line and the horizontal, then $\alpha = \frac{\pi}{2} - \beta$, so $\cot\alpha = \cot\left(\frac{\pi}{2} - \beta\right) = \tan\beta = y'(x)$. It follows that

$$\sin\alpha = \frac{\sin\alpha}{\sqrt{\sin^2\alpha + \cos^2\alpha}} = \frac{1}{\sqrt{1+\cot^2\alpha}} = \frac{1}{\sqrt{1+y'(x)^2}}.$$

Therefore the mechanical condition that $\frac{\sin\alpha}{v}$ be a (positive) constant with $v = \sqrt{2gy}$ implies that $\frac{1}{\sqrt{2gy}\sqrt{1+(y')^2}}$ is constant, so that $y\left[1+(y')^2\right] = 2a$ for some positive constant *a*. Noting that $y' > 0$ because the bead is falling (and hence moving in the direction of increasing *x*), we readily solve the latter equation for the desired differential equation $y' = \frac{dy}{dx} = \sqrt{\frac{2a-y}{y}}$.

(b) The substitution $y = 2a\sin^2 t$, $dy = 4a\sin t\cos t\,dt$ now gives

$$4a\sin t\cos t\,dt = \sqrt{\frac{2a-2a\sin^2 t}{2a\sin^2 t}}\,dx = \frac{\cos t}{\sin t}dx, \quad dx = 4a\sin^2 t\,dt$$

Integration now gives

$$x = \int 4a\sin^2 t\,dt = 2a\int 1-\cos 2t\,dt = 2a\left(t - \frac{1}{2}\sin 2t\right) + C = a(2t - \sin 2t) + C,$$

and we recall that $y = 2a\sin^2 t = a(1-\cos 2t)$. The requirement that $x = 0$ when $t = 0$ implies that $C = 0$. Finally, the substitution $\theta = 2t$ yields the desired parametric equations

$$x = a(\theta - \sin\theta), \quad y = a(1-\cos\theta).$$

of the cycloid that is generated by a point on the rim of a circular wheel of radius *a* as it rolls along the *x*-axis. [See Example 5 in Section 9.4 of Edwards and Penney, *Calculus: Early Transcendentals*, 7th edition (Upper Saddle River, NJ: Pearson, 2008).]

69. Substitution of $v = dy/dx$ in the differential equation for $y = y(x)$ gives $a\frac{dv}{dx} = \sqrt{1+v^2}$, and separation of variables then yields $\int \frac{1}{\sqrt{1+v^2}}dv = \int \frac{1}{a}dx$, or $\sinh^{-1} v = \frac{x}{a} + C_1$, or $\frac{dy}{dx} = \sinh\left(\frac{x}{a} + C_1\right)$. The fact that $y'(0) = 0$ implies that $C_1 = 0$, so it follows that $\frac{dy}{dx} = \sinh\left(\frac{x}{a}\right)$, or $y(x) = a\cosh\left(\frac{x}{a}\right) + C$. Of course the (vertical) position of the *x*-axis can be adjusted so that $C = 0$, and the units in which *T* and ρ are measured may be adjusted so that $a = 1$. In essence, then, the shape of the hanging cable is the hyperbolic cosine graph $y = \cosh x$.

SECTION 1.5

LINEAR FIRST-ORDER EQUATIONS

1. An integrating factor is given by $\rho = \exp\left(\int 1\,dx\right) = e^x$, and multiplying the differential equation by ρ gives $e^x y' + e^x y = 2e^x$, or $D_x\left(e^x \cdot y\right) = 2e^x$. Integrating then leads to $e^x \cdot y = \int 2e^x\,dx = 2e^x + C$, and thus to the general solution $y = 2 + Ce^{-x}$. Finally, the initial condition $y(0) = 0$ implies that $C = -2$, so the corresponding particular solution is $y(x) = 2 - 2e^{-x}$.

2. An integrating factor is given by $\rho = \exp\left(\int -2\,dx\right) = e^{-2x}$, and multiplying the differential equation by gives $e^{-2x}y' - 2e^{-2x}y = 3$, or $D_x\left(e^{-2x} \cdot y\right) = 3$. Integrating then leads to $e^{-2x} \cdot y = 3x + C$, and thus to the general solution $y = 3xe^{2x} + Ce^{2x}$. Finally, the initial condition $y(0) = 0$ implies that .., so the corresponding particular solution is $y(x) = 3xe^{2x}$.

3. An integrating factor is given by $\rho = \exp\left(\int 3\,dx\right) = e^{3x}$, and multiplying the differential equation by ρ gives $D_x\left(y \cdot e^{3x}\right) = 2x$. Integrating then leads to $y \cdot e^{3x} = x^2 + C$, and thus to the general solution $y(x) = \left(x^2 + C\right)e^{-3x}$.

4. An integrating factor is given by $\rho = \exp\left(\int -2x\,dx\right) = e^{-x^2}$, and multiplying the differential equation by ρ gives $D_x\left(y \cdot e^{-x^2}\right) = 1$. Integrating then leads to $y \cdot e^{-x^2} = x + C$, and thus to the general solution $y(x) = (x + C)e^{x^2}$.

5. We first rewrite the differential equation for $x > 0$ as $y' + \dfrac{2}{x}y = 3$. An integrating factor is given by $\rho = \exp\left(\int \dfrac{2}{x}dx\right) = e^{2\ln x} = x^2$, and multiplying the equation by ρ gives $x^2 \cdot y' + 2xy = 3$, or $D_x\left(y \cdot x^2\right) = 3x^2$. Integrating then leads to $y \cdot x^2 = x^3 + C$, and thus to the general solution $y(x) = x + \frac{C}{x^2}$. Finally, the initial condition $y(1) = 5$ implies that $C = 4$, so the corresponding particular solution is $y(x) = x + \frac{4}{x^2}$.

6. We first rewrite the differential equation for $x > 0$ as $y' + \frac{5}{x}y = 7x$. An integrating factor is given by $\rho = \exp\left(\int \frac{5}{x}dx\right) = e^{5\ln x} = x^5$, and multiplying the equation by ρ gives $x^5 \cdot y' + 5x^4 y = 7x^6$, or $D_x\left(y \cdot x^5\right) = 7x^6$. Integrating then leads to $y \cdot x^5 = x^7 + C$, and thus to the general solution $y(x) = x^2 + \frac{C}{x^5}$. Finally, the initial condition $y(2) = 5$ implies that $C = 32$, so the corresponding particular solution is $y(x) = x^2 + \frac{32}{x^5}$.

7. We first rewrite the differential equation for $x > 0$ as $y' + \frac{1}{2x}y = \frac{5}{\sqrt{x}}$. An integrating factor is given by $\rho = \exp\left(\int \frac{1}{2x}dx\right) = e^{(\ln x)/2} = \sqrt{x}$, and multiplying the equation by ρ gives $\sqrt{x} \cdot y' + \frac{1}{2\sqrt{x}}y = 5$, or $D_x\left(y \cdot \sqrt{x}\right) = 5$. Integrating then leads to $y \cdot \sqrt{x} = 5x + C$, and thus to the general solution $y(x) = 5\sqrt{x} + \frac{C}{\sqrt{x}}$.

8. We first rewrite the differential equation for $x > 0$ as $y' + \frac{1}{3x}y = 4$. An integrating factor is given by $\rho = \exp\left(\int \frac{1}{3x}dx\right) = e^{(\ln x)/3} = \sqrt[3]{x}$, and multiplying the equation by ρ gives $\sqrt[3]{x} \cdot y' + \frac{1}{3}x^{-2/3}y = 4\sqrt[3]{x}$, or $D_x\left(y \cdot \sqrt[3]{x}\right) = 4\sqrt[3]{x}$. Integrating then leads to $y \cdot \sqrt[3]{x} = 3x^{4/3} + C$, and thus to the general solution $y(x) = 3x + Cx^{-1/3}$.

9. We first rewrite the differential equation for $x > 0$ as $y' - \frac{1}{x}y = 1$. An integrating factor is given by $\rho = \exp\left(\int -\frac{1}{x}dx\right) = \frac{1}{x}$, and multiplying the equation by ρ gives $\frac{1}{x}y' - \frac{1}{x^2}y = \frac{1}{x}$, or $D_x\left(y \cdot \frac{1}{x}\right) = \frac{1}{x}$. Integrating then leads to $y \cdot \frac{1}{x} = \ln x + C$, and thus to the general solution $y(x)=x\ln x+Cx$. Finally, the initial condition $y(1)=7$ implies that $C=7$, so the corresponding particular solution is $y(x)=x\ln x+7x$.

10. We first rewrite the differential equation for $x>0$ as $y'-\frac{3}{2x}y=\frac{9}{2}x^2$. An integrating factor is given by $\rho=\exp\left(\int\frac{3}{2x}dx\right)=e^{-3(\ln x)/2}=x^{-3/2}$, and multiplying by ρ gives $x^{-3/2}\cdot y'-\frac{3}{2}x^{-5/2}y=\frac{9}{2}x^{1/2}$, or $D_x\left(y\cdot x^{-3/2}\right)=\frac{9}{2}x^{1/2}$. Integrating then leads to $y\cdot x^{-3/2}=3x^{3/2}+C$, and thus to the general solution $y(x)=3x^3+Cx^{3/2}$.

11. We first collect terms and rewrite the differential equation for $x>0$ as $y'+\left(\frac{1}{x}-3\right)y=0$. An integrating factor is given by

$$\rho=\exp\left[\int\left(\frac{1}{x}-3\right)dx\right]=e^{\ln x-3x}=xe^{-3x},$$

and multiplying by ρ gives $xe^{-3x}\cdot y'+\left(e^{-3x}-3xe^{-3x}\right)y=0$, or $D_x\left(y\cdot xe^{-3x}\right)=0$. Integrating then leads to $y\cdot xe^{-3x}=C$, and thus to the genral solution $y(x)=Cx^{-1}e^{3x}$. Finally, the initial condition $y(1)=0$ implies that $C=0$, so the corresponding particular solution is $y(x)\equiv 0$, that is, the solution is the zero function.

12. We first rewrite the differential equation for $x>0$ as $y'+\frac{3}{x}y=2x^4$. An integrating factor is given by $\rho=\exp\left(\int\frac{3}{x}dx\right)=e^{3\ln x}=x^3$, and multiplying by ρ gives $x^3\cdot y'+3x^2y=2x^7$, or $D_x\left(y\cdot x^3\right)=2x^7$. Integrating then leads to $y\cdot x^3=\frac{1}{4}x^8+C$, and thus to the general solution $y(x)=\frac{1}{4}x^5+Cx^{-3}$. Finally, the initial condition $y(2)=1$ implies that $C=-56$, so the corresponding particular solution is $y(x)=\frac{1}{4}x^5-56x^{-3}$.

13. An integrating factor is given by $\rho=\exp\left(\int 1\,dx\right)=e^x$, and multiplying by ρ gives $e^x\cdot y'+e^xy=e^{2x}$, or $D_x\left(y\cdot e^x\right)=e^{2x}$. Integrating then leads to $y\cdot e^x=\frac{1}{2}e^{2x}+C$, and thus to the general solution $y(x)=\frac{1}{2}e^x+Ce^{-x}$. Finally, the initial condition $y(0)=1$

implies that $C=\frac{1}{2}$, so the corresponding particular solution is $y(x)=\frac{1}{2}e^{x}+\frac{1}{2}e^{-x}$, that is, $y=\cosh x$.

14. We first rewrite the differential equation for $x>0$ as $y'-\frac{3}{x}y=x^2$. An integrating factor is given by $\rho=\exp\left(\int -\frac{3}{x}dx\right)=x^{-3}$, and multiplying by ρ gives $x^{-3}\cdot y'-\frac{3}{x^4}y=x^{-1}$, or $D_x\left(y\cdot x^{-3}\right)=x^{-1}$. Integrating then leads to $y\cdot x^{-3}=\ln x+C$, and thus to the general solution $y(x)=x^3\ln x+Cx^3$. Finally, the initial condition $y(1)=10$ implies that $C=10$, so the corresponding particular solution is $y(x)=x^3\ln x+10x^3$.

15. An integrating factor is given by $\rho=\exp\left(\int 2x\,dx\right)=e^{x^2}$, and multiplying by ρ gives $e^{x^2}\cdot y'+2xe^{x^2}y=xe^{x^2}$, or $D_x\left(y\cdot e^{x^2}\right)=xe^{x^2}$. Integrating then leads to $y\cdot e^{x^2}=\frac{1}{2}e^{x^2}+C$, and thus to the general solution $y(x)=\frac{1}{2}+Ce^{-x^2}$. Finally, the initial condition $y(0)=-2$ implies that $C=-\frac{5}{2}$, so the corresponding particular solution is $y(x)=\frac{1}{2}-\frac{5}{2}e^{-x^2}$.

16. We first rewrite the differential equation as $y'+(\cos x)y=\cos x$. An integrating factor is given by $\rho=\exp\left(\int \cos x\,dx\right)=e^{\sin x}$, and multiplying by ρ gives $e^{\sin x}\cdot y'+e^{\sin x}(\cos x)y=e^{\sin x}\cos x$, or $D_x\left(y\cdot e^{\sin x}\right)=e^{\sin x}\cos x$. Integrating then leads to $y\cdot e^{\sin x}=e^{\sin x}+C$, and thus to the general solution $y(x)=1+Ce^{-\sin x}$. Finally, the initial condition $y(\pi)=2$ implies that $C=1$, so the corresponding particular solution is $y(x)=1+e^{-\sin x}$.

17. We first rewrite the differential equation for $x>-1$ as $y'+\frac{1}{1+x}y=\frac{\cos x}{1+x}$. An integrating factor is given by $\rho=\exp\left(\int \frac{1}{1+x}dx\right)=1+x$, and multiplying by ρ gives $(1+x)y'+y=\cos x$ (which happens to be the original differential equation), or $D_x\left[y\cdot(1+x)\right]=\cos x$. Integrating then leads to $y\cdot(1+x)=\sin x+C$, and thus to the general solution $y(x)=\dfrac{\sin x+C}{1+x}$. Finally, the initial condition $y(0)=1$ implies that $C=1$, so the corresponding particular solution is $y(x)=\dfrac{1+\sin x}{1+x}$.

18. We first rewrite the differential equation for $x>0$ as $y'-\dfrac{2}{x}y=x^2\cos x$. An integrating factor is given by $\rho=\exp\left(\int -\dfrac{2}{x}dx\right)=x^{-2}$, and multiplying by ρ gives $x^{-2}\cdot y'-\dfrac{2}{x^3}y=\cos x$, or $D_x\left(y\cdot x^{-2}\right)=\cos x$. Integrating then leads to $y\cdot x^{-2}=\sin x+C$, and thus to the general solution $y(x)=x^2(\sin x+C)$.

19. For $x>0$ an integrating factor is given by $\rho=\exp\left(\int \cot x\,dx\right)=e^{\ln(\sin x)}=\sin x$, and multiplying by ρ gives $(\sin x)\cdot y'+(\cos x)y=\sin x\cos x$, or $D_x(y\cdot\sin x)=\sin x\cos x$. Integrating then leads to $y\cdot\sin x=\dfrac{1}{2}\sin^2 x+C$, and thus to the general solution $y(x)=\dfrac{1}{2}\sin x+C\csc x$.

20. We first rewrite the differential equation as $y'-(1+x)y=1+x$. An integrating factor is given by $\rho=\exp\left(-\int 1+x\,dx\right)=e^{-x-\frac{x^2}{2}}$, and multiplying by ρ gives $e^{-x-\frac{x^2}{2}}\cdot y'-(1+x)e^{-x-\frac{x^2}{2}}y=(1+x)e^{-x-\frac{x^2}{2}}$, or $D_x\left(y\cdot e^{-x-\frac{x^2}{2}}\right)=(1+x)e^{-x-\frac{x^2}{2}}$. Integrating then leads to $y\cdot e^{-x-\frac{x^2}{2}}=-e^{-x-\frac{x^2}{2}}+C$, and thus to the general solution $y(x)=-1+Ce^{x+\frac{x^2}{2}}$. Finally, the initial condition $y(0)=0$ implies that $C=1$, so the corresponding particular solution is $y(x)=-1+e^{x+\frac{x^2}{2}}$.

21. We first rewrite the differential equation for $x>0$ as $y'-\dfrac{3}{x}y=x^3\cos x$. An integrating factor is given by $\rho=\exp\left(\int -\dfrac{3}{x}dx\right)=e^{-3\ln x}=x^{-3}$, and multiplying by ρ gives $x^{-3}\cdot y'-3x^{-4}y=\cos x$, or $D_x\left(y\cdot x^{-3}\right)=\cos x$. Integrating then leads to

$y \cdot x^{-3} = \sin x + C$, and thus to the general solution $y(x) = x^3 \sin x + Cx^3$. Finally, the initial condition $y(2\pi) = 0$, so the corresponding particular solution is $y(x) = x^3 \sin x$.

22. We first rewrite the differential equation as $y' - 2xy = 3x^2 e^{x^2}$. An integrating factor is given by $\rho = \exp\left(\int -2x\,dx\right) = e^{-x^2}$, and multiplying by ρ gives $e^{-x^2} \cdot y' - 2xe^{-x^2} y = 3x^2$, or $D_x\left(y \cdot e^{-x^2}\right) = 3x^2$. Integrating then leads to $y \cdot e^{-x^2} = x^3 + C$, and thus to the general solution $y(x) = \left(x^3 + C\right)e^{+x^2}$. Finally, the initial condition $y(0) = 5$ implies that $C = 5$, so the corresponding particular solution is $y(x) = \left(x^3 + 5\right)e^{+x^2}$.

23. We first rewrite the differential equation for $x > 0$ as $y' + \left(2 - \frac{3}{x}\right)y = 4x^3$. An integrating factor is given by $\rho = \exp\left(\int 2 - \frac{3}{x}dx\right) = \exp(2x - 3\ln x) = x^{-3}e^{2x}$, and multiplying by ρ gives $x^{-3}e^{2x} \cdot y' + \left(2x^{-3} - 3x^{-4}\right)e^{2x} y = 4e^{2x}$, or $D_x\left(y \cdot x^{-3}e^{2x}\right) = 4e^{2x}$. Integrating then leads to $y \cdot x^{-3}e^{2x} = 2e^{2x} + C$, and thus to the general solution $y(x) = 2x^3 + Cx^3e^{-2x}$.

24. We first rewrite the differential equation as $y' + \frac{3x}{x^2+4}y = \frac{x}{x^2+4}$. An integrating factor is given by $\rho = \exp\left(\int \frac{3x}{x^2+4}dx\right) = \exp\left[\frac{3}{2}\ln\left(x^2+4\right)\right] = \left(x^2+4\right)^{3/2}$, and multiplying by ρ gives $\left(x^2+4\right)^{3/2} \cdot y' + 3x\left(x^2+4\right)^{1/2} y = x\left(x^2+4\right)^{1/2}$, or
$D_x\left[y \cdot \left(x^2+4\right)^{3/2}\right] = x\left(x^2+4\right)^{1/2}$. Integrating then leads to
$y \cdot \left(x^2+4\right)^{3/2} = \frac{1}{3}\left(x^2+4\right)^{3/2} + C$, and thus to the general solution
$y(x) = \frac{1}{3} + C\left(x^2+4\right)^{-3/2}$. Finally, the initial condition $y(0) = 1$ implies that $C = \frac{16}{3}$, so the corresponding particular solution is $y(x) = \frac{1}{3}\left[1 + 16\left(x^2+4\right)^{-3/2}\right]$.

25. We first rewrite the differential equation as $y' + \frac{3x^3}{x^2+1}y = \frac{6x}{x^2+1}e^{-\frac{3}{2}x^2}$. An integrating factor is given by $\rho = \exp\left(\int \frac{3x^3}{x^2+1}dx\right)$. Long division of polynomials shows that
$\frac{3x^3}{x^2+1} = 3x - \frac{3x}{x^2+1}$, and so

$$\rho = \exp\left(\int 3x - \frac{3x}{x^2+1}\,dx\right) = \exp\left[\frac{3}{2}x^2 - \frac{3}{2}\ln\left(x^2+1\right)\right] = \left(x^2+1\right)^{-3/2} e^{\frac{3}{2}x^2}.$$

Multiplying by ρ gives

$$\left(x^2+1\right)^{-3/2} e^{\frac{3}{2}x^2} \cdot y' + 3x^3\left(x^2+1\right)^{-5/2} e^{\frac{3}{2}x^2} y = 6x\left(x^2+1\right)^{-5/2},$$

or (as can be verified using the product rule twice, together with some algebra) $D_x\left[y\cdot\left(x^2+1\right)^{-3/2} e^{\frac{3}{2}x^2}\right] = 6x\left(x^2+1\right)^{-5/2}$. Integrating then leads to

$$y\cdot\left(x^2+1\right)^{-3/2} e^{\frac{3}{2}x^2} = \int 6x\left(x^2+1\right)^{-5/2} dx = -2\left(x^2+1\right)^{-3/2} + C,$$

and thus to the general solution $y = \left[-2 + C\left(x^2+1\right)^{3/2}\right] e^{-\frac{3}{2}x^2}$. Finally, the initial condition $y(0) = 1$ implies that $C = 3$, so the corresponding particular solution is $y = \left[-2 + 3\left(x^2+1\right)^{3/2}\right] e^{-\frac{3}{2}x^2}$.

The strategy in each of Problems 26-28 is to use the inverse function theorem to conclude that at points (x, y) where $\frac{dy}{dx} \neq 0$, *x* is locally a function of *y* with $\frac{dx}{dy}\cdot\frac{dy}{dx} = 1$. Thus the given differential equation is equivalent to one in which *x* is the dependent variable and *y* as the independent variable, and this latter equation may be easier to solve than the one originally given. It may not be feasible, however, to solve the resulting solution for the original dependent variable *y*.

26. At points (x, y) with $1 - 4xy^2 \neq 0$ and $y \neq 0$, rewriting the differential equation as $\frac{dy}{dx} = \frac{y^3}{1 - 4xy^2}$ shows that $\frac{dx}{dy} = \frac{1 - 4xy^2}{y^3}$, or (putting x' for $\frac{dx}{dy}$) $x' + \frac{4}{y}x = \frac{1}{y^3}$, a linear equation for the dependent variable *x* as a function of the independent variable *y*. For $y > 0$, an integrating factor is given by $\rho = \exp\left(\int \frac{4}{y}dy\right) = y^4$, and multiplying by ρ gives $y^4 \cdot x' + 4y^3 x = y$, or $D_y\left(x\cdot y^4\right) = y$. Integrating then leads to $x\cdot y^4 = \frac{1}{2}y^2 + C$, and thus to the general (implicit) solution $x(y) = \frac{1}{2y^2} + \frac{C}{y^4}$.

27. At points (x, y) with $x + ye^y \neq 0$, rewriting the differential equation as $\frac{dy}{dx} = \frac{1}{x + ye^y}$ shows that $\frac{dx}{dy} = x + ye^y$, or (putting x' for $\frac{dx}{dy}$) $x' - x = ye^y$, a linear equation for the dependent variable *x* as a function of the independent variable *y*. An integrating factor is given by $\rho = \exp\left(\int -1\,dy\right) = e^{-y}$, and multiplying by ρ gives $e^{-y} \cdot x' - e^{-y}x = y$, or $D_y\left(x \cdot e^{-y}\right) = y$. Integrating then leads to $x \cdot e^{-y} = \frac{1}{2}y^2 + C$, and thus to the general (implicit) solution $x(y) = \left(\frac{1}{2}y^2 + C\right)e^{y}$.

28. At points (x, y) with $1 + 2xy \neq 0$, rewriting the differential equation as $\frac{dy}{dx} = \frac{1+y^2}{1+2xy}$ shows that $\frac{dx}{dy} = \frac{1+2xy}{1+y^2}$, or (putting x' for $\frac{dx}{dy}$) $x' - \frac{2y}{1+y^2}x = \frac{1}{1+y^2}$, a linear equation for the dependent variable *x* as a function of the independent variable *y*. An integrating factor is given by $\rho = \exp\left(\int -\frac{2y}{1+y^2}\,dy\right) = \exp\left[-\ln\left(1+y^2\right)\right] = \frac{1}{1+y^2}$, and multiplying by ρ gives $\frac{1}{1+y^2} \cdot x' - \frac{2y}{\left(1+y^2\right)^2}x = \frac{1}{\left(1+y^2\right)^2}$, or $D_y\left(\frac{x}{1+y^2}\right) = \frac{1}{\left(1+y^2\right)^2}$. Integrating, by means of either the initial substitution $y = \tan\theta$ or the use of an integral table, then leads to

$$\frac{x}{1+y^2} = \int \frac{1}{\left(1+y^2\right)^2}\,dy = \frac{1}{2}\left(\frac{y}{1+y^2} + \tan^{-1} y + C\right),$$

and thus to the (implicit) general solution $x(y) = \frac{1}{2}\left[y + \left(1+y^2\right)\left(\tan^{-1} y + C\right)\right]$.

29. We first rewrite the differential equation as $y' - 2xy = 1$. An integrating factor is given by $\rho = \exp\left(\int -2x\,dx\right) = e^{-x^2}$, and multiplying by ρ gives $e^{-x^2} \cdot y' - 2xe^{-x^2}y = e^{-x^2}$, or $D_x\left(y \cdot e^{-x^2}\right) = e^{-x^2}$. Integrating then leads to $y \cdot e^{-x^2} = \int e^{-x^2}\,dx$. Any antiderivative of e^{-x^2} differs by a constant (call it *C*) from the definite integral $\int_0^x e^{-t^2}\,dt$, and so we can write $y \cdot e^{-x^2} = \int_0^x e^{-t^2}\,dt + C$. The definition of $\operatorname{erf}(x)$ then gives $y \cdot e^{-x^2} = \frac{\sqrt{\pi}}{2}\operatorname{erf}(x) + C$, and thus the general solution $y(x) = e^{x^2}\left[\frac{\sqrt{\pi}}{2}\operatorname{erf}(x) + C\right]$.

30. We first rewrite the differential equation for $x > 0$ as $y' - \frac{1}{2x}y = \cos x$. An integrating factor is given by $\rho = \exp\left(\int -\frac{1}{2x}\,dx\right) = x^{-1/2}$, and multiplying by ρ gives

$x^{-1/2}\cdot y' - \frac{1}{2}x^{-3/2}y = x^{-1/2}\cos x$, or $D_x\left(x^{-1/2}\cdot y\right) = x^{-1/2}\cos x$. Integrating then leads to $x^{-1/2}\cdot y = \int x^{-1/2}\cos x\,dx$. Any antiderivative of $x^{-1/2}\cos x$ differs by a constant (call it *C*) from the definite integral $\int_1^x t^{-1/2}\cos t\,dt$, and so we can write $x^{-1/2}\cdot y = \int_1^x t^{-1/2}\cos t\,dt + C$, which gives the general solution $y(x) = x^{1/2}\left[\int_1^x t^{-1/2}\cos t\,dt + C\right]$. Finally, the initial condition $y(1) = 0$ implies that $C = 0$, and so the desired particular solution is given by $y(x) = x^{1/2}\int_1^x t^{-1/2}\cos t\,dt$.

31. **(a)** The fundamental theorem of calculus implies, for any value of *C,* that

$$y_c'(x) = Ce^{-\int P(x)dx}\left[-P(x)\right] = -P(x)y_c(x),$$

and thus that $y_c'(x) + P(x)y_c(x) = 0$. Therefore y_c is a general solution of $\frac{dy}{dx} + P(x)y = 0$.

(b) The product rule and the fundamental theorem of calculus imply that

$$\begin{aligned} y_p'(x) &= e^{-\int P(x)dx}\cdot Q(x)e^{\int P(x)dx} + e^{-\int P(x)dx}\left[-P(x)\right]\cdot\int\left(Q(x)e^{\int P(x)dx}\right)dx \\ &= Q(x) - P(x)e^{-\int P(x)dx}\int\left(Q(x)e^{\int P(x)dx}\right)dx \\ &= Q(x) - P(x)y_p(x), \end{aligned}$$

and thus that $y_p'(x) + P(x)y_p(x) = Q(x)$. Therefore y_p is a particular solution of $\frac{dy}{dx} + P(x)y = Q(x)$.

(c) The stated assumptions imply that

$$\begin{aligned} y'(x) + P(x)y &= y_c'(x) + y_p'(x) + P(x)\left[y_c(x) + y_p(x)\right] \\ &= \left[y_c'(x) + P(x)y_c(x)\right] + \left[y_p'(x) + P(x)y_p(x)\right] \\ &= 0 + Q(x) \\ &= Q(x), \end{aligned}$$

proving that $y(x)$ is a general solution of $\frac{dy}{dx} + P(x)y = Q(x)$.

32. **(a)** Substituting $y_p(x)$ into the given differential equation gives

$$(A\cos x - B\sin x) + (A\sin x + B\cos x) = 2\sin x,$$

that is

$$(A-B)\sin x+(A+B)\cos x=2\sin x,$$

for all x. It follows that $A-B=2$ and $A+B=0$, and solving this system gives $A=1$ and $B=-1$. Thus $y_p(x)=\sin x-\cos x$.

(b) The result of Problem 31(a), applied with $P(x)\equiv 1$, implies that $y_c(x)=Ce^{-\int 1dx}=Ce^{-x}$ is a general solution of $\dfrac{dy}{dx}+y=0$. Part (b) of this problem implies that $y_p(x)=\sin x-\cos x$ is a particular solution of $\dfrac{dy}{dx}+y=2\sin x$. It follows from Problem 31(c), then, that a general solution of $\dfrac{dy}{dx}+y=2\sin x$ is given by $y(x)=y_c(x)+y_p(x)=Ce^{-x}+\sin x-\cos x$.

(c) The initial condition $y(0)=1$ implies that $1=C-1$, that is, $C=2$; thus the desired particular solution is $y(x)=2e^{-x}+\sin x-\cos x$.

33. Let $x(t)$ denote the amount of salt (in kg) in the tank after t seconds. We want to know when $x(t)=10$. In the notation of Equation (18) of the text, the differential equation for $x(t)$ is

$$\frac{dx}{dt}=r_ic_i-\frac{r_o}{V}x=(5\,\mathrm{L/s})(0\,\mathrm{kg/L})-\frac{5\,\mathrm{L/s}}{1000\,\mathrm{L}}\cdot x\,\mathrm{kg},$$

or $\dfrac{dx}{dt}=-\dfrac{x}{200}$. Separating variables gives the general solution $x(t)=Ce^{-t/200}$, and the initial condition $x(0)=100$ implies that $C=100$, and so $x(t)=100e^{-t/200}$. Setting $x(t)=10$ gives $10=100e^{-t/200}$, or $t=200\ln 10\approx 461$ sec, that is, about 7 min 41 sec.

34. Let $x(t)$ denote the amount of pollutants in the reservoir after t days, measured in millions of cubic feet (mft^3). The volume of the reservoir is 8000 mft^3, and the initial amount $x(0)$ of pollutants is $(0.25\%)(8000)=20\,\mathrm{mft}^3$. We want to know when $x(t)=(0.10\%)(8000)=8\,\mathrm{mft}^3$. In the notation of Equation (18) of the text, the differential equation for $x(t)$ is

$$\frac{dx}{dt}=r_ic_i-\frac{r_o}{V}x=\left(500\,\mathrm{mft}^3/\mathrm{day}\right)(0.05\%)-\frac{500\,\mathrm{mft}^3/\mathrm{day}}{8000\,\mathrm{mft}^3}\cdot x\,\mathrm{mft}^3=\frac{1}{4}-\frac{x}{16},$$

or $\frac{dx}{dt}+\frac{1}{16}x=\frac{1}{4}$. An integrating factor is given by $\rho=e^{t/16}$, and multiplying the differential equation by ρ gives $e^{t/16}\cdot\frac{dx}{dt}+\frac{1}{16}e^{t/16}x=\frac{1}{4}e^{t/16}$, or $D_t\left(e^{t/16}\cdot x\right)=\frac{1}{4}e^{t/16}$. Integrating then leads to $e^{t/16}\cdot x=4e^{t/16}+C$, and thus to the general solution $x=4+Ce^{-t/16}$. The initial condition $x(0)=20$ implies that $C=16$, and so $x(t)=4+16e^{-t/16}$. Finally, we find that $x=8$ when $t=16\ln 4\approx 22.2$ days.

35. The only difference from the Example 4 solution in the textbook is that $V=1640\,\text{km}^3$ and $r=410\,\text{km}^3/\text{yr}$ for Lake Ontario, so the time required is

$$t=\frac{V}{r}\ln 4=4\ln 4\approx 5.5452 \text{ years.}$$

36. **(a)** Let $x(t)$ denote the amount of salt (in kg) in the tank after *t* minutes. Because the volume of liquid in the tank is decreasing by 1 gallon each minute, the volume after *t* min is $60-t$ gallons. Thus in the notation of Equation (18) of the text, the differential equation for $x(t)$ is

$$\frac{dx}{dt}=r_ic_i-\frac{r_o}{V}x=(2\,\text{gal/min})(1\,\text{lb/gal})-\frac{3\,\text{gal/min}}{(60-t)\,\text{gal}}x\,\text{lb},$$

or $\frac{dx}{dt}+\frac{3}{60-t}x=2$. An integrating factor is given by $\rho=\exp\left(\int\frac{3}{60-t}dt\right)=(60-t)^{-3}$, and multiplying the differential equation by ρ gives

$$(60-t)^{-3}\frac{dx}{dt}+3(60-t)^{-4}x=2(60-t)^{-3},$$

or $D_t\left[(60-t)^{-3}\cdot x\right]=2(60-t)^{-3}$. Integrating then leads to

$$(60-t)^{-3}\cdot x=\int 2(60-t)^{-3}\,dt=(60-t)^{-2}+C,$$

and thus to the general solution $x(t)=(60-t)+C(60-t)^3$. The initial condition $x(0)=0$ implies that $C=\frac{1}{3600}$, so the desired particular solution is

$$x(t)=(60-t)-\frac{1}{3600}(60-t)^3.$$

(b) By part (a), $x'(t)=-1+\frac{3}{3600}(60-t)^2$, which is zero when $t=60\pm 20\sqrt{3}$. We ignore $t=60+20\sqrt{3}$ because the tank is empty after 60 min. The facts that $x''(t)=\frac{-6}{3600}(60-t)<0$ for $0<t<60$ and that $t=60-20\sqrt{3}$ is the lone critical point of $x(t)$ over this interval imply that $x(t)$ reaches its absolute maximum at $t = 60 - 20\sqrt{3} \approx 25.36$ min $\approx 25\,\text{min}\,22\,\text{s}$. It follows that the maximum amount of salt ever in the tank is

$$x\left(60 - 20\sqrt{3}\right) = 20\sqrt{3} - \frac{1}{3600}\left(20\sqrt{3}\right)^3 = \frac{40\sqrt{3}}{3} \approx 23.09 \text{ lb}.$$

37. Let $x(t)$ denote the amount of salt (in lb) after *t* seconds. Because the volume of liquid in the tank is increasing by 2 gallon each minute, the volume after *t* sec is $100 + 2t$ gallons. Thus in the notation of Equation (18) of the text, the differential equation for $x(t)$ is

$$\frac{dx}{dt} = r_i c_i - \frac{r_o}{V}x = (5\,\text{gal/s})(1\,\text{lb/gal}) - \frac{3\,\text{gal/s}}{(100 + 2t)\,\text{gal}} \cdot x\,\text{lb},$$

or $\frac{dx}{dt} + \frac{3}{100 + 2t}x = 5$. An integrating factor is given by $\rho = \exp\left(\int \frac{3}{100 + 2t}dt\right) = (100 + 2t)^{3/2}$, and multiplying the differential equation by ρ gives

$$(100 + 2t)^{3/2} \cdot \frac{dx}{dt} + 3(100 + 2t)^{1/2}\,x = 5(100 + 2t)^{3/2},$$

or $D_t\left[(100 + 2t)^{3/2} \cdot x\right] = 5(100 + 2t)^{3/2}$. Integrating then leads to

$$(100 + 2t)^{3/2} \cdot x = \int 5(100 + 2t)^{3/2}\,dt = (100 + 2t)^{5/2} + C,$$

and thus to the general solution $x(t) = 100 + 2t + C(100 + 2t)^{-3/2}$. The initial condition $x(0) = 50$ implies that $50 = 100 + C \cdot 100^{-3/2}$, or $C = -50000$, and so the desired particular solution is $x(t) = 100 + 2t - \frac{50000}{(100 + 2t)^{3/2}}$. Finally, because the tank starts out with 300 gallons of excess capacity and the volume of its contents increases at $2\,\text{gal/s}$, the tank is full when $t = \frac{300\,\text{gal}}{2\,\text{gal/s}} = 150\,\text{s}$. At this time the tank contains $x(150) = 400 - \frac{50000}{(400)^{3/2}} = 393.75\,\text{lb}$ of salt.

38. **(a)** In the notation of Equation (16) of the text, the differential equation for $x(t)$ is

$$\frac{dx}{dt} = r_i c_i - r_o c_o = (5\,\text{gal/min})(0\,\text{lb/gal}) - \frac{5\,\text{gal/min}}{100\,\text{gal}} \cdot x\,\text{lb},$$

or $\dfrac{dx}{dt} = -\dfrac{1}{20}x$. Separating variables leads to the general solution $x(t) = Ce^{-t/20}$, and the initial condition $x(0) = 50$ implies that $C = 50$. Thus $x(t) = 50e^{-t/20}$.

(b) In the same way, the differential equation for $y(t)$ is

$$\frac{dy}{dt} = (5\,\text{gal/min})\left(\frac{x}{100}\,\text{lb/gal}\right) - (5\,\text{gal/min})\left(\frac{y}{200}\,\text{lb/gal}\right) = \frac{5x}{100} - \frac{5y}{200},$$

in light of the (constant) volumes of liquid in the two tanks. Substituting the result of part (a) gives $\dfrac{dy}{dt} + \dfrac{1}{40}y = \dfrac{5}{2}e^{-t/20}$. An integrating factor is given by $\rho = \exp\left(\int \dfrac{1}{40}dt\right) = e^{t/40}$, and multiplying the differential equation by ρ gives $e^{t/40}\cdot\dfrac{dy}{dt} + \dfrac{1}{40}e^{t/40}y = \dfrac{5}{2}e^{-t/40}$, or $D_t\left[e^{t/40}\cdot y\right] = \dfrac{5}{2}e^{-t/40}$. Integrating then leads to $e^{t/40}\cdot y = \int \dfrac{5}{2}e^{-t/40}\,dt = -100e^{-t/40} + C$, and thus to the general solution $y(t) = -100e^{-t/20} + Ce^{-t/40}$. The initial condition $y(0) = 50$ implies that $C = 150$, so that $y(t) = -100e^{-t/20} + 150e^{-t/40}$.

(c) By part (b), $y'(t) = 5e^{-t/20} - \dfrac{15}{4}e^{-t/40} = 5e^{-t/20}\left(1 - \dfrac{3}{4}e^{t/40}\right)$, from which we see that $y'(t) = 0$ when $t = 40\ln\dfrac{4}{3}$. Furthermore, $y'(t) > 0$ for $0 < t < 40\ln\dfrac{4}{3}$ and $y'(t) < 0$ for $t > 40\ln\dfrac{4}{3}$, which implies that $y(t)$ reaches its absolute maximum at $t = 40\ln\dfrac{4}{3} \approx 11.51$ min. The maximum amount of salt in tank 2 is therefore

$$y\left(40\ln\frac{4}{3}\right) = -100\left(\frac{3}{4}\right)^2 + 150\cdot\frac{3}{4} = \frac{3}{4}\cdot 75 = 56.25\,\text{lb}.$$

39. **(a)** In the notation of Equation (18) of the text, the differential equation for $x(t)$ is

$$\frac{dx}{dt} = r_i c_i - \frac{r_o}{V}x = (10\,\text{gal/min})(0) - (10\,\text{gal/min})\left(\frac{x}{100}\right),$$

or $\dfrac{dx}{dt} + \dfrac{1}{10}x = 0$. Separating variables leads to the general solution $x(t) = Ce^{-t/10}$, and the initial condition $x(0) = 100$ implies that $C = 100$. Thus $x(t) = 100e^{-t/10}$. In the same way, the differential equation for $y(t)$ is

$$\frac{dy}{dt} = (10\,\text{gal/min})\left(\frac{x}{100}\right) - (10\,\text{gal/min})\left(\frac{y}{100}\right),$$

because the volume of liquid in each tank remains constant at 2 gal. Substituting the result of part (a) gives $\frac{dy}{dt} + \frac{1}{10} y = 10e^{-t/10}$. An integrating factor is given by $\rho = \exp\left(\int \frac{1}{10} dt \right) = e^{t/10}$, and multiplying the differential equation by ρ gives $e^{t/10} \cdot \frac{dy}{dt} + \frac{1}{10} e^{t/10} y = 10$, or $D_t\left(e^{t/10} \cdot y\right) = 10$. Integrating then leads to $e^{t/10} \cdot y = 10t + C$, and thus to the general solution $y(t) = (10t + C)e^{-t/10}$. The initial condition $y(0) = 0$ implies that $C = 0$, so that $y(t) = 10te^{-t/10}$.

(b) By Part (a), $y'(t) = 10\left(-\frac{t}{10} e^{-t/10} + e^{-t/10} \right) = e^{-t/10}(10 - t)$, which is zero for $t = 10$. Furthermore, $y'(t) > 0$ for $0 < t < 10$, and $y'(t) < 0$ for $t > 10$, which implies that $y(t)$ reaches its absolute maximum at $t = 10$ min. The maximum amount of ethanol in tank 2 is therefore $y(10) = 100e^{-1} \approx 36.79$ gal.

40. **(a)** In the notation of Equation (16) of the text, the differential equation for $x_0(t)$ is

$$\frac{dx_0}{dt} = r_i c_i - r_o c_o = (1\,\text{gal/min})(0) - (1\,\text{gal/min})\left(\frac{x_0}{2} \right),$$

or $\frac{dx_0}{dt} = -\frac{x_0}{2}$. Separating variables leads to the general solution $x_0(t) = Ce^{-t/20}$, and the initial condition $x(0) = 1$ implies that $C = 1$. Thus $x_0(t) = e^{-t/20}$.

(b) First, when $n = 0$, the proposed formula predicts that $x_0(t) = e^{-t/2}$, which was verified in part (a). Next, for a fixed positive value of *n* we assume the inductive hypothesis $x_n(t) = \frac{t^n e^{-t/2}}{n!2^n}$ and seek to show that $x_{n+1}(t) = \frac{t^{n+1} e^{-t/2}}{(n+1)!2^{n+1}}$; this will prove by mathematical induction that the proposed formula holds for all $n \geq 0$.

The differential equation for $x_{n+1}(t)$ is

$$\frac{dx_{n+1}}{dt} = (1\,\text{gal/min})\left(\frac{x_n}{2} \right) - (1\,\text{gal/min})\left(\frac{x_{n+1}}{2} \right) = \frac{x_n}{2} - \frac{x_{n+1}}{2},$$

because the volume of liquid in each tank remains constant at 2 gal. Our inductive hypothesis then gives

$$\frac{dx_{n+1}}{dt} = \frac{1}{2} \frac{t^n e^{-t/2}}{n!2^n} - \frac{x_{n+1}}{2} = \frac{t^n e^{-t/2}}{n!2^{n+1}} - \frac{1}{2} x_{n+1},$$

or $\frac{dx_{n+1}}{dt} + \frac{1}{2} x_{n+1} = \frac{t^n e^{-t/2}}{n!2^{n+1}}$. An integrating factor is given by $\rho = \exp\left(\int \frac{1}{2} dt \right) = e^{t/2}$, and multiplying the differential equation by ρ gives

$$e^{t/2}\cdot\frac{dx_{n+1}}{dt}+\frac{1}{2}e^{t/2}x_{n+1}=\frac{t^n e^{-t/2}}{n!2^{n+1}}e^{t/2}=\frac{t^n}{n!2^{n+1}},$$

or $D_t\left(e^{t/2}\cdot x_{n+1}\right)=\dfrac{t^n}{n!2^{n+1}}$. Integrating then leads to

$$e^{t/2}\cdot x_{n+1}=\frac{t^{n+1}}{(n+1)\cdot n!2^{n+1}}+C=\frac{t^{n+1}}{(n+1)!2^{n+1}}+C,$$

and thus to the general solution $x_{n+1}(t)=\left[\dfrac{t^{n+1}}{(n+1)!2^{n+1}}+C\right]e^{t/2}$. The initial condition $x_{n+1}(0)=0$ implies that $C=0$, so that

$$x_{n+1}(t)=\frac{t^{n+1}}{(n+1)!2^{n+1}}e^{t/2}=\frac{t^{n+1}e^{t/2}}{(n+1)!2^{n+1}},$$

as desired.

(c) Part (b) implies that

$$x_n'(t)=\frac{1}{n!2^n}\left(-\frac{1}{2}t^n e^{-t/2}+nt^{n-1}e^{-t/2}\right)=\frac{t^{n-1}e^{-t/2}}{n!2^{n+1}}(2n-t),$$

which is zero for $t=2n$. Further, $x_n'(t)>0$ for $0<t<2n$ and $x_n'(t)<0$ for $t>2n$, which means that $x_n(t)$ achieves its absolute maximum when $t=2n$. It follows that $M_n=x_n(2n)=\dfrac{(2n)^n e^{-n}}{n!2^n}=\dfrac{n^n e^{-n}}{n!}$.

(d) Substituting Sterling's approximation into the result of part (c) gives $M_n\approx\dfrac{n^n e^{-n}}{n^n e^{-n}\sqrt{2\pi n}}=\dfrac{1}{\sqrt{2\pi n}}$.

41. **(a)** Between time *t* and time $t+\Delta t$, the amount $A(t)$ (in thousands of dollars) increases by a deposit of $0.12S(t)\Delta t$ (12% per year of annual salary) as well as interest earnings of $0.06A(t)\Delta t$ (6% per year of current balance). It follows that

$$\Delta A\approx 0.12S(t)\Delta t+0.06A(t)\Delta t,$$

leading to the linear differential equation $\dfrac{dA}{dt}=0.12S+0.06A=3.6e^{t/20}+0.06A$, or $\dfrac{dA}{dt}-0.06A=3.6e^{t/20}$.

(b) An integrating factor is given by $\rho=\exp\left(\int -0.06\,dt\right)=e^{-0.06t}$, and multiplying the differential equation by ρ gives $e^{-0.06t}\cdot\dfrac{dA}{dt}-0.06e^{-0.06t}A=3.6e^{t/20}e^{-0.06t}=3.6e^{-0.01t}$, or

$D_t\left(e^{-0.06t}\cdot A\right)=3.6e^{-0.01t}$. Integrating then leads to $e^{-0.06t}\cdot A=-360e^{-0.01t}+C$, and thus to the general solution $A(t)=-360e^{0.05t}+Ce^{0.06t}$. The initial condition $A(0)=0$ implies that $C=360$, so that $A(t)=360\left(e^{0.06t}-e^{0.05t}\right)$. At age 70 she will have $A(40)\approx 1308.283$ thousand dollars, that is, $1,308,283.

42. Since both *m* and *v* vary with time, Newton's second law and the product rule give $m\dfrac{dv}{dt}+v\dfrac{dm}{dt}=mg$. Now since the hailstone is of uniform density 1, its mass $m(t)$ equals its volume $\dfrac{4\pi}{3}r^3=\dfrac{4\pi}{3}(kt)^3=\dfrac{4\pi k^3}{3}t^3$, which means that $\dfrac{dm}{dt}=4\pi k^3t^2$. Thus the velocity $v(t)$ of the hailstone satisfies the linear differential equation $\dfrac{4\pi k^3}{3}t^3\dfrac{dv}{dt}+4\pi k^3t^2v=\dfrac{4\pi k^3}{3}t^3g$, or $\dfrac{dv}{dt}+\dfrac{3}{t}v=g$. An integrating factor is given by $\rho=\exp\left(\int\dfrac{3}{t}dt\right)=t^3$, and multiplying the differential equation by ρ gives $t^3\cdot\dfrac{dv}{dt}+3t^2v=gt^3$, or $D_t\left(t^3\cdot v\right)=gt^3$. Integrating then leads to $t^3\cdot v=\dfrac{g}{4}t^4+C$, and thus to the general solution $v(t)=\dfrac{g}{4}t+Ct^{-3}$. The initial condition $v(0)=0$ implies that $C=0$, so that $v(t)=\dfrac{g}{4}t$, and therefore $\dfrac{dv}{dt}=\dfrac{g}{4}$.

43. **(a)** First we rewrite the differential equation as $y'+y=x$. An integrating factor is given by $\rho=\exp\left(\int 1dx\right)=e^x$, and multiplying the differential equation by ρ gives $e^x\cdot y'+e^xy=xe^x$, or $D_x\left(e^x\cdot y\right)=xe^x$. Integrating (by parts) then leads to $e^x\cdot y=\int xe^x\,dx=xe^x-e^x+C$, and thus to the general solution $y(x)=x-1+Ce^{-x}$. Then the fact that $\lim_{x\to\infty}e^{-x}=0$ implies that every solution curve approaches the straight line $y=x-1$ as $x\to\infty$.

(b) The initial condition $y(-5)=y_0$ imposed upon the general solution in part (a) implies that $y_0=-5-1+Ce^5$, and thus that $C=e^{-5}\left(y_0+6\right)$. Hence the solution of the initial value problem $y'=x-y$, $y(-5)=y_0$ is $y(x)=x-1+\left(y_0+6\right)e^{-x-5}$. Substituting $x=5$, we therefore solve the equation $4+(y_0+6)e^{-10}=y_1$ with

$$y_1=3.998, 3.999, 4, 4.001, 4.002$$

for the desired initial values

$$y_0=-50.0529, -28.0265, -6.0000, 16.0265, 38.0529,$$

respectively.

44. **(a)** First we rewrite the differential equation as $y' - y = x$. An integrating factor is given by $\rho = \exp\left(\int -1\,dx\right) = e^{-x}$, and multiplying the differential equation by ρ gives $e^{-x} \cdot y' - e^{-x} y = xe^{-x}$, or $D_x\left(e^{-x} \cdot y\right) = xe^{-x}$. Integrating (by parts) then leads to $e^{-x} \cdot y = \int xe^{-x}\,dx = -xe^{-x} - e^{-x} + C$, and thus to the general solution $y(x) = -x - 1 + Ce^{x}$. Then the fact that $\lim_{x \to -\infty} e^{x} = 0$ implies that every solution curve approaches the straight line $y = -x - 1$ as $x \to -\infty$.

(b) The initial condition $y(-5) = y_0$ imposed upon the general solution in part (a) implies that $y_0 = 5 - 1 + Ce^{-5}$, and thus that $C = e^{5}\left(y_0 - 4\right)$. Hence the solution of the initial value problem $y' = x + y$, $y(-5) = y_0$ is $y(x) = -x - 1 + \left(y_0 - 4\right)e^{x+5}$. Substituting $x = 5$, we therefore solve the equation $-6 + \left(y_0 - 4\right)e^{10} = y_1$ with

$$y_1 = -10, -5, 0, 5, 10$$

for the desired initial values

$$y_0 = 3.99982, 4.00005, 4.00027, 4.00050, 4.00073,$$

respectively.

45. The volume of the reservoir (in millions of cubic meters, denoted m-m^3) is 2. In the notation of Equation (18) of the text, the differential equation for $x(t)$ is

$$\frac{dx}{dt} = r_i c_i - \frac{r_o}{V} x = \left(0.2\,\text{m-m}^3/\text{month}\right)\left(10\,\text{L}/\text{m}^3\right) - \left(0.2\,\text{m-m}^3/\text{month}\right)\left(\frac{x}{2}\,\text{L}/\text{m}^3\right),$$

or $\frac{dx}{dt} + \frac{1}{10}x = 2$. An integrating factor is given by $\rho = e^{t/10}$, and multiplying the differential equation by ρ gives $e^{t/10} \cdot \frac{dx}{dt} + \frac{1}{10}e^{t/10}x = 2e^{t/10}$, or $D_t\left(e^{t/10} \cdot x\right) = 2e^{t/10}$. Integrating then leads to $e^{t/10} \cdot x = 20e^{t/10} + C$, and thus to the general solution $x(t) = 20 + Ce^{-t/10}$. The initial condition $x(0) = 0$ implies that $C = -20$, and so $x(t) = 20\left(1 - e^{-t/10}\right)$, which shows that indeed $\lim_{t \to \infty} x(t) = 20$ (million liters). This was to be expected because the reservoir's pollutant concentration should ultimately match that of the incoming water, namely $10\,\text{L}/\text{m}^3$. Finally, since the volume of reservoir remains constant at 2 m-m^3, a pollutant concentration of $5\,\text{L}/\text{m}^3$ is reached when $\frac{x(t)}{2} = 5$, that is, when $10 = 20\left(1 - e^{-t/10}\right)$, or $t = 10\ln 2 \approx 6.93\,\text{months}$.

46. The volume of the reservoir (in millions of cubic meters, denoted m-m^3) is 2. In the notation of Equation (18) of the text, the differential equation for $x(t)$ is

$$\begin{aligned}\frac{dx}{dt} &= r_i c_i - \frac{r_o}{V}x \\ &= \left(0.2\,\text{m-m}^3/\text{month}\right)\left[10(1+\cos t)\,\text{L/m}^3\right] - \left(0.2\,\text{m-m}^3/\text{month}\right)\left(\frac{x}{2}\,\text{L/m}^3\right),\end{aligned}$$

or $\dfrac{dx}{dt} + \dfrac{1}{10}x = 2(1+\cos t)$. An integrating factor is given by $\rho = e^{t/10}$, and multiplying the differential equation by ρ gives $e^{t/10}\cdot\dfrac{dx}{dt} + \dfrac{1}{10}e^{t/10}x = 2e^{t/10}(1+\cos t)$, or $D_t\left(e^{t/10}\cdot x\right) = 2e^{t/10}(1+\cos t)$. Integrating (by parts twice, or using an integral table) then leads to

$$e^{t/10}\cdot x = \int 2e^{t/10} + 2e^{t/10}\cos t\,dt = 20e^{t/10} + \frac{2e^{t/10}}{\left(\frac{1}{10}\right)^2 + 1^2}\left(\frac{1}{10}\cos t + \sin t\right) + C,$$

and thus to the general solution

$$x(t) = 20 + \frac{200}{101}\left(\frac{1}{10}\cos t + \sin t\right) + Ce^{-t/10}.$$

The initial condition $x(0) = 0$ implies that $C = -20 - \dfrac{20}{101} = -20\cdot\dfrac{102}{101}$, and so

$$\begin{aligned}x(t) &= 20 + \frac{200}{101}\left(\frac{1}{10}\cos t + \sin t\right) - 20\cdot\frac{102}{101}e^{-t/10} \\ &= \frac{20}{101}\left(101 - 102e^{-t/10} + \cos t + 10\sin t\right).\end{aligned}$$

This shows that as $t\to\infty$, $x(t)$ is more and more like $20 + \dfrac{200}{101}\left(\dfrac{1}{10}\cos t + \sin t\right)$, and thus oscillates around 20 (million liters). This was to be expected because the reservoir's pollutant concentration should ultimately match that of the incoming water, which oscillates around $10\,\text{L/m}^3$. Finally, since the volume of reservoir remains constant at 2 m-m^3, a pollutant concentration of $5\,\text{L/m}^3$ is reached when $\dfrac{x(t)}{2} = 5$, that is, when

$$10 = \frac{20}{101}\left(101 - 102e^{-t/10} + \cos t + 10\sin t\right).$$

To solve this equation for *t* requires technology. For instance, the *Mathematica* commands

```
x = (20/101)(101 - 102 Exp[-t/10] + Cos[t] + 10Sin[t]);
FindRoot[ x == 10, {t, 7}]
```

yield $t \approx 6.47$ months.

SECTION 1.6

SUBSTITUTION METHODS AND EXACT EQUATIONS

It is traditional for every elementary differential equations text to include the particular types of equation that are found in this section. However, no one of them is vitally important solely in its own right. Their main purpose (at this point in the course) is to familiarize students with the technique of transforming a differential equation by substitution. The subsection on airplane flight trajectories (together with Problems 56–59) is included as an application, but is optional material and may be omitted if the instructor desires.

The differential equations in Problems 1-15 are homogeneous, and so we solve by means of the substitution $v = y/x$ indicated in Equation (8) of the text. In some cases we present solutions by other means, as well.

1. For $x \neq 0$ and $x + y \neq 0$ we rewrite the differential equation as $\dfrac{dy}{dx} = \dfrac{x-y}{x+y} = \dfrac{1-\frac{y}{x}}{1+\frac{y}{x}}$.

Substituting $v = \dfrac{y}{x}$ then gives $v + x\dfrac{dv}{dx} = \dfrac{1-v}{1+v}$, or $x\dfrac{dv}{dx} = \dfrac{1-v}{1+v} - v = \dfrac{1-2v-v^2}{1+v}$. Separating variables leads to $\displaystyle\int \frac{v+1}{v^2+2v-1}\,dv = -\int \frac{1}{x}\,dx$, or $\dfrac{1}{2}\ln\left|v^2+2v-1\right| = -\ln|x| + C$, or $\left|v^2+2v-1\right| = Cx^{-2}$, where C is an arbitrary positive constant, or finally $v^2 + 2v - 1 = Cx^{-2}$, where C is an arbitrary nonzero constant. Back-substituting $\dfrac{y}{x}$ for v then gives the solution $\left(\dfrac{y}{x}\right)^2 + 2\dfrac{y}{x} - 1 = Cx^{-2}$, or $y^2 + 2xy - x^2 = C$.

2. For $x, y \neq 0$ we rewrite the differential equation as $\dfrac{dy}{dx} = \dfrac{1}{2}\cdot\dfrac{x}{y} + \dfrac{y}{x}$. Substituting $v = \dfrac{y}{x}$ then gives $v + x\dfrac{dv}{dx} = \dfrac{1}{2v} + v$, or $x\dfrac{dv}{dx} = \dfrac{1}{2v}$. Separating variables leads to $\displaystyle\int 2v\,dv = \int \frac{1}{x}\,dx$, or $v^2 = \ln|x| + C$, where C is an arbitrary constant. Back-substituting $\dfrac{y}{x}$ for v then gives the solution $y^2 = x^2\left(\ln|x| + C\right)$.

Alternatively, the substitution $v = y^2$, which implies that $v' = 2y\cdot y'$, gives $xv' = x^2 + 2v$, or $v' - \dfrac{2}{x}v = x$, a linear equation in v as a function of x. An integrating factor is given by

$\rho = \exp\left(\int -\frac{2}{x}dx\right) = \frac{1}{x^2}$, and multiplying by ρ gives $\frac{1}{x^2}\cdot v' - \frac{2}{x^3}v = \frac{1}{x}$, or $D_x\left(\frac{1}{x^2}\cdot v\right) = \frac{1}{x}$. Integrating then gives $\frac{1}{x^2}\cdot v = \ln|x| + C$, or $v = x^2\left(\ln|x| + C\right)$, or finally $y^2 = x^2\left(\ln|x| + C\right)$, as determined above.

3. For x, y with $xy > 0$ we rewrite the differential equation as $\frac{dy}{dx} = \frac{y}{x} + 2\sqrt{\frac{y}{x}}$. Substituting $v = \frac{y}{x}$ then gives $v + x\frac{dv}{dx} = v + 2\sqrt{v}$, or $x\frac{dv}{dx} = 2\sqrt{v}$. Separating variables leads to $\int \frac{1}{\sqrt{v}}dv = \int \frac{2}{x}dx$, or $2\sqrt{v} = 2\ln|x| + C$, or $v = \left(\ln|x| + C\right)^2$. Back-substituting $\frac{y}{x}$ for v then gives the solution $y = x\left(\ln|x| + C\right)^2$.

4. For $x \neq 0$ and $x - y \neq 0$ we rewrite the differential equation as $\frac{dy}{dx} = \frac{x+y}{x-y} = \frac{1+\frac{y}{x}}{1-\frac{y}{x}}$. substituting $v = \frac{y}{x}$ then gives $v + x\frac{dv}{dx} = \frac{1+v}{1-v}$, or $x\frac{dv}{dx} = \frac{1+v^2}{1-v}$. Separating variables leads to $\int \frac{1-v}{1+v^2}dv = \int \frac{1}{x}dx$, or $\tan^{-1}v - \frac{1}{2}\ln\left(1+v^2\right) = \ln|x| + C$. Back-substituting $\frac{y}{x}$ for v then gives $\tan^{-1}\frac{y}{x} - \frac{1}{2}\ln\left[1 + \left(\frac{y}{x}\right)^2\right] = \ln|x| + C$.

5. For $x \neq 0$ and $x + y \neq 0$ we rewrite the differential equation as $\frac{dy}{dx} = \frac{y}{x}\cdot\frac{x-y}{x+y} = \frac{y}{x}\cdot\frac{1-\frac{y}{x}}{1+\frac{y}{x}}$. Substituting $v = \frac{y}{x}$ then gives $v + x\frac{dv}{dx} = v\cdot\frac{1-v}{1+v}$, or $x\frac{dv}{dx} = v\cdot\frac{1-v}{1+v} - v = \frac{-2v^2}{1+v}$. Separating variables leads to $\int \frac{1+v}{v^2}dv = -2\int \frac{1}{x}dx$, or $-\frac{1}{v} + \ln|v| = -2\ln|x| + C$. Back-substituting $\frac{y}{x}$ for v then gives $-\frac{x}{y} + \ln\left|\frac{y}{x}\right| = -2\ln|x| + C$, or $\ln\left|\frac{y}{x}\right| + 2\ln|x| = \frac{x}{y} + C$, or $\ln|xy| = \frac{x}{y} + C$.

6. For $x+2y \neq 0$ and $x \neq 0$ we rewrite the differential equation as $\dfrac{dy}{dx} = \dfrac{y}{x+2y} = \dfrac{\dfrac{y}{x}}{1+2\dfrac{y}{x}}$.

Substituting $v = \dfrac{y}{x}$ then gives $v + x\dfrac{dv}{dx} = \dfrac{v}{1+2v}$, or $x\dfrac{dv}{dx} = \dfrac{v}{1+2v} - v = \dfrac{-2v^2}{1+2v}$. Separating variables leads to $\int \dfrac{1+2v}{v^2}\,dv = -\int \dfrac{2}{x}\,dx$, or $-v^{-1} + 2\ln|v| = -2\ln|x| + C$. Back-substituting $\dfrac{y}{x}$ for v then gives $-\dfrac{x}{y} + 2\ln\left|\dfrac{y}{x}\right| = -2\ln|x| + C$, or $-x + 2y\ln|y| = Cy$.

7. For $x, y \neq 0$ we rewrite the differential equation as $\dfrac{dy}{dx} = \left(\dfrac{x}{y}\right)^2 + \dfrac{y}{x}$. Substituting $v = \dfrac{y}{x}$ then gives $v + x\dfrac{dv}{dx} = \left(\dfrac{1}{v}\right)^2 + v$, or $x\dfrac{dv}{dx} = \left(\dfrac{1}{v}\right)^2$. Separating variables leads to $\int v^2\,dv = \int \dfrac{1}{x}\,dx$, or $v^3 = 3\ln|x| + C$. Back-substituting $\dfrac{y}{x}$ for v then gives $\left(\dfrac{y}{x}\right)^3 = 3\ln|x| + C$, or $y^3 = x^3\left(3\ln|x| + C\right)$.

Alternatively, the substitution $v = y^3$, which implies that $v' = 3y^2 y'$, gives $\dfrac{1}{3}xv' = x^3 + v$, or $v' - \dfrac{3}{x}v = 3x^2$, a linear equation in v as a function of x. An integrating factor is given by $\rho = \exp\left(-\int \dfrac{3}{x}\,dx\right) = x^{-3}$, and multiplying the differential equation by ρ gives $x^{-3} \cdot v' - 3x^{-4}v = 3x^{-1}$, or $D_x\left(x^{-3} \cdot v\right) = 3x^{-1}$. Integrating then gives $x^{-3} \cdot v = 3\ln|x| + C$, and finally back-substituting y^3 for v yields $y^3 = x^3\left(3\ln|x| + C\right)$, as determined above.

8. For $x \neq 0$ we rewrite the differential equation as $\dfrac{dy}{dx} = \dfrac{y}{x} + e^{y/x}$. Substituting $v = \dfrac{y}{x}$ then gives $v + x\dfrac{dv}{dx} = v + e^v$, or $x\dfrac{dv}{dx} = e^v$. Separating variables leads to $\int e^{-v}\,dv = \int \dfrac{1}{x}\,dx$, or $-e^{-v} = \ln|x| + C$, that is, $v = -\ln\left(C - \ln|x|\right)$. Back-substituting $\dfrac{y}{x}$ for v then gives the solution $y = -x\ln\left(C - \ln|x|\right)$.

9. For $x \neq 0$ we rewrite the differential equation as $\dfrac{dy}{dx} = \dfrac{y}{x} + \left(\dfrac{y}{x}\right)^2$. Substituting $v = \dfrac{y}{x}$ then gives $v + x\dfrac{dv}{dx} = v + v^2$, or $x\dfrac{dv}{dx} = v^2$. Separating variables leads to $\int \dfrac{1}{v^2}dv = \int \dfrac{1}{x}dx$, or $-\dfrac{1}{v} = \ln|x| + C$. Back-substituting $\dfrac{y}{x}$ for v then gives the solution $y = \dfrac{x}{C - \ln|x|}$.

10. For $x, y \neq 0$ we rewrite the differential equation as $\dfrac{dy}{dx} = \dfrac{x}{y} + 3\dfrac{y}{x}$. Substituting $v = \dfrac{y}{x}$ then gives $v + x\dfrac{dv}{dx} = \dfrac{1}{v} + 3v$, or $x\dfrac{dv}{dx} = \dfrac{1}{v} + 2v = \dfrac{1+2v^2}{v}$. Separating variables leads to $\int \dfrac{v}{1+2v^2}dv = \int \dfrac{1}{x}dx$, or $\dfrac{1}{4}\ln\left(1+2v^2\right) = \ln|x| + C$, or $2v^2 = Cx^4 - 1$. Back-substituting $\dfrac{y}{x}$ for v then gives the solution $2y^2 = Cx^6 - x^2$.

Alternatively, the substitution $v = y^2$, which implies that $v' = 2y \cdot y'$, gives $\dfrac{1}{2}x \cdot v' = x^2 + 3v$, or $v' - \dfrac{6}{x}v = 2x$, a linear equation in v as a function of x. An integrating factor is given by $\rho = \exp\left(\int -\dfrac{6}{x}dx\right) = x^{-6}$, and multiplying the differential equation by ρ gives $x^{-6} \cdot v' - 6x^{-7}v = 2x^{-5}$, or $D_x\left(x^{-6} \cdot v\right) = 2x^{-5}$. Integrating then gives $x^{-6} \cdot v = -\dfrac{1}{2}x^{-4} + C$, or $2v = -x^2 + Cx^6$. Finally, back-substituting y^2 for v then gives the solution $2y^2 = -x^2 + Cx^6$, as determined above.

11. For $x^2 - y^2 \neq 0$ and $x \neq 0$ we rewrite the differential equation as

$\dfrac{dy}{dx} = \dfrac{2xy}{x^2 - y^2} = \dfrac{2\frac{y}{x}}{1 - \left(\frac{y}{x}\right)^2}$. Substituting $v = \dfrac{y}{x}$ then gives $v + x\dfrac{dv}{dx} = \dfrac{2v}{1-v^2}$, or

$x\dfrac{dv}{dx} = \dfrac{2v}{1-v^2} - v = \dfrac{v+v^3}{1-v^2}$. Separating variables leads to $\int \dfrac{1-v^2}{v+v^3}dv = \int \dfrac{1}{x}dx$, or (after decomposing into partial fractions) $\int \dfrac{1}{v} - \dfrac{2v}{v^2+1}dv = \int \dfrac{1}{x}dx$, or

$\ln|v| - \ln\left(v^2+1\right) = \ln|x| + C$, or $\dfrac{v}{v^2+1} = Cx$. Back-substituting $\dfrac{y}{x}$ for v then gives the solution $\dfrac{y}{x} = Cx\left[\left(\dfrac{y}{x}\right)^2 + 1\right]$, or finally $y = C\left(x^2 + y^2\right)$.

12. For $x, y > 0$ we rewrite the differential equation as

$$\frac{dy}{dx} = \frac{y}{x} + \frac{\sqrt{4x^2 + y^2}}{y} = \frac{y}{x} + \sqrt{4\left(\frac{x}{y}\right)^2 + 1}\,.$$

Substituting $v = \frac{y}{x}$ then gives $v + x\frac{dv}{dx} = v + \sqrt{\frac{4}{v^2} + 1}$, or $x\frac{dv}{dx} = \sqrt{\frac{4}{v^2} + 1} = \frac{\sqrt{4 + v^2}}{v}$. Separating variables leads to $\int \frac{v}{\sqrt{4 + v^2}}\,dv = \int \frac{1}{x}\,dx$, or $\left(4 + v^2\right)^{1/2} = \ln|x| + C$, or $4 + v^2 = \left(\ln|x| + C\right)^2$. Back-substituting $\frac{y}{x}$ for v then gives the solution $4x^2 + y^2 = \left(\ln|x| + C\right)^2$.

13. For $x > 0$ we rewrite the differential equation as

$$\frac{dy}{dx} = \frac{y}{x} + \frac{\sqrt{x^2 + y^2}}{x} = \frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2}\,.$$

Substituting $v = \frac{y}{x}$ then gives $v + x\frac{dv}{dx} = v + \sqrt{1 + v^2}$, or $x\frac{dv}{dx} = \sqrt{1 + v^2}$. Separating variables leads to $\int \frac{1}{\sqrt{1 + v^2}}\,dv = \int \frac{1}{x}\,dx$, or (by means of either the substitution $v = \tan\theta$ or an integral table) $\ln\left(v + \sqrt{v^2 + 1}\right) = \ln|x| + C$, or finally $v + \sqrt{v^2 + 1} = Cx$. Back-substituting $\frac{y}{x}$ for v then gives the solution $y + \sqrt{y^2 + x^2} = Cx^2$.

14. For $x \neq 0$ and $y > 0$ we rewrite the differential equation as

$$\frac{dy}{dx} = -\frac{x}{y} + \frac{\sqrt{x^2 + y^2}}{y} = -\frac{x}{y} + \sqrt{\left(\frac{x}{y}\right)^2 + 1}\,.$$

Substituting $v = \frac{y}{x}$ then gives $v + x\frac{dv}{dx} = -\frac{1}{v} + \sqrt{\frac{1}{v^2} + 1} = \frac{\sqrt{v^2 + 1} - 1}{v}$, or $x\frac{dv}{dx} = \frac{\sqrt{v^2 + 1} - \left(1 + v^2\right)}{v}$. Separating variables leads to $\int \frac{v}{\sqrt{v^2 + 1} - \left(1 + v^2\right)}\,dv = \int \frac{1}{x}\,dx$. The substitution $u = 1 + v^2$ gives $\int \frac{v}{\sqrt{v^2 + 1} - \left(1 + v^2\right)}\,dv = \frac{1}{2}\int \frac{1}{\sqrt{u}\left(1 - \sqrt{u}\right)}\,du$, which under the further substitution $w = 1 - \sqrt{u}$ becomes

$$-\int \frac{1}{w}\,dw = -\ln|w| = -\ln\left|1 - \sqrt{u}\right| + C = -\ln\left|1 - \sqrt{1 + v^2}\right|.$$

Thus $-\ln\left|1-\sqrt{1+v^2}\right| = \ln|x| + C$, or $x\left(1-\sqrt{1+v^2}\right) = C$. Back-substituting $\frac{y}{x}$ for v then gives the solution $x\left[1-\sqrt{1+\left(\frac{y}{x}\right)^2}\right] = C$, or $x - \sqrt{x^2+y^2} = C$.

Alternatively, the substitution $v = x^2 + y^2$, which implies that $v' = 2x + 2y \cdot y'$, gives $\frac{1}{2}v' = \sqrt{v}$. Separating variables leads to $\int \frac{1}{\sqrt{v}}dv = \int 2\,dx$, or $2\sqrt{v} = 2x + C$. Back-substituting $x^2 + y^2$ for v then gives the solution $\sqrt{x^2+y^2} = x + C$, as determined above.

15. For $x \neq 0$ and $x + y \neq 0$ we rewrite the differential equation as

$$\frac{dy}{dx} = \frac{-y(3x+y)}{x(x+y)} = -\frac{y}{x}\cdot\frac{3+\frac{y}{x}}{1+\frac{y}{x}}.$$

Substituting $v = \frac{y}{x}$ then gives $v + x\frac{dv}{dx} = -v\frac{3+v}{1+v} = -\frac{3v+v^2}{1+v}$, or

$$x\frac{dv}{dx} = -\frac{3v+v^2}{1+v} - v = \frac{-4v-2v^2}{1+v}.$$

Separating variables leads to $\int \frac{1+v}{4v+2v^2}dv = -\int \frac{1}{x}dx$, or $\frac{1}{4}\ln\left|4v+2v^2\right| = -\ln|x| + C$, or $x^4\left(4v+2v^2\right) = C$, or simply $x^4\left(2v+v^2\right) = C$. Back-substituting $\frac{y}{x}$ for v then gives the solution $x^2\left(2xy+y^2\right) = C$.

The differential equations in Problems 16-18 rely upon substitutions that are generally suggested by the equations themselves.

16. The expression $\sqrt{x+y+1}$ suggests the substitution $v = x + y + 1$, which implies that $y = v - x - 1$, and thus that $y' = v' - 1$. Substituting gives $v' - 1 = \sqrt{v}$, or $v' = \sqrt{v} + 1$, a separable equation for v as a function of x. Separating variables gives $\int \frac{1}{\sqrt{v}+1}dv = \int dx$. Under the substitution $v = u^2$ the integral on the right becomes $\int \frac{2u}{1+u}du$, which after long division is

$$\int 2 - \frac{2}{1+u}du = 2u - 2\ln(1+u) = 2\sqrt{v} - 2\ln\left(1+\sqrt{v}\right).$$

Finally, back-substituting $x + y + 1$ for v leads to the solution

$$2\sqrt{x+y+1} - 2\ln\left(1+\sqrt{x+y+1}\right) = x + C.$$

17. The expression $4x + y$ suggests the substitution $v = 4x + y$, which implies that $y = v - 4x$, and thus that $y' = v' - 4$. Substituting gives $v' - 4 = v^2$, or $v' = v^2 + 4$, a separable equation for v as a function of x. Separating variables gives $\int \frac{1}{v^2+4}\,dv = \int dx$, or $\frac{1}{2}\tan^{-1}\frac{v}{2} = x + C$, or $v = 2\tan(2x + C)$. Finally, back-substituting $4x + y$ for v leads to the solution $y = 2\tan(2x + C) - 4x$.

18. The expression $x + y$ suggests the substitution $v = x + y$, which implies that $y = v - x$, and thus that $y' = v' - 1$. Substituting gives $v(v' - 1) = 1$, or $v' = \frac{1}{v} + 1 = \frac{v+1}{v}$, a separable equation for v as a function of x. Separating variables gives $\int \frac{v}{v+1}\,dv = \int dx$, or (by long division) $\int 1 - \frac{1}{v+1}\,dv = \int dx$, or $v - \ln|v+1| = x + C$. Finally, back-substituting $x + y$ for v gives $y - \ln|x + y + 1| = C$.

The differential equations in Problems 19-25 are Bernoulli equations, and so we solve by means of the substitution $v = y^{1-n}$ indicated in Equation (10) of the text. (Problem 25 also admits of another solution.)

19. We first rewrite the differential equation for $x, y > 0$ as $y' + \frac{2}{x}y = \frac{5}{x^2}y^3$, a Bernoulli equation with $n = 3$. The substitution $v = y^{1-3} = y^{-2}$ implies that $y = v^{-1/2}$ and thus that $y' = -\frac{1}{2}v^{-3/2}v'$. Substituting gives $-\frac{1}{2}v^{-3/2}v' + \frac{2}{x}v^{-1/2} = \frac{5}{x^2}v^{-3/2}$, or $v' - \frac{4}{x}v = -\frac{10}{x^2}$, a linear equation for v as a function of x. An integrating factor is given by $\rho = \exp\left(\int -\frac{4}{x}dx\right) = x^{-4}$, and multiplying the differential equation by ρ gives $\frac{1}{x^4}v' - \frac{4}{x^5}v = -\frac{10}{x^6}$, or $D_x\left(\frac{1}{x^4}\cdot v\right) = -\frac{10}{x^6}$. Integrating then leads to $\frac{1}{x^4}\cdot v = \frac{2}{x^5} + C$, or $v = \frac{2}{x} + Cx^4 = \frac{2 + Cx^5}{x}$. Finally, back-substituting y^{-2} for v gives the general solution $y^{-2} = \frac{2 + Cx^5}{x}$, or $y^2 = \frac{x}{2 + Cx^5}$.

20. We first rewrite the differential equation for $y>0$ as $y'+2xy=6xy^{-2}$, a Bernoulli equation with $n=-2$. The substitution $v=y^{1-(-2)}=y^3$ implies that $y=v^{1/3}$ and thus that $y'=\frac{1}{3}v^{-2/3}v'$. Substituting gives $\frac{1}{3}v^{-2/3}v'+2xv^{1/3}=6xv^{-2/3}$, or $v'+6xv=18x$, a linear equation for v as a function of x. An integrating factor is given by $\rho=\exp\left(\int 6x\,dx\right)=e^{3x^2}$, and multiplying the differential equation by ρ gives $e^{3x^2}\cdot v'+6xe^{3x^2}v=18xe^{3x^2}$, or $D_x\left(e^{3x^2}\cdot v\right)=18xe^{3x^2}$. Integrating then leads to $e^{3x^2}\cdot v=3e^{3x^2}+C$, or $v=3+Ce^{-3x^2}$. Finally, back-substituting y^3 for v gives the general solution $y^3=3+Ce^{-3x^2}$.

21. We first rewrite the differential equation as $y'-y=y^3$, a Bernoulli equation with $n=3$. The substitution $v=y^{1-3}=y^{-2}$ implies that $y=v^{-1/2}$ and thus that $y'=-\frac{1}{2}v^{-3/2}v'$. Substituting gives $-\frac{1}{2}v^{-3/2}v'-v^{-1/2}=v^{-3/2}$, or $v'+2v=-2$, a linear equation for v as a function of x. An integrating factor is given by $\rho=\exp\left(\int 2\,dx\right)=e^{2x}$, and multiplying the differential equation by ρ gives $e^{2x}\cdot v'+2e^{2x}v=-2e^{2x}$, or $D_x\left(e^{2x}\cdot v\right)=-2e^{2x}$. Integrating then leads to $e^{2x}\cdot v=-e^{2x}+C$, or $v=-1+Ce^{-2x}$. Finally, back-substituting y^{-2} for v gives the general solution $y^{-2}=-1+Ce^{-2x}$, or $y^2=\dfrac{1}{Ce^{-2x}-1}$.

22. We first rewrite the differential equation for $x>0$ as $y'+\frac{2}{x}y=\frac{5}{x^2}y^4$, a Bernoulli equation with $n=4$. The substitution $v=y^{1-4}=y^{-3}$ implies that $y=v^{-1/3}$ and thus that $y'=-\frac{1}{3}v^{-4/3}v'$. Substituting gives $-\frac{1}{3}v^{-4/3}v'+\frac{2}{x}v^{-1/3}=\frac{5}{x^2}v^{-4/3}$, or $v'-\frac{6}{x}v=-\frac{15}{x^2}$, a linear equation for v as a function of x. An integrating factor is given by $\rho=\exp\left(\int -\frac{6}{x}dx\right)=\frac{1}{x^6}$, and multiplying the differential equation by ρ gives $\frac{1}{x^6}\cdot v'-\frac{6}{x^7}v=-\frac{15}{x^8}$, or $D_x\left(\frac{1}{x^6}\cdot v\right)=-15x^{-8}$. Integrating then leads to $\frac{1}{x^6}\cdot v=\frac{15}{7}x^{-7}+C$, or $v=\frac{15}{7x}+Cx^6=\frac{Cx^7+15}{7x}$. Finally, back-substituting y^{-3} for v gives the general solution $y^{-3}=\dfrac{Cx^7+15}{7x}$, or $y^3=\dfrac{7x}{Cx^7+15}$.

23. We first rewrite the differential equation for $x > 0$ as $y' + \frac{6}{x} y = 3y^{4/3}$, a Bernoulli equation with $n = 4/3$. The substitution $v = y^{1-(4/3)} = y^{-1/3}$ implies that $y = v^{-3}$ and thus that $y' = -3v^{-4}v'$. Substituting gives $-3v^{-4}v' + \frac{6}{x}v^{-3} = 3v^{-4}$, or $v' - \frac{2}{x}v = -1$, a linear equation for v as a function of x. An integrating factor is given by $\rho = \exp\left(\int -\frac{2}{x} dx \right) = \frac{1}{x^2}$, and multiplying the differential equation by ρ gives $\frac{1}{x^2}v' - \frac{2}{x^3}v = -\frac{1}{x^2}$, or $D_x\left(\frac{1}{x^2}v \right) = -\frac{1}{x^2}$. Integrating then leads to $\frac{1}{x^2}v = \frac{1}{x} + C$, or $v = x + Cx^2$. Finally, back-substituting $y^{-1/3}$ for v gives the general solution $y^{-1/3} = x + Cx^2$, or $y = \left(x + Cx^2\right)^{-3}$.

24. We first rewrite the differential equation for $x > 0$ as $y' - y = -\frac{e^{-2x}}{2x} y^3$, a Bernoulli equation with $n = 3$. The substitution $v = y^{1-3} = y^{-2}$ implies that $y = v^{-1/2}$, and thus that $y' = -\frac{1}{2}v^{-3/2}v'$. Substituting gives $-\frac{1}{2}v^{-3/2}v' - v^{-1/2} = -\frac{e^{-2x}}{2x}v^{-3/2}$, or $v' + 2v = \frac{e^{-2x}}{x}$, a linear equation for v as a function of x. An integrating factor is given by $\rho = \exp\left(\int 2\,dx \right) = e^{2x}$, and multiplying the differential equation by ρ gives $e^{2x} \cdot v' + 2e^{2x}v = \frac{1}{x}$, or $D_x\left(e^{2x} \cdot v\right) = \frac{1}{x}$. Integrating then leads to $e^{2x} \cdot v = \ln x + C$, or $v = \left(C + \ln x\right)e^{-2x}$. Finally, back-substituting y^{-2} for v gives the general solution $y^{-2} = \left(C + \ln x\right)e^{-2x}$, or $y^2 = \frac{e^{2x}}{\left(C + \ln x\right)}$.

25. We first rewrite the differential equation for $x, y > 0$ as $y' + \frac{1}{x} y = \frac{1}{\left(1 + x^4\right)^{1/2}} y^{-2}$, a Bernoulli equation with $n = -2$. The substitution $v = y^{1-(-2)} = y^3$ implies that $y = v^{1/3}$ and thus that $y' = \frac{1}{3}v^{-2/3}v'$. Substituting gives $\frac{1}{3}v^{-2/3}v' + \frac{1}{x}v^{1/3} = \frac{1}{\left(1 + x^4\right)^{1/2}}v^{-2/3}$, or

$v' + \frac{3}{x}v = \frac{3}{\left(1 + x^4\right)^{1/2}}$, a linear equation for v as a function of x. An integrating factor is given by $\rho = \exp\left(\int \frac{3}{x} dx \right) = x^3$, and multiplying the differential equation by ρ gives

$x^3 \cdot v' + 3x^2 v = \dfrac{3x^3}{\left(1+x^4\right)^{1/2}}$, or $D_x\left(x^3 \cdot v\right) = \dfrac{3x^4}{\left(1+x^4\right)^{1/2}}$. Integrating then leads to

$x^3 \cdot v = \dfrac{3}{2}\left(1+x^4\right)^{1/2} + C$, or $v = \dfrac{3\left(1+x^4\right)^{1/2} + C}{2x^3}$. Finally, back-substituting y^3 for v gives

the general solution $y^3 = \dfrac{3\left(1+x^4\right)^{1/2} + C}{2x^3}$.

Alternatively, for $x \neq 0$, the substitution $v = xy$, which implies that $v' = xy' + y$ and that

$y = \dfrac{v}{x}$, gives $\dfrac{v^2}{x^2} v'\left(1+x^4\right)^{1/2} = x$. Separating variables leads to $\int v^2\, dv = \int \dfrac{x^3}{\left(1+x^4\right)^{1/2}} dx$,

or $\dfrac{1}{3} v^3 = \dfrac{1}{2}\left(1+x^4\right)^{1/2} + C$, or $v^3 = \dfrac{3\left(1+x^4\right)^{1/2} + C}{2}$. Back-substituting xy for v then gives

the solution $y^3 = \dfrac{3\left(1+x^4\right)^{1/2} + C}{2x^3}$, as determined above.

As with Problems 16-18, the differential equations in Problems 26-30 rely upon substitutions that are generally suggested by the equations themselves. Two of these equations are also Bernoulli equations.

26. The substitution $v = y^3$, which implies that $v' = 3y^2 y'$, gives $v' + v = e^{-x}$, a linear equation for v as a function of x. An integrating factor is given by $\rho = \exp\left(\int 1\, dx\right) = e^x$, and multiplying the differential equation by ρ gives $e^x \cdot v' + e^x v = 1$, or $D_x\left(e^x \cdot v\right) = 1$. Integrating then leads to $e^x \cdot v = x + C$, or $v = \left(x + C\right)e^{-x}$. Finally, back-substituting y^3 for v gives the general solution $y^3 = \left(x + C\right)e^{-x}$.

Alternatively, for $y \neq 0$ we can first rewrite the differential equation as

$y' + \dfrac{1}{3} y = \dfrac{1}{3} e^{-x} y^{-2}$, a Bernoulli equation with $n = -2$. This leads to the substitution

$v = y^{1-(-2)} = y^3$ used above.

27. The substitution $v = y^3$, which implies that $v' = 3y^2 y'$, gives $xv' - v = 3x^4$, or (for $x > 0$) $v' - \dfrac{1}{x} v = 3x^3$, a linear equation for v as a function of x. An integrating factor is given by

$\rho = \exp\left(\int -\dfrac{1}{x} dx\right) = \dfrac{1}{x}$, and multiplying the differential equation by ρ gives

$\dfrac{1}{x} \cdot v' - \dfrac{1}{x^2} v = 3x^2$, or $D_x\left(\dfrac{1}{x} \cdot v\right) = 3x^2$. Integrating then leads to $\dfrac{1}{x} \cdot v = x^3 + C$, or

$v = x^4 + Cx$. Finally, back-substituting y^3 for v gives the general solution $y^3 = x^4 + Cx$, or $y = \left(x^4 + Cx\right)^{1/3}$.

Alternatively, for $x, y > 0$ we can first rewrite the differential equation as $y' - \frac{1}{3x} y = x^3 y^{-2}$, a Bernoulli equation with $n = -2$. This leads to the substitution $v = y^{1-(-2)} = y^3$ used above.

28. The substitution $v = e^y$, which implies that $v' = e^y y'$, gives $xv' = 2\left(v + x^3 e^{2x}\right)$, or $v' - \frac{2}{x} v = 2x^2 e^{2x}$, a linear equation for v as a function of x. An integrating factor is given by $\rho = \exp\left(\int -\frac{2}{x} dx\right) = \frac{1}{x^2}$, and multiplying the differential equation by ρ gives $\frac{1}{x^2} \cdot v' - \frac{2}{x^3} v = 2e^{2x}$, or $D_x\left(\frac{1}{x^2} \cdot v\right) = 2e^{2x}$. Integrating then leads to $\frac{1}{x^2} \cdot v = e^{2x} + C$, or $v = x^2 e^{2x} + Cx^2$. Finally, back-substituting e^y for v gives the general solution $e^y = x^2 e^{2x} + Cx^2$, or $y = \ln\left(x^2 e^{2x} + Cx^2\right)$.

29. The substitution $v = \sin^2 y$, which implies that $v' = \left(2 \sin y \cos y\right) y'$, gives $xv' = 4x^2 + v$, or (for $x > 0$) $v' - \frac{1}{x} v = 4x$, a linear equation for v as a function of x. An integrating factor is given by $\rho = \exp\left(\int -\frac{1}{x} dx\right) = \frac{1}{x}$, and multiplying the differential equation by ρ gives $\frac{1}{x} \cdot v' - \frac{1}{x^2} v = 4$, or $D_x\left(\frac{1}{x} \cdot v\right) = 4$. Integrating then leads to $\frac{1}{x} \cdot v = 4x + C$, or $v = 4x^2 + Cx$. Finally, back-substituting $\sin^2 y$ for v gives the general solution $\sin^2 y = 4x^2 + Cx$.

30. It is easiest first to multiply each side of the given equation by e^y, giving $\left(x + e^y\right) e^y y' = x - e^y$. This suggests the substitution $v = e^y$, which implies that $v' = e^y y'$, and leads to $\left(x + v\right) v' = x - v$, which is identical to the homogeneous equation in Problem 1. The solution found there is $v^2 + 2xv - x^2 = C$. Back-substituting e^y for v then gives the general solution $e^{2y} + 2xe^y - x^2 = C$

Each of the differential equations in Problems 31–42 is of the form $M\,dx + N\,dy = 0$, and the exactness condition $\partial M/\partial y = \partial N/\partial x$ is routine to verify. For each problem we give the principal steps in the calculation corresponding to the method of Example 9 in this section.

31. The condition $F_x = M$ implies that $F(x,y) = \int 2x + 3y\,dx = x^2 + 3xy + g(y)$, and then the condition $F_y = N$ implies that $3x + g'(y) = 3x + 2y$, or $g'(y) = 2y$, or $g(y) = y^2$. Thus the solution is given by $x^2 + 3xy + y^2 = C$.

32. The condition $F_x = M$ implies that $F(x,y) = \int 4x - y\,dx = 2x^2 - xy + g(y)$, and then the condition $F_y = N$ implies that $-x + g'(y) = 6y - x$, or $g'(y) = 6y$, or $g(y) = 3y^2$. Thus the solution is given by $2x^2 - xy + 3y^2 = C$

33. The condition $F_x = M$ implies that $F(x,y) = \int 3x^2 + 2y^2\,dx = x^3 + xy^2 + g(y)$, and then the condition $F_y = N$ implies that $4xy + g'(y) = 4xy + 6y^2$, or $g'(y) = 6y^2$, or $g(y) = 2y^3$. Thus the solution is given by $x^3 + 2xy^2 + 2y^3 = C$.

34. The condition $F_x = M$ implies that $F(x,y) = \int 2xy^2 + 3x^2\,dx = x^3 + x^2y^2 + g(y)$, and then the condition $F_y = N$ implies that $2x^2y + g'(y) = 2x^2y + 4y^3$, or $g'(y) = 4y^3$, or $g(y) = y^4$. Thus the solution is given by $x^3 + x^2y^2 + y^4 = C$.

35. The condition $F_x = M$ implies that $F(x,y) = \int x^3 + \frac{y}{x}\,dx = \frac{1}{4}x^4 + y\ln x + g(y)$, and then the condition $F_y = N$ implies that $\ln x + g'(y) = y^2 + \ln x$, or $g'(y) = y^2$, or $g(y) = \frac{1}{3}y^3$. Thus the solution is given by $\frac{1}{4}x^4 + \frac{1}{3}y^3 + y\ln x = C$.

36. The condition $F_x = M$ implies that $F(x,y) = \int 1 + ye^{xy}\,dx = x + e^{xy} + g(y)$, and then the condition $F_y = N$ implies that $xe^{xy} + g'(y) = 2y + xe^{xy}$, or $g'(y) = 2y$, or $g(y) = y^2$. Thus the solution is given by $x + e^{xy} + y^2 = C$.

37. The condition $F_x = M$ implies that $F(x,y) = \int \cos x + \ln y\,dx = \sin x + x\ln y + g(y)$, and then the condition $F_y = N$ implies that $\frac{x}{y} + g'(y) = \frac{x}{y} + e^y$, or $g'(y) = e^y$, or $g(y) = e^y$. Thus the solution is given by $\sin x + x\ln y + e^y = C$.

38. The condition $F_x = M$ implies that $F(x,y) = \int x + \tan^{-1} y\,dx = \frac{1}{2}x^2 + x\tan^{-1} y + g(y)$, and then the condition $F_y = N$ implies that $\frac{x}{1+y^2} + g'(y) = \frac{x+y}{1+y^2}$, or $g'(y) = \frac{y}{1+y^2}$, or $g(y) = \frac{1}{2}\ln(1+y^2)$. Thus the solution is given by $\frac{1}{2}x^2 + x\tan^{-1} y + \frac{1}{2}\ln(1+y^2) = C$.

39. The condition $F_x = M$ implies that $F(x,y) = \int 3x^2y^3 + y^4\,dx = x^3y^3 + xy^4 + g(y)$, and then the condition $F_y = N$ implies that $3x^3y^2 + 4xy^3 + g'(y) = 3x^3y^2 + y^4 + 4xy^3$, or $g'(y) = y^4$, or $g(y) = \frac{1}{5}y^5$. Thus the solution is given by $x^3y^3 + xy^4 + \frac{1}{5}y^5 = C$.

40. The condition $F_x = M$ implies that

$$F(x,y) = \int e^x \sin y + \tan y\,dx = e^x \sin y + x\tan y + g(y),$$

and then the condition $F_y = N$ implies that

$$e^x \cos y + x\sec^2 y + g'(y) = e^x \cos y + x\sec^2 y,$$

or $g'(y) = 0$, or $g(y) = 0$. Thus the solution is given by $e^x \sin y + x\tan y = C$.

41. The condition $F_x = M$ implies that $F(x,y) = \int \frac{2x}{y} - \frac{3y^2}{x^4}\,dx = \frac{x^2}{y} + \frac{y^2}{x^3} + g(y)$, and then the condition $F_y = N$ implies that $-\frac{x^2}{y^2} + \frac{2y}{x^3} + g'(y) = -\frac{x^2}{y^2} + \frac{2y}{x^3} + \frac{1}{\sqrt{y}}$, or $g'(y) = \frac{1}{\sqrt{y}}$, or $g(y) = 2\sqrt{y}$. Thus the solution is given by $\frac{x^2}{y} + \frac{y^2}{x^3} + 2\sqrt{y} = C$.

42. The condition $F_x = M$ implies that

$$F(x,y) = \int y^{-2/3} - \frac{3}{2}x^{-5/2}y\,dx = xy^{-2/3} + x^{-3/2}y + g(y),$$

and then the condition $F_y = N$ implies that $-\frac{2}{3}xy^{-5/3} + x^{-3/2} + g'(y) = x^{-3/2} - \frac{2}{3}xy^{-5/3}$, or $g'(y) = 0$, or $g(y) = 0$. Thus the solution is given by $xy^{-2/3} + x^{-3/2}y = C$.

In Problems 43-48 either the dependent variable y or the independent variable x (or both) is missing, and so we use the substitutions in equations (34) and/or (36) of the text to reduce the given differential equation to a first-order equation for $p = y'$.

43. Since the dependent variable y is missing, we can substitute $y' = p$ and $y'' = p'$ as in Equation (34) of the text. This leads to $xp' = p$, a separable equation for p as a function of x. Separating variables gives $\int \frac{dp}{p} = \int \frac{dx}{x}$, or $\ln p = \ln x + \ln C$, or $p = Cx$, that is, $y' = Cx$. Finally, integrating gives the solution $y(x) = \frac{1}{2}Cx^2 + B$, which we rewrite as $y(x) = Ax^2 + B$.

44. Since the independent variable x is missing, we can substitute $y' = p$ and $y'' = p\frac{dp}{dy}$ as in Equation (36) of the text. This leads to $yp\frac{dp}{dy} + p^2 = 0$, or , a separable equation for p as a function of y. Separating variables gives $\int \frac{dp}{p} = -\int \frac{dy}{y}$, or $\ln p = -\ln y + \ln C$, or $p = \frac{C}{y}$, that is, $\frac{dy}{dx} = \frac{C}{y}$. Separating variables once again leads to $\int y\,dy = \int C\,dx$, or $\frac{1}{2}y^2 = Cx + D$, or $x(y) = \frac{1}{2C}y^2 - \frac{D}{C}$, which we rewrite as $x(y) = Ay^2 + B$.

45. Since the independent variable x is missing, we can substitute $y' = p$ and $y'' = p\frac{dp}{dy}$ as in Equation (36) of the text. This leads to $p\frac{dp}{dy} + 4y = 0$, or $\int p\,dp = -\int 4y\,dy$, or $\frac{1}{2}p^2 = -2y^2 + C$, or $p = \sqrt{2C - 4y^2} = 2\sqrt{C - y^2}$ (replacing $\frac{C}{2}$ simply with C in the last step). Thus $\frac{dy}{dx} = 2\sqrt{C - y^2}$. Separating variables once again yields $\int \frac{dy}{2\sqrt{C - y^2}} = \int dx$, or $\int \frac{dy}{2\sqrt{k^2 - y^2}} = \int dx$, upon replacing C with k^2. Integrating gives $x = \int \frac{dy}{2\sqrt{k^2 - y^2}} = \frac{1}{2}\sin^{-1}\frac{y}{k} + D$; solving for y leads to the solution

$$y(x) = k\sin(2x - 2D) = k(\sin 2x\cos 2D - \cos 2x\sin 2D),$$

or simply $y(x) = A\cos 2x + B\sin 2x$. (A much easier method of solution for this equation will be introduced in Chapter 3.)

46. Since the dependent variable y is missing, we can substitute $y' = p$ and $y'' = p'$ as in Equation (34) of the text. This leads to $xp' + p = 4x$, a linear equation for p as a function of x which we can rewrite as $D_x(x\cdot p)=4x$ (thus, no integrating factor is needed), or $x\cdot p=2x^2+A$, or $p=2x+\frac{A}{x}$, that is, $\frac{dy}{dx}=2x+\frac{A}{x}$. Finally, integrating gives the solution $y(x)=x^2+A\ln x+B$.

47. Since the dependent variable y is missing, we can substitute $y'=p$ and $y''=p'$ as in Equation (34) of the text. This leads to $p'=p^2$, a separable equation for p as a function of x. Separating variables gives $\int\frac{dp}{p^2}=\int x\,dx$, or $-\frac{1}{p}=x+B$, or $p=-\frac{1}{x+B}$, that is, $\frac{dy}{dx}=-\frac{1}{x+B}$. Finally, integrating gives the solution $y(x)=A-\ln|x+B|$.

Alternatively, since the independent variable x is also missing, we can instead substitute $y'=p$ and $y''=p\frac{dp}{dy}$ as in Equation (36) of the text. This leads to $p\frac{dp}{dy}=p^2$, or $\int\frac{dp}{p}=\int dy$, or $\ln p=y+C$, or $p=Ce^y$, that is, $\frac{dy}{dx}=Ce^y$. Separating variables once again leads to $\int e^{-y}\,dy=C\int dx$, or $-e^{-y}=Cx+D$, or

$$y=-\ln(Cx+D)=-\ln\left[C\left(x+\frac{D}{C}\right)\right]=-\ln C-\ln\left(x+\frac{D}{C}\right).$$

Putting $A=-\ln C$ and $B=\frac{D}{C}$ gives the same solution as found above.

48. Since the dependent variable y is missing, we can substitute $y'=p$ and $y''=p'$ as in Equation (34) of the text. This leads to $x^2p'+3xp=2$, a linear equation for p as a function of x. We rewrite this equation as $p'+\frac{3}{x}p=\frac{2}{x^2}$, showing that an integrating factor is given by $\rho=\exp\left(\int\frac{3}{x}dx\right)=x^3$. Multiplying by ρ gives $x^3\cdot p'+3x^2p=2x$, or $D_x(x^3\cdot p)=2x$, or $x^3\cdot p=x^2+C$, or $p=\frac{1}{x}+\frac{C}{x^3}$, that is, $\frac{dy}{dx}=\frac{1}{x}+\frac{C}{x^3}$. Finally, integrating gives the solution $y(x)=\ln x-\frac{C}{2x^2}+D$, which we rewrite as $y(x)=\ln x+\frac{A}{x^2}+B$.

49. Since the independent variable x is missing, we can substitute $y' = p$ and $y'' = p\dfrac{dp}{dy}$ as in Equation (36) of the text. This leads to $py\dfrac{dp}{dy} + 4p^2 = yp$, or $y\dfrac{dp}{dy} + p = y$, a linear equation for p as a function as a function of y which we can rewrite as $D_y\left(y \cdot p\right) = y$, or $y \cdot p = \dfrac{1}{2}y^2 + C$, or $p = \dfrac{y^2 + C}{2y}$, that is, $\dfrac{dy}{dx} = \dfrac{y^2 + C}{2y}$. Separating variables leads to $\int \dfrac{2y}{y^2 + C}dy = \int dx$, or $x = \int \dfrac{2y\,dy}{y^2 + C} = \ln\left(y^2 + C\right) + B$. Solving for y leads to the solution $y^2 + C = e^{x+B} = Be^x$, or finally $y\left(x\right) = \pm\sqrt{A + Be^x}$.

50. Since the dependent variable y is missing, we can substitute $y' = p$ and $y'' = p'$ as in Equation (34) of the text. This leads to $p' = \left(x + p\right)^2$, a first-order equation for p as a function of x which is neither linear nor separable. However, the further substitution $v = x + p$, which implies that $p' = v' - 1$, yields $v' - 1 = v^2$, or $\dfrac{dv}{dx} = 1 + v^2$, a separable equation for v as a function of x. Separating variables gives $\int \dfrac{dv}{1 + v^2} = \int dx$, or $\arctan v = x + A$, or $v = \tan\left(x + A\right)$. Back-substituting $x + p$ for v then leads to $p = \tan\left(x + A\right) - x$, or $\dfrac{dy}{dx} = \tan\left(x + A\right) - x$. Finally, integrating gives the solution $y\left(x\right) = \ln\left|\sec\left(x + A\right)\right| - \dfrac{1}{2}x^2 + B$

51. Since the independent variable x is missing, we can substitute $y' = p$ and $y'' = p\dfrac{dp}{dy}$ as in Equation (36) of the text. This leads to $pp' = 2yp^3$, or $p' = 2yp^2$, or $\int \dfrac{1}{p^2}dp = \int 2y\,dy$, or $-\dfrac{1}{p} = y^2 + C$, or $p = -\dfrac{1}{y^2 + C}$, that is, $\dfrac{dy}{dx} = -\dfrac{1}{y^2 + C}$. Separating variables once again leads to $\int y^2 + C\,dy = -\int dx$, or $\dfrac{1}{3}y^3 + Cy = -x + D$, or finally the solution $y^3 + 3x + Ay + B = 0$.

52. Since the independent variable x is missing, we can substitute $y' = p$ and $y'' = p\dfrac{dp}{dy}$ as in Equation (36) of the text. This leads to $y^3pp' = 1$, or $\int p\,dp = \int \dfrac{1}{y^3}dy$, or

$\frac{1}{2}p^2 = -\frac{1}{2y^2} + A$, or $p^2 = \frac{Ay^2-1}{y^2}$, or $p = \frac{\sqrt{Ay^2-1}}{y}$, that is, $\frac{dy}{dx} = \frac{\sqrt{Ay^2-1}}{y}$. Separating variables once again yields $\int \frac{y}{\sqrt{Ay^2-1}}dv = \int dx$, or $x = \frac{1}{A}\sqrt{Ay^2-1} + C$, which we rewrite as $Ax + B = \sqrt{Ay^2-1}$, leading to the solution $Ay^2 - (Ax+B)^2 = 1$.

53. Since the independent variable *x* is missing, we can substitute $y' = p$ and $y'' = p\frac{dp}{dy}$ as in Equation (36) of the text. This leads to $pp' = 2yp$, or $\int dp = \int 2y\,dy$, or $p = y^2 + A$, that is, $\frac{dy}{dx} = y^2 + A$. Separating variables once again yields $\int \frac{1}{y^2+A}dy = \int dx$, or $A\arctan\frac{y}{A} = x + C$, or $\frac{y}{A} = \tan(Ax+B)$, or finally the solution $y(x) = A\tan(Ax+B)$.

54. Since the independent variable *x* is missing, we can substitute $y' = p$ and $y'' = p\frac{dp}{dy}$ as in Equation (36) of the text. This leads to $ypp' = 3p^2$, or $\int \frac{1}{p}dp = \int \frac{3}{y}dy$, or $\ln p = 3\ln y + C$, or $p = Cy^3$, that is, $\frac{dy}{dx} = Cy^3$. Separating variables once again yields the solution $\int \frac{1}{y^3}dy = \int C\,dx$, or $-\frac{1}{2y^2} = Cx + D$, or $-1 = 2y^2(Cx+D)$, which we rewrite as $y^2(B-x) = 1$.

55. The proposed substitution $v = ax + by + c$ implies that $y = \frac{1}{b}(v - ax - c)$, so that $y' = \frac{1}{b}(v' - a)$. Substituting into the given differential equation gives $\frac{1}{b}(v'-a) = F(v)$, that is $\frac{dv}{dx} = bF(v) + a$, a separable equation for *v* as a function of *x*.

56. The proposed substitution $v = y^{1-n}$ implies that $y = v^{1/(1-n)}$ and thus that $\frac{dy}{dx} = \frac{1}{1-n}v^{\frac{1}{1-n}-1}\frac{dv}{dx} = \frac{1}{1-n}v^{n/(1-n)}\frac{dv}{dx}$. Substituting into the given Bernoulli equation yields $\frac{1}{1-n}v^{n/(1-n)}\frac{dv}{dx} + P(x)v^{1/(1-n)} = Q(x)v^{n/(1-n)}$, and multiplication by $\frac{1-n}{v^{n/(1-n)}}$ then leads to the linear differential equation $v' + (1-n)P(x)v = (1-n)Q(x)v$.

Problems 57-62 illustrate additional substitutions that are helpful in solving certain types of first-order differential equation.

57. The proposed substitution $v = \ln y$ implies that $y = e^v$, and thus that $\frac{dy}{dx} = e^v \frac{dv}{dx}$. Substituting into the given equation yields $e^v \frac{dv}{dx} + P(x)e^v = Q(x)ve^v$. Cancellation of the factor e^v then yields the linear differential equation $\frac{dv}{dx} - Q(x)v = P(x)$.

58. By Problem 57, substituting $v = \ln y$ into the given equation yields the linear equation $xv' + 2v = 4x^2$, which we rewrite (for $x > 0$) as $v' + \frac{2}{x}v = 4x$. An integrating factor is given by $\rho = \exp\left(\int \frac{2}{x}dx\right) = x^2$, and multiplying by ρ gives $x^2 \cdot v' + 2xv = 4x^3$, or $D\left(x^2 \cdot v\right) = 4x^3$. Integrating then leads to $x^2 \cdot v = x^4 + C$, or $v = x^2 + \frac{C}{x^2}$. Finally, back-substituting $\ln y$ for *v* gives the solution $\ln y = x^2 + \frac{C}{x^2}$, or $y = \exp\left(x^2 + \frac{C}{x^2}\right)$.

59. The substitution $y = v + k$ implies that $\frac{dy}{dx} = \frac{dv}{dx}$, leading to

$$\frac{dv}{dx} = \frac{x - (v+k) - 1}{x + (v+k) + 3} = \frac{x - v - (k+1)}{x + v + (k+3)}.$$

Likewise the substitution $x = u + h$ implies that $u = x - h$ and thus that $\frac{dv}{dx} = \frac{dv}{du}\frac{du}{dx} = \frac{dv}{du}$ (since $\frac{du}{dx} = 1$), giving

$$\frac{dv}{du} = \frac{(u+h) - (v+k) - 1}{(u+h) + (v+k) + 3} = \frac{u - v + (h - k - 1)}{u + v + (h + k + 3)}.$$

Thus *h* and *k* must be chosen to satisfy the system

$$\begin{aligned} h - k - 1 &= 0 \\ h + k + 3 &= 0 \end{aligned},$$

which means that $h = -1$ and $k = -2$. These choices for *h* and *k* lead to the homogeneous equation

$$\frac{dv}{du} = \frac{u - v}{u + v} = \frac{1 - \frac{v}{u}}{1 + \frac{v}{u}},$$

which calls for the further substitution $p = \frac{v}{u}$, so that $v = pu$ and thus $\frac{dv}{du} = p + u\frac{dp}{du}$.

Substituting gives $p + u\frac{dp}{du} = \frac{1-p}{1+p}$, or

$$u\frac{dp}{du} = \frac{1-p}{1+p} - \frac{p+p^2}{1+p} = \frac{1-2p-p^2}{1+p}.$$

Separating variables yields $\int \frac{1+p}{1-2p-p^2}\,dp = \int \frac{1}{u}\,du$, or $-\frac{1}{2}\ln\left(1-2p-p^2\right) = \ln u + C$,

or $\left(p^2+2p-1\right)u^2 = C$. Back-substituting $\frac{v}{u}$ for p leads to $\left[\left(\frac{v}{u}\right)^2 + 2\frac{v}{u} - 1\right]u^2 = C$, or

$v^2 + 2uv - u^2 = C$. Finally, back-substituting $x+1$ for u and $y+2$ for v gives the implicit solution

$$(y+2)^2 + 2(x+1)(y+2) - (x+1)^2 = C,$$

which reduces to $y^2 + 2xy - x^2 + 2x + 6y = C$.

60. As in Problem 59, the substitutions $x = u + h$, $y = v + k$ give

$$\frac{dv}{du} = \frac{2(v+k)-(u+h)+7}{4(u+h)-3(v+k)-18} = \frac{2v-u+(2k-h+7)}{4u-3v+(4h-3k-18)}.$$

Thus h and k must be chosen to satisfy the system

$$\begin{aligned}-h+2k+7&=0\\4h-3k-18&=0\end{aligned},$$

which means that $h = 3$ and $k = -2$. These choices for h and k lead to the homogeneous equation

$$\frac{dv}{du} = \frac{2v-u}{4u-3v} = \frac{2\frac{v}{u}-1}{4-3\frac{v}{u}},$$

which calls for the further substitution $p = \frac{v}{u}$, so that $v = pu$ and thus $\frac{dv}{du} = p + u\frac{dp}{du}$.

Substituting gives $p + u\frac{dp}{du} = \frac{2p-1}{4-3p}$, or

$$u\frac{dp}{du} = \frac{2p-1}{4-3p} - \frac{4p-3p^2}{4-3p} = \frac{3p^2-2p-1}{4-3p}.$$

Separating variables yields $\int \frac{4-3p}{3p^2-2p-1}\,dp = \int \frac{1}{u}\,du$. Now the method of partial fractions gives

$$\int \frac{4-3p}{3p^2-2p-1}dp = \frac{1}{4}\int \frac{1}{p-1} - \frac{15}{3p+1}dp = \frac{1}{4}\left[\ln(p-1) - 5\ln(3p+1) + \ln C\right],$$

so the solution is given by

$$\ln(p-1) - 5\ln(3p+1) + \ln C = 4\ln u,$$

or $u^4 = \dfrac{C(p-1)}{(3p+1)^5}$. Back-substituting $\dfrac{v}{u}$ for p gives

$$u^4 = \frac{C\left(\frac{v}{u}-1\right)}{\left(3\frac{v}{u}+1\right)^5} = \frac{Cu^4(v-u)}{(3v+u)^5},$$

or $(3v+u)^5 = C(v-u)$, and finally, back-substituting $x-3$ for u and $y+2$ for v yields the implicit solution

$$(3y+x+3)^5 = C(y-x+5).$$

61. The expression $x-y$ appearing on the right-hand side suggests that we try the substitution $v = x-y$, which implies that $y = x-v$, and thus that $\dfrac{dy}{dx} = 1-\dfrac{dv}{dx}$. This gives the separable equation $1-\dfrac{dv}{dx} = \sin v$, or $\dfrac{dv}{dx} = 1-\sin v$. Separating variables leads to $\int \dfrac{1}{1-\sin v}dv = \int dx$. The left-hand integral is carried out with the help of the trigonometric identities

$$\frac{1}{1-\sin v} = \frac{1+\sin v}{\cos^2 v} = \sec^2 v + \sec v \tan v;$$

the solution is given by $\int \sec^2 v + \sec v \tan v\, dv = \int dx$, or $x = \tan v + \sec v + C$. Finally, back-substituting $x-y$ for v gives the implicit solution $x = \tan(x-y) + \sec(x-y) + C$. However, for no value of the constant C does this general solution include the "basic" solution $y(x) = x - \dfrac{\pi}{2}$. The reason is that for this solution, $v = x-y$ is the constant $\dfrac{\pi}{2}$, so that the expression $1-\sin v$ (by which we divided above) is identically zero. Thus the solution $y(x) = x-\dfrac{\pi}{2}$ is singular for this solution procedure.

62. First we note that the given differential equation is homogeneous; for $x > 0$ we have

$$\frac{dy}{dx} = -\frac{y\left(2x^3 - y^3\right)}{x\left(2y^3 - x^3\right)} = -\frac{y}{x}\cdot\frac{2 - \left(\frac{y}{x}\right)^3}{2\left(\frac{y}{x}\right)^3 - 1},$$

and substituting $v = \frac{y}{x}$ as usual leads to

$$v + x\frac{dv}{dx} = -v\cdot\frac{2 - v^3}{2v^3 - 1} = \frac{v^4 - 2v}{2v^3 - 1},$$

or

$$x\frac{dv}{dx} = \frac{v^4 - 2v}{2v^3 - 1} - v\frac{2v^3 - 1}{2v^3 - 1} = \frac{-v^4 - v}{2v^3 - 1},$$

or $\int \frac{2v^3 - 1}{v^4 + v}\,dv = -\int \frac{1}{x}\,dx$, after separating variables. By the method of partial fractions,

$$\int \frac{2v^3 - 1}{v^4 + v}\,dv = \int \frac{2v - 1}{v^2 - v + 1} - \frac{1}{v} + \frac{1}{v+1}\,dv = \ln\left(v^2 - v + 1\right) - \ln v + \ln\left(v + 1\right),$$

yielding the implicit solution

$$\ln\left(v^2 - v + 1\right) - \ln v + \ln\left(v + 1\right) = -\ln x + C,$$

or $x\left(v^2 - v + 1\right)\left(v + 1\right) = Cv$, that is, $x\left(v^3 + 1\right) = Cv$. Finally, back-substituting $\frac{y}{x}$ for v gives the solution $x\left[\left(\frac{y}{x}\right)^3 + 1\right] = C\frac{y}{x}$, or $x^3 + y^3 = Cxy$.

63. The substitution $y = y_1 + \frac{1}{v}$, which implies that $\frac{dy}{dx} = y_1' - \frac{1}{v^2}\frac{dv}{dx}$, gives

$$y_1' - \frac{1}{v^2}\frac{dv}{dx} = A(x)\left(y_1 + \frac{1}{v}\right)^2 + B(x)\left(y_1 + \frac{1}{v}\right) + C(x),$$

which upon expanding becomes

$$\begin{aligned} \underline{y_1'} - \frac{1}{v^2}\frac{dv}{dx} &= A(x)\left(y_1^2 + 2\frac{y_1}{v} + \frac{1}{v^2}\right) + B(x)y_1 + B(x)\frac{1}{v} + C(x) \\ &= \underline{A(x)y_1^2 + B(x)y_1 + C(x)} + A(x)\left(2\frac{y_1}{v} + \frac{1}{v^2}\right) + B(x)\frac{1}{v}. \end{aligned}$$

The underlined terms cancel because y_1 is a solution of the given equation $\frac{dy}{dx} = A(x)y^2 + B(x)y + C(x)$, resulting in

$$-\frac{1}{v^2}\frac{dv}{dx} = A(x)\left(2\frac{y_1}{v} + \frac{1}{v^2}\right) + B(x)\frac{1}{v},$$

or $\dfrac{dv}{dx} = -A(x)(2vy_1 + 1) - B(x)v$, that is, $\dfrac{dv}{dx} + (B + 2Ay_1)v = -A$, a linear equation for v as a function of x.

In Problems 64 and 65 we outline the application of the method of Problem 63 to the given Riccati equation.

64. Here $A(x) = -1$, $B(x) = 0$, and $C(x) = 1 + x^2$. Thus the substitution $y = y_1 + \dfrac{1}{v} = x + \dfrac{1}{v}$ leads to the linear equation $\dfrac{dv}{dx} - 2xv = 1$. An integrating factor is given by $\rho = \exp\left(\int -2x\,dx\right) = e^{-x^2}$, and multiplying by ρ gives $e^{-x^2} \cdot \dfrac{dv}{dx} - 2xe^{-x^2}v = e^{-x^2}$, or $D_x\left(e^{-x^2} \cdot v\right) = e^{-x^2}$. In Problem 29 of Section 1.5 we saw that the general solution of this linear equation is $v(x) = e^{x^2}\left[C + \dfrac{\sqrt{\pi}}{2}\operatorname{erf}(x)\right]$, expressed in terms of the *error function* $\operatorname{erf}(x)$ introduced there. Hence the general solution of our Riccati equation is given by

$$y(x) = x + e^{-x^2}\left[C + \frac{\sqrt{\pi}}{2}\operatorname{erf}(x)\right]^{-1}.$$

65. Here $A(x) = 1$, $B(x) = -2x$, and $C(x) = 1 + x^2$. Thus the substitution $y = y_1 + \dfrac{1}{v} = x + \dfrac{1}{v}$ yields the trivial linear equation $\dfrac{dv}{dx} = -1$, with immediate solution $v(x) = C - x$. Hence the general solution of our Riccati equation is given by

$$y(x) = x + \frac{1}{C - x}.$$

66. Substituting $y(x) = Cx + g(C)$ into the given differential equation leads to $Cx + g(C) = Cx + g(C)$, a true statement. Thus the one-parameter family $y(x) = Cx + g(C)$ is a general solution of the equation.

67. First, the line $y = Cx - \dfrac{1}{4}C^2$ has slope C and passes through the point $\left(\tfrac{1}{2}C, \tfrac{1}{4}C^2\right)$; the same is true of the parabola $y = x^2$ at the point $\left(\tfrac{1}{2}C, \tfrac{1}{4}C^2\right)$, because $\dfrac{dy}{dx} = 2x = 2 \cdot \tfrac{1}{2}C = C$. Thus the line is tangent to the parabola at this point. It follows that $y = x^2$ is in fact a solution to the differential equation, since for each x, the parabola has the same values of y and y' as the known solution $y = Cx - \frac{1}{4}C^2$. Finally, $y = x^2$ is a singular solution with respect to the general solution $y = Cx - \frac{1}{4}C^2$, since for no value of C does $Cx - \frac{1}{4}C^2$ equal x^2 for all x.

68. Substituting $C = k\ln a$ into $\ln\left(v + \sqrt{1+v^2}\right) = -k\ln x + C$ gives

$$\ln\left(v + \sqrt{1+v^2}\right) = -k\ln x + k\ln a = \ln\left(x/a\right)^{-k},$$

or $v + \sqrt{1+v^2} = \left(x/a\right)^{-k}$, or $\left[\left(x/a\right)^{-k} - v\right]^2 = 1 + v^2$, or $\left(x/a\right)^{-2k} - 2v\left(x/a\right)^{-k} + v^2 = 1 + v^2$, that is

$$v = \frac{1}{2}\left[\left(\frac{x}{a}\right)^{-2k} - 1\right]\Big/\left(\frac{x}{a}\right)^{-k} = \frac{1}{2}\left[\left(\frac{x}{a}\right)^{-k} - \left(\frac{x}{a}\right)^{k}\right].$$

69. With $a = 100$ and $k = \frac{1}{10}$, Equation (19) in the text is $y = 50\left[\left(\frac{x}{100}\right)^{9/10} - \left(\frac{x}{100}\right)^{11/10}\right]$. We find the maximum northward displacement of plane by setting

$$y'(x) = 50\left[\frac{9}{10}\left(\frac{x}{100}\right)^{-1/10} - \frac{11}{10}\left(\frac{x}{100}\right)^{1/10}\right] = 0,$$

which yields $\left(\frac{x}{100}\right)^{1/10} = \left(\frac{9}{11}\right)^{1/2}$. Because

$$y''(x) = 50\left[\frac{-9}{100}\left(\frac{x}{100}\right)^{-11/10} - \frac{11}{100}\left(\frac{x}{100}\right)^{-9/10}\right] < 0$$

for all x, this critical point in fact represents the absolute maximum value of y. Substituting this value of x into $y(x)$ gives $y_{\max} = 50\left[\left(\frac{9}{11}\right)^{9/2} - \left(\frac{9}{11}\right)^{11/2}\right] \approx 3.68\,\text{mi}$.

70. With $k = \frac{w}{v_0} = \frac{10}{500} = \frac{1}{10}$, Equation (16) in the text gives $\ln\left(v + \sqrt{1+v^2}\right) = -\frac{1}{10}\ln x + C$, where v denotes $\frac{y}{x}$. Substitution of $x = 200$, $y = 150$, and $v = \frac{3}{4}$ yields $C = \ln\left(2 \cdot 200^{1/10}\right)$, which gives

$$\ln\left(\frac{y}{x}+\sqrt{1+\frac{y^2}{x^2}}\right)=-\frac{1}{10}\ln x+\ln\left(2\cdot 200^{1/10}\right).$$

Exponentiation and then multiplication of the resulting equation by x finally leads to $y+\sqrt{x^2+y^2}=2\left(200x^9\right)^{1/10}$, as desired. Note that if $x=0$, then this equation yields $y=0$, confirming that the airplane reaches the airport at the origin.

71. Equations (12)-(19) apply to this situation as with the airplane in flight.

(a) With $a=100$ and $k=\frac{w}{v_0}=\frac{2}{4}=\frac{1}{2}$, the solution given by Equation (19) is $y=50\left[\left(\frac{x}{100}\right)^{1/2}-\left(\frac{x}{100}\right)^{3/2}\right]$. The fact that $y(0)=0$ means that this trajectory goes through the origin where the tree is located.

(b) With $k=\frac{4}{4}=1$, the solution is $y=50\left[1-\left(\frac{x}{100}\right)^2\right]$, and we see that the dog hits the bank at a distance $y(0)=50\,\text{ft}$ north of the tree.

(c) With $k=\frac{6}{4}=\frac{3}{2}$, the solution is $y=50\left[\left(\frac{x}{100}\right)^{-1/2}-\left(\frac{x}{100}\right)^{5/2}\right]$. This trajectory is asymptotic to the positive x-axis, so we see that the dog never reaches the west bank of the river.

72. We note that the dependent variable y is missing in the given differential equation $ry''=\left[1+(y')^2\right]^{3/2}$, leading us to substitute $y'=\rho$, and $y''=\rho'$. This results in $rp'=\left(1+\rho^2\right)^{3/2}$, a separable first-order differential equation for ρ as a function of x. Separating variables gives $\int\frac{r}{\left(1+\rho^2\right)^{3/2}}d\rho=\int dx$, and then integral formula #52 in the back of our favorite calculus textbook gives $\frac{r\rho}{\sqrt{1+\rho^2}}=x-a$, that is, $r^2\rho^2=\left(1+\rho^2\right)(x-a)^2$. We solve readily for $\rho^2=\frac{(x-a)^2}{r^2-(x-a)^2}$, so that $\frac{dy}{dx}=\rho^2=\sqrt{\frac{x-a}{r^2-(x-a)^2}}$. Finally, a further integration gives

$$y=\int\frac{(x-a)}{\sqrt{r^2-(x-a)^2}}dx=-\sqrt{r^2-(x-a)^2}+b,$$

which leads to $(x-a)^2+(y-b)^2=r^2$, as desired.

CHAPTER 1 Review Problems

The main objective of this set of review problems is practice in the identification of the different types of first-order differential equations discussed in this chapter. In each of Problems 1–36 we identify the type of the given equation and indicate one or more appropriate method(s) of solution.

1. We first rewrite the differential equation for $x>0$ as $y'-\frac{3}{x}y=x^2$, showing that the equation is *linear*. An integrating factor is given by $\rho=\exp\left(-\int\frac{3}{x}dx\right)=e^{-3\ln x}=x^{-3}$, and multiplying the equation by ρ gives $x^{-3}\cdot y'-3x^{-4}y=x^{-1}$, or $D_x\left(x^{-3}\cdot y\right)=x^{-1}$. Integrating then leads to $x^{-3}\cdot y=\ln x+C$, and thus to the general solution $y=x^3\left(\ln x+C\right)$.

2. We first rewrite the differential equation for $x,y>0$ as $\frac{y'}{y^2}=\frac{x+3}{x^2}$, showing that the equation is *separable*. Separating variables yields $-\frac{1}{y}=\ln x-\frac{3}{x}+C$, and thus the general solution $y=\frac{-1}{\ln x-\frac{3}{x}+C}=\frac{x}{3-x\left(C+\ln x\right)}$.

3. Rewriting the differential equation for $x\neq 0$ as $y'=\frac{xy+y^2}{x^2}=\frac{y}{x}+\left(\frac{y}{x}\right)^2$ shows that the equation is *homogeneous*. Actually the equation is identical to Problem 9 in Section 1.6; the general solution found there is $y=\frac{x}{C-\ln|x|}$.

4. Rewriting the differential equation in differential form gives

$$M\,dx+N\,dy=\left(2xy^3+e^x\right)dx+\left(3x^2y^2+\sin y\right)dy=0,$$

and because $\frac{\partial}{\partial y}M(x,y)=6xy^2=\frac{\partial}{\partial x}N(x,y)$, the given equation is *exact*. Thus we apply the method of Example 9 in Section 1.6 to find a solution of the form $F(x,y)=C$. First, the condition $F_x=M$ implies that $F(x,y)=\int 2xy^3+e^x\,dx=x^2y^3+e^x+g(y)$, and then the condition $F_y = N$ implies that $3x^2y^2 + g'(y) = 3x^2y^2 + \sin y$, or $g'(y) = \sin y$, or $g(y) = -\cos y$. Thus the solution is given by $x^2y^3 + e^x - \cos y = C$.

5. We first rewrite the differential equation for $x, y \neq 0$ as $\dfrac{y'}{y} = \dfrac{2x-3}{x^4}$, showing that the equation is *separable*. Separating variables yields $\int \dfrac{1}{y}\,dy = \int \dfrac{2x-3}{x^4}\,dx$, or $\ln|y| = -\dfrac{1}{x^2} + \dfrac{1}{x^3} + C = \dfrac{1-x}{x^3} + C$, leading to the general solution $y = C\exp\left(\dfrac{1-x}{x^3}\right)$, where C is an arbitrary nonzero constant.

6. We first rewrite the differential equation for $x, y > 0$ as $\dfrac{y'}{y^2} = \dfrac{1-2x}{x^2}$, showing that the equation is *separable*. Separating variables yields $\int \dfrac{1}{y^2}\,dy = \int \dfrac{1-2x}{x^2}\,dx$, or $-\dfrac{1}{y} = -\dfrac{1}{x} - 2\ln x + C$, that is $\dfrac{1}{y} = \dfrac{1 + 2x\ln x + Cx}{x}$, leading to the general solution $y = \dfrac{x}{1 + 2x\ln x + Cx}$.

7. We first rewrite the differential equation for $x > 0$ as $y' + \dfrac{2}{x}y = \dfrac{1}{x^3}$, showing that the equation is *linear*. An integrating factor is given by $\rho = \exp\left(\int \dfrac{2}{x}\,dx\right) = e^{2\ln x} = x^2$, and multiplying the equation by ρ gives $x^2 \cdot y' + 2xy = \dfrac{1}{x}$, or $D_x\left(x^2 \cdot y\right) = \dfrac{1}{x}$. Integrating then leads to $x^2 \cdot y = \ln x + C$, and thus to the general solution $y = \dfrac{\ln x + C}{x^2}$.

8. We first rewrite the differential equation for $x \neq 0$ as $\dfrac{dy}{dx} = \left(\dfrac{y}{x}\right)^2 - 2\dfrac{y}{x}$, showing that is it *homogeneous*. Substituting $v = \dfrac{y}{x}$ then gives $v + x\dfrac{dv}{dx} = v^2 - 2v$, or $x\dfrac{dv}{dx} = v^2 - 3v$. Separating variables leads to $\int \dfrac{1}{v^2 - 3v}\,dv = \int \dfrac{1}{x}\,dx$. Upon partial fraction decomposition the solution takes the form $-\dfrac{1}{3}\ln|v| + \dfrac{1}{3}\ln|v-3| = \ln|x| + C$, or $\left|\dfrac{v-3}{v}\right| = C|x|^3$, where C is an arbitrary positive constant, or $v - 3 = Cvx^3$, where C is an arbitrary nonzero constant.

Back-substituting $\frac{y}{x}$ for v then gives the solution $\frac{y}{x} - 3 = C\frac{y}{x}x^3$, or $y - 3x = Cyx^3$, or finally $y = \frac{3x}{1 + Cx^3}$.

Alternatively, writing the given equation as $\frac{dy}{dx} + \frac{2}{x}y = \frac{1}{x^2}y^2$ shows that it is a *Bernoulli* equation with $n = 2$. The substitution $v = y^{1-2} = y^{-1}$ implies that $y = v^{-1}$ and thus that $y' = -v^{-2}v'$. Substituting gives $-v^{-2}v' + \frac{2}{x}v^{-1} = \frac{1}{x^2}v^{-2}$, or $v' - \frac{2}{x}v = -\frac{1}{x^2}$, a linear equation for v as a function of x. An integrating factor is given by $\rho = \exp\left(\int -\frac{2}{x}dx\right) = x^{-2}$, and multiplying the differential equation by ρ gives $\frac{1}{x^2}v' - \frac{2}{x^3}v = -\frac{1}{x^4}$, or $D_x\left(\frac{1}{x^2} \cdot v\right) = -\frac{1}{x^4}$. Integrating then leads to $\frac{1}{x^2} \cdot v = \frac{1}{3x^3} + C$, or $v = \frac{1}{3x} + Cx^2$. Finally, back-substituting y^{-1} for v gives the general solution $\frac{1}{y} = \frac{1}{3x} + Cx^2$, or $y = \frac{1}{\frac{1}{3x} + Cx^2} = \frac{3x}{1 + Cx^3}$, as found above.

9. We first rewrite the differential equation for $x, y > 0$ as $y' + \frac{2}{x}y = 6xy^{1/2}$, showing that it is a *Bernoulli* equation with $n = 1/2$. The substitution $v = y^{1/2}$ implies that $y = v^2$ and thus that $y' = 2vv'$. Substituting gives $2vv' + \frac{2}{x}v^2 = 6xv$, or $v' + \frac{1}{x}v = 3x$, a linear equation for v as a function of x. An integrating factor is given by $\rho = \exp\left(\int \frac{1}{x}dx\right) = x$, and multiplying the differential equation by ρ gives $xv' + v = 3x^2$, or $D_x(xv) = 3x^2$. Integrating then leads to $xv = x^3 + C$, or $v = x^2 + \frac{C}{x}$. Finally, back-substituting $y^{1/2}$ for v gives the general solution $y^{1/2} = x^2 + \frac{C}{x}$, or $y = \left(x^2 + \frac{C}{x}\right)^2$.

10. Factoring the right-hand side gives $\frac{dy}{dx} = (1+x^2)(1+y^2)$, showing that the equation is *separable*. Separating variables gives $\int \frac{1}{1+y^2}\,dy = \int 1+x^2\,dx$, or $\tan^{-1} y = x + \frac{1}{3}x^3 + C$, or finally $y = \tan\left(x + \frac{1}{3}x^3 + C\right)$.

11. We first rewrite the differential equation for $x, y > 0$ as $\frac{dy}{dx} = \frac{y}{x} + 3\left(\frac{y}{x}\right)^2$, showing that it is *homogeneous*. Substituting $v = \frac{y}{x}$ then gives $v + x\frac{dv}{dx} = v + 3v^2$, or $x\frac{dv}{dx} = 3v^2$. Separating variables leads to $\int \frac{1}{v^2}\,dv = \int \frac{3}{x}\,dx$, or $-\frac{1}{v} = 3\ln x + C$, or $v = \frac{1}{C - 3\ln x}$. Back-substituting $\frac{y}{x}$ for *v* then gives the solution $\frac{y}{x} = \frac{1}{C - 3\ln x}$, or $y = \frac{x}{C - 3\ln x}$.

Alternatively, writing the equation in the form $y' - \frac{1}{x}y = \frac{3}{x^2}y^2$ for $x, y > 0$ shows that it is also a *Bernoulli* equation with $n = 2$. The substitution $v = y^{-1}$ implies that $y = v^{-1}$ and thus that $y' = -v^{-2}v'$. Substituting gives $-v^{-2}v' - \frac{1}{x}v^{-1} = \frac{3}{x^2}v^{-2}$, or $v' + \frac{1}{x}v = -\frac{3}{x^2}$, a linear equation for *v* as a function of *x*. An integrating factor is given by $\rho = \exp\left(\int \frac{1}{x}\,dx\right) = x$, and multiplying the differential equation by ρ gives $xv' + v = -\frac{3}{x}$, or $D_x(xv) = -\frac{3}{x}$. Integrating then leads to $xv = -3\ln x + C$, or $v = \frac{-3\ln x + C}{x}$. Finally, back-substituting y^{-1} for *v* gives the same general solution as found above.

12. Rewriting the differential equation in differential form gives

$$(6xy^3 + 2y^4)\,dx + (9x^2y^2 + 8xy^3)\,dy = 0,$$

and because $\frac{\partial}{\partial y}(6xy^3 + 2y^4) = 18xy^2 + 8y^3 = \frac{\partial}{\partial x}(9x^2y^2 + 8xy^3)$, the given equation is *exact*. We apply the method of Example 9 in Section 1.6 to find a solution in the form $F(x,y) = C$. First, the condition $F_x = M$ implies that

$$F(x,y) = \int 6xy^3 + 2y^4\,dx = 3x^2y^3 + 2xy^4 + g(y),$$

and then the condition $F_y = N$ implies that $9x^2y^2 + 8xy^3 + g'(y) = 9x^2y^2 + 8xy^3$, or $g'(y) = 0$, that is, $g(y)$ is constant. Thus the solution is given by $3x^2y^3 + 2xy^4 = C$.

13. We first rewrite the differential equation for $y > 0$ as $\dfrac{y'}{y^2} = 5x^4 - 4x$, showing that the equation is *separable*. Separating variables yields $\int \dfrac{1}{y^2}\,dy = \int 5x^4 - 4x\,dx$, or $-\dfrac{1}{y} = x^5 - 2x^2 + C$, leading to the general solution $y = \dfrac{1}{C + 2x^2 - x^5}$.

14. We first rewrite the differential equation for $x, y > 0$ as $\dfrac{dy}{dx} = \dfrac{y}{x} - \left(\dfrac{y}{x}\right)^3$, showing that it is *homogeneous*. Substituting $v = \dfrac{y}{x}$ then gives $v + x\dfrac{dv}{dx} = v - v^3$, or $x\dfrac{dv}{dx} = -v^3$. Separating variables leads to $\int \dfrac{1}{v^3}\,dv = \int -\dfrac{1}{x}\,dx$, or $\dfrac{1}{2v^2} = \ln x + C$, or $v^2 = \dfrac{1}{C + 2\ln x}$. Back-substituting $\dfrac{y}{x}$ for v then gives the solution $\left(\dfrac{y}{x}\right)^2 = \dfrac{1}{C + 2\ln x}$, or $y^2 = \dfrac{x^2}{C + 2\ln x}$.

Alternatively, writing the equation in the form $y' - \dfrac{1}{x}y = -\dfrac{1}{x^3}y^3$ for $x, y > 0$ shows that it is also a *Bernoulli* equation with $n = 3$. The substitution $v = y^{-2}$ implies that $y = v^{-1/2}$ and thus that $y' = -\dfrac{1}{2}v^{-3/2}v'$. Substituting gives $-\dfrac{1}{2}v^{-3/2}v' - \dfrac{1}{x}v^{-1/2} = -\dfrac{1}{x^3}v^{-3/2}$, or $v' + \dfrac{2}{x}v = \dfrac{2}{x^3}$, a linear equation for v as a function of x. An integrating factor is given by $\rho = \exp\left(\int \dfrac{2}{x}\,dx\right) = x^2$, and multiplying the differential equation by ρ gives $x^2 \cdot v' + 2x \cdot v = \dfrac{2}{x}$, or $D_x\left(x^2 \cdot v\right) = \dfrac{2}{x}$. Integrating then leads to $x^2 v = 2\ln x + C$, or $v = \dfrac{2\ln x + C}{x^2}$. Finally, back-substituting y^{-2} for v gives the same general solution as found above.

15. This is a *linear* differential equation. An integrating factor is given by $\rho = \exp\left(\int 3\,dx\right) = e^{3x}$, and multiplying the equation by ρ gives $e^{3x} \cdot y' + 3e^{3x}y = 3x^2$, or $D_x\left(e^{3x} \cdot y\right) = 3x^2$. Integrating then leads to $e^{3x} \cdot y = x^3 + C$, and thus to the general solution $y = \left(x^3 + C\right)e^{-3x}$.

16. Rewriting the differential equation as $y' = (x - y)^2$ suggests the substitution $v = x - y$, which implies that $y = x - v$, and thus that $y' = 1 - v'$. Substituting gives $1 - v' = v^2$, or $v' = 1 - v^2$, a separable equation for v as a function of x. Separating variables gives

$\int \frac{1}{1-v^2}\,dv = \int dx$, or (*via* the method of partial fractions) $\frac{1}{2}\left(\ln|1+v| - \ln|1-v|\right) = x + C$, or $\ln\left|\frac{1+v}{1-v}\right| = 2x + C$, or $1+v = Ce^{2x}(1-v)$. Finally, back-substituting $x-y$ for *v* gives the implicit solution $1+x-y = Ce^{2x}(1-x+y)$.

17. Rewriting the differential equation in differential form gives

$$\left(e^x + ye^{xy}\right)dx + \left(e^y + xe^{xy}\right)dy = 0,$$

and because $\frac{\partial}{\partial y}\left(e^x + ye^{xy}\right) = xye^{xy} = \frac{\partial}{\partial x}\left(e^y + xe^{xy}\right)$, the given equation is *exact*. We apply the method of Example 9 in Section 1.6 to find a solution in the form $F(x,y) = C$. First, the condition $F_x = M$ implies that

$$F(x,y) = \int e^x + ye^{xy}\,dx = e^x + e^{xy} + g(y),$$

and then the condition $F_y = N$ implies that $xe^{xy} + g'(y) = e^y + xe^{xy}$, or $g'(y) = e^y$, or $g(y) = e^y$. Thus the solution is given by $e^x + e^{xy} + e^y = C$.

18. We first rewrite the differential equation for $x, y > 0$ as $\frac{dy}{dx} = 2\frac{y}{x} - \left(\frac{y}{x}\right)^3$, showing that it is *homogeneous*. Substituting $v = \frac{y}{x}$ then gives $v + x\frac{dv}{dx} = 2v - v^3$, or $x\frac{dv}{dx} = v - v^3$. Separating variables leads to $\int \frac{1}{v-v^3}\,dv = \int \frac{1}{x}\,dx$, or (after decomposing into partial fractions) $\int \frac{2}{v} + \frac{1}{1-v} - \frac{1}{1+v}\,dv = \int \frac{2}{x}\,dx$, or $\ln\left|\frac{v^2}{(1-v)(1+v)}\right| = 2\ln|x| + C$, or $\frac{v^2}{(1-v)(1+v)} = Cx^2$, or $v^2 = Cx^2(1-v)(1+v)$. Back-substituting $\frac{y}{x}$ for *v* then gives the solution $\left(\frac{y}{x}\right)^2 = Cx^2\left(1-\frac{y}{x}\right)\left(1+\frac{y}{x}\right)$, or finally $y^2 = Cx^2(x-y)(x+y) = Cx^2\left(x^2-y^2\right)$.

Alternatively, rewriting the differential equation for $x > 0$ as $y' - \frac{2}{x}y = -\frac{1}{x^3}y^3$ shows that it is *Bernoulli* with $n = 3$. The substitution $v = y^{1-3} = y^{-2}$ implies that $y = v^{-1/2}$, and thus that $y' = -\frac{1}{2}v^{-3/2}v'$. Substituting gives $-\frac{1}{2}v^{-3/2}v' - \frac{2}{x}v^{-1/2} = -\frac{1}{x^3}v^{-3/2}$, or $v' + \frac{4}{x}v = \frac{2}{x^3}$, a linear equation for *v* as a function of *x*. An integrating factor is given by

$\rho = \exp\left(\int \frac{4}{x}dx\right) = x^4$, and multiplying the differential equation by ρ gives $x^4 \cdot v' + 4x^3 v = 2x$, or $D_x\left(x^4 \cdot v\right) = 2x$. Integrating then leads to $x^4 \cdot v = x^2 + C$, or $v = \frac{1}{x^4}\left(x^2 + C\right)$. Finally, back-substituting y^{-2} for v gives the general solution $\frac{1}{y^2} = \frac{1}{x^4}\left(x^2 + C\right) = \frac{1}{x^2} + \frac{C}{x^4}$, or $C = x^4\left(\frac{1}{y^2} - \frac{1}{x^2}\right) = \frac{x^2}{y^2}\left(x^2 - y^2\right)$, or (for $C \neq 0$) $y^2 = \frac{1}{C}x^2\left(x^2 - y^2\right) = Cx^2\left(x^2 - y^2\right)$, the same general solution found above. (Note that the case $C = 0$ in this latter solution corresponds to the solutions $y = \pm x$, which are singular for the first solution method, since they cause $v - v^3$ to equal zero.)

19. We first rewrite the differential equation for $x, y \neq 0$ as $\frac{y'}{y^2} = 2x^{-3} - 3x^2$, showing that the equation is *separable*. Separating variables yields $\int \frac{1}{y^2}dy = \int 2x^{-3} - 3x^2\,dx$, or $-\frac{1}{y} = -x^{-2} - x^3 + C$, leading to the general solution $y = \frac{1}{x^{-2} + x^3 + C} = \frac{x^2}{x^5 + Cx^2 + 1}$.

20. We first rewrite the differential equation for $x > 0$ as $y' + \frac{3}{x}y = 3x^{-5/2}$, showing that the equation is *linear*. An integrating factor is given by $\rho = \exp\left(\int \frac{3}{x}dx\right) = e^{3\ln x} = x^3$, and multiplying the equation by ρ gives $x^3 \cdot y' + 3x^2 y = 3x^{1/2}$, or $D_x\left(x^3 \cdot y\right) = 3x^{1/2}$. Integrating then leads to $x^3 \cdot y = 2x^{3/2} + C$, and thus to the general solution $y = 2x^{-3/2} + Cx^{-3}$.

21. We first rewrite the differential equation for $x > 1$ as $y' + \frac{1}{x+1}y = \frac{1}{x^2 - 1}$, showing that the equation is *linear*. An integrating factor is given by $\rho = \exp\left(\int \frac{1}{x+1}dx\right) = x + 1$, and multiplying the equation by ρ gives $(x+1)y' + y = \frac{1}{x-1}$, or $D_x\left[(x+1)y\right] = \frac{1}{x-1}$. Integrating then leads to $(x+1)y = \ln(x-1) + C$, and thus to the general solution $y = \frac{1}{x+1}\left[\ln(x-1) + C\right]$.

22. Writing the given equation for $x > 0$ as $\frac{dy}{dx} - \frac{6}{x}y = 12x^3y^{2/3}$ shows that it is a *Bernoulli* equation with $n = 2/3$. The substitution $v = y^{1-2/3} = y^{1/3}$ implies that $y = v^3$ and thus that $y' = 3v^2v'$. Substituting gives $3v^2v' - \frac{6}{x}v^3 = 12x^3v^2$, or $v' - \frac{2}{x}v = 4x^3$, a linear equation for *v* as a function of *x*. An integrating factor is given by $\rho = \exp\left(\int -\frac{2}{x}dx\right) = x^{-2}$, and multiplying the differential equation by ρ gives $x^{-2} \cdot v' - 2x^{-3}v = 4x$, or $D_x\left(x^{-2} \cdot v\right) = 4x$. Integrating then leads to $x^{-2} \cdot v = 2x^2 + C$, or $v = 2x^4 + Cx^2$. Finally, back-substituting $y^{1/3}$ for *v* gives the general solution $y^{1/3} = 2x^4 + Cx^2$, or $y = \left(2x^4 + Cx^2\right)^3$.

23. Rewriting the differential equation in differential form gives

$$\left(e^y + y\cos x\right)dx + \left(xe^y + \sin x\right)dy = 0,$$

and because $\frac{\partial}{\partial y}\left(e^y + y\cos x\right) = e^y + \cos x = \frac{\partial}{\partial x}\left(xe^y + \sin x\right)$, the given equation is exact. We apply the method of Example 9 in Section 1.6 to find a solution in the form $F(x, y) = C$. First, the condition $F_x = M$ implies that

$$F(x, y) = \int e^y + y\cos x\,dx = xe^y + y\sin x + g(y),$$

and then the condition $F_y = N$ implies that $xe^y + \sin x + g'(y) = xe^y + \sin x$, or $g'(y) = 0$, that is, *g* is constant. Thus the solution is given by $xe^y + y\sin x = C$.

24. We first rewrite the differential equation for $x, y > 0$ as $\frac{y'}{y^2} = x^{-3/2} - 9x^{1/2}$, showing that the equation is *separable*. Separating variables yields $\int \frac{1}{y^2}dy = \int x^{-3/2} - 9x^{1/2}\,dx$, or $-\frac{1}{y} = -2x^{-1/2} - 6x^{3/2} + C$, leading to the general solution $y = \frac{x^{1/2}}{6x^2 + Cx^{1/2} + 2}$.

25. We first rewrite the differential equation for $x > -1$ as $y' + \frac{2}{x+1}y = 3$, showing that the equation is *linear*. An integrating factor is given by $\rho = \exp\left(\int \frac{2}{x+1}dx\right) = (x+1)^2$, and multiplying the equation by ρ gives $(x+1)^2 y' + 2(x+1)y = 3(x+1)^2$, or

$D_x\left[(x+1)^2\cdot y\right]=3(x+1)^2$. Integrating then leads to $(x+1)^2\cdot y=(x+1)^3+C$, and thus to the general solution $y=x+1+\dfrac{C}{(x+1)^2}$.

26. Rewriting the differential equation in differential form gives

$$\left(9x^{1/2}y^{4/3}-12x^{1/5}y^{3/2}\right)dx+\left(8x^{3/2}y^{1/3}-15x^{6/5}y^{1/2}\right)dy=0,$$

and because

$$\frac{\partial}{\partial y}\left(9x^{1/2}y^{4/3}-12x^{1/5}y^{3/2}\right)=12x^{1/2}y^{4/3}-18x^{1/5}y^{1/2}=\frac{\partial}{\partial x}\left(8x^{3/2}y^{1/3}-15x^{6/5}y^{1/2}\right),$$

the given equation is *exact*. We apply the method of Example 9 in Section 1.6 to find a solution in the form $F(x,y)=C$. First, the condition $F_x=M$ implies that

$$F(x,y)=\int 9x^{1/2}y^{4/3}-12x^{1/5}y^{3/2}\,dx=6x^{3/2}y^{4/3}-10x^{6/5}y^{3/2}+g(y),$$

and then the condition $F_y=N$ implies that

$$8x^{3/2}y^{1/3}-15x^{6/5}y^{1/2}+g'(y)=8x^{3/2}y^{1/3}-15x^{6/5}y^{1/2},$$

or $g'(y)=0$, that is, g is constant. Thus the solution is given by $6x^{3/2}y^{4/3}-10x^{6/5}y^{3/2}=C$.

27. Writing the given equation for $x>0$ as $\dfrac{dy}{dx}+\dfrac{1}{x}y=-\dfrac{x^2}{3}y^4$ shows that it is a *Bernoulli* equation with $n=4$. The substitution $v=y^{-3}$ implies that $y=v^{-1/3}$ and thus that $y'=-\dfrac{1}{3}v^{-4/3}v'$. Substituting gives $-\dfrac{1}{3}v^{-4/3}v'+\dfrac{1}{x}v^{-1/3}=-\dfrac{x^2}{3}v^{-4/3}$, or $v'-\dfrac{3}{x}v=x^2$, a linear equation for v as a function of x. An integrating factor is given by $\rho=\exp\left(\int-\dfrac{3}{x}dx\right)=x^{-3}$, and multiplying the differential equation by ρ gives $x^{-3}\cdot v'-3x^{-4}v=x^{-1}$, or $D_x\left(x^{-3}\cdot v\right)=x^{-1}$. Integrating then leads to $x^{-3}\cdot v=\ln x+C$, or $v=x^3(\ln x+C)$. Finally, back-substituting y^{-3} for v gives the general solution $y=x^{-1}(\ln x+C)^{-1/3}$.

28. We first rewrite the differential equation for $x>0$ as $y'+\dfrac{1}{x}y=\dfrac{2e^{2x}}{x}$, showing that the equation is *linear*. An integrating factor is given by $\rho=\exp\left(\int\dfrac{1}{x}dx\right)=x$, and multiplying the equation by ρ gives $x \cdot y' + y = 2e^{2x}$, or $D_x(x \cdot y) = 2e^{2x}$. Integrating then leads to $x \cdot y = e^{2x} + C$, and thus to the general solution $y = x^{-1}\left(e^{2x} + C\right)$.

29. We first rewrite the differential equation for $x > -\frac{1}{2}$ as $y' + \frac{1}{2x+1} y = (2x+1)^{1/2}$, showing that the equation is linear. An integrating factor is given by
$\rho = \exp\left(\int \frac{1}{2x+1} dx\right) = (2x+1)^{1/2}$, and multiplying the equation by ρ gives
$(2x+1)^{1/2} y' + (2x+1)^{-1/2} y = 2x+1$, or $D_x\left[(2x+1)^{1/2} \cdot y\right] = 2x+1$. Integrating then leads to $(2x+1)^{1/2} \cdot y = x^2 + x + C$, and thus to the general solution
$y = \left(x^2 + x + C\right)(2x+1)^{-1/2}$.

30. The expression $x + y$ suggests the substitution $v = x + y$, which implies that $y = v - x$, and thus that $y' = v' - 1$. Substituting gives $v' - 1 = \sqrt{v}$, or $v' = \sqrt{v} + 1$, a separable equation for v as a function of x. Separating variables gives $\int \frac{1}{\sqrt{v}+1} dv = \int dx$. The further substitution $v = u^2$ (so that $dv = 2u\,du$) and long division give

$$\int \frac{1}{\sqrt{v}+1} dv = \int \frac{2u}{u+1} du = \int 2 - \frac{2}{u+1} du = 2u - 2\ln(u+1) = 2\sqrt{v} - 2\ln\left(\sqrt{v}+1\right),$$

leading to $2\sqrt{v} - 2\ln\left(\sqrt{v}+1\right) = x + C$. Finally, back-substituting $x + y$ for *v* leads to the implicit general solution $x = 2\sqrt{x+y} - 2\ln\left(\sqrt{x+y}+1\right) + C$.

31. Rewriting the differential equation as $y' - 3x^2 y = 21x^2$ shows that it is *linear*. An integrating factor is given by $\rho = \exp\left(\int -3x^2\, dx\right) = e^{-x^3}$, and multiplying the equation by ρ gives $e^{-x^3} \cdot y' - 3x^2 e^{-x^3} y = 21x^2 e^{-x^3}$, or $D_x\left(e^{-x^3} \cdot y\right) = 21x^2 e^{-x^3}$. Integrating then leads to $e^{-x^3} \cdot y = -7e^{-x^3} + C$, and thus to the general solution $y = -7 + Ce^{x^3}$.

Alternatively, writing the equation for $y > -7$ as $\frac{dy}{y+7} = 3x^2 dx$ shows that it is *separable*. Integrating yields the general solution $\ln(y+7) = x^3 + C$, that is, $y = Ce^{x^3} - 7$, as found above.

(Note that the restriction $y > -7$ in the second solution causes no loss of generality. The general solution as found by the first method shows that either $y < -7$ for all *x* or $y > -7$ for all *x*. Of course, the second solution could be carried out under the assumption $y < -7$ as well.)

32. We first rewrite the differential equation as $\dfrac{dy}{dx} = x\left(y^3 - y\right)$, showing that the equation is *separable*. For $y > 1$ separating variables gives $\int \dfrac{1}{y^3 - y}\,dy = \int x\,dx$, and the method of partial fractions yields

$$\int \frac{1}{y^3 - y}\,dy = \int -\frac{1}{y} + \frac{1}{2(y+1)} + \frac{1}{2(y-1)}\,dy = \ln\frac{\sqrt{y^2-1}}{y},$$

leading to the solution $\ln\dfrac{\sqrt{y^2-1}}{y} = \dfrac{1}{2}x^2 + C$, or $y^2 - 1 = Cy^2 e^{x^2}$, or finally $y = \dfrac{1}{\sqrt{Ce^{x^2} + 1}}$.

Alternatively, writing the given equation as $\dfrac{dy}{dx} + xy = xy^3$ shows that it is a *Bernoulli* equation with $n = 3$. The substitution $v = y^{-2}$ for $y > 0$ implies that $y = v^{-1/2}$ and thus that $y' = -\dfrac{1}{2}v^{-3/2}v'$. Substituting gives $-\dfrac{1}{2}v^{-3/2}v' + xv^{-1/2} = xv^{-3/2}$, or $v' - 2xv = -2x$, a linear equation for v as a function of x. An integrating factor is given by $\rho = \exp\left(\int -2x\,dx\right) = e^{-x^2}$, and multiplying the differential equation by ρ gives $e^{-x^2}\cdot v' - 2xe^{-x^2}v = -2xe^{-x^2}$, or $D_x\left(e^{-x^2}\cdot v\right) = -2xe^{-x^2}$. Integrating then leads to $e^{-x^2}\cdot v = e^{-x^2} + C$, or $v = Ce^{x^2} + 1$. Finally, back-substituting y^{-2} for v gives the same general solution as found above.

33. Rewriting the differential equation for $x, y > 0$ in differential form gives

$$\left(3x^2 + 2y^2\right)dx + 4xy\,dy = 0,$$

and because $\dfrac{\partial}{\partial y}\left(3x^2 + 2y^2\right) = 4y = \dfrac{\partial}{\partial x}4xy$, the given equation is *exact*. We apply the method of Example 9 in Section 1.6 to find a solution in the form $F(x,y) = C$. First, the condition $F_x = M$ implies that

$$F(x,y) = \int 3x^2 + 2y^2\,dx = x^3 + 2xy^2 + g(y),$$

and then the condition $F_y = N$ implies that $4xy + g'(y) = 4xy$, or $g'(y) = 0$, that is, g is constant. Thus the solution is given by $x^3 + 2xy^2 = C$.

Alternatively, rewriting the given equation for $x, y > 0$ as $\dfrac{dy}{dx} = -\dfrac{3}{4}\dfrac{x}{y} - \dfrac{1}{2}\dfrac{y}{x}$ shows that it is *homogeneous*. Substituting $v = \dfrac{y}{x}$ then gives $v + x\dfrac{dv}{dx} = -\dfrac{3}{4v} - \dfrac{1}{2}v$, or

$x\dfrac{dv}{dx} = -\dfrac{3}{4v} - \dfrac{3}{2}v = -\dfrac{3+6v^2}{4v}$. Separating variables leads to $\int \dfrac{4v}{6v^2+3}dv = -\int \dfrac{1}{x}dx$, or $\ln\left(6v^2+3\right) = -3\ln x + C$, or $\left(2v^2+1\right)x^3 = C$. Back-substituting $\dfrac{y}{x}$ for v then gives the solution $\left[2\left(\dfrac{y}{x}\right)^2+1\right]x^3 = C$, or finally $2y^2x + x^3 = C$, as found above.

Still another solution arises from writing the differential equation for $x, y > 0$ as $\dfrac{dy}{dx} + \dfrac{1}{2x}y = -\dfrac{3x}{4}y^{-1}$, which shows that it is *Bernoulli* with $n = -1$. The substitution $v = y^2$ implies that $y = v^{1/2}$ and thus that $y' = \dfrac{1}{2}v^{-1/2}v'$. Substituting gives $\dfrac{1}{2}v^{-1/2}v' + \dfrac{1}{2x}v^{1/2} = -\dfrac{3x}{4}v^{-1/2}$, or $v' + \dfrac{1}{x}v = -\dfrac{3x}{2}$, a linear equation for v as a function of x. An integrating factor is given by $\rho = \exp\left(\int \dfrac{1}{x}dx\right) = x$, and multiplying the differential equation by ρ gives $x\cdot v' + v = -\dfrac{3x^2}{2}$, or $D_x\left(x\cdot v\right) = -\dfrac{3x^2}{2}$. Integrating then leads to $x\cdot v = -\dfrac{x^3}{2} + C$. Finally, back-substituting y^2 for v leads to the general solution $x\cdot y^2 = -\dfrac{x^3}{2} + C$, that is, $2xy^2 + x^3 = C$, as found above.

34. Rewriting the differential equation in differential form gives

$$\left(x+3y\right)dx + \left(3x-y\right)dy = 0,$$

and because $\dfrac{\partial}{\partial y}\left(x+3y\right) = 3 = \dfrac{\partial}{\partial x}\left(3x-y\right)$, the given equation is exact. We apply the method of Example 9 in Section 1.6 to find a solution in the form $F\left(x,y\right) = C$. First, the condition $F_x = M$ implies that

$$F\left(x,y\right) = \int x + 3y\,dx = \frac{1}{2}x^2 + 3xy + g\left(y\right),$$

and then the condition $F_y = N$ implies that $3x + g'\left(y\right) = 3x - y$, or $g'\left(y\right) = -y$, or $g\left(y\right) = -\dfrac{1}{2}y^2$. Thus the solution is given by $\dfrac{1}{2}x^2 + 3xy - \dfrac{1}{2}y^2 = C$, or $x^2 + 6xy - y^2 = C$.

Alternatively, rewriting the given equation for $x, y > 0$ as $\dfrac{dy}{dx} = \dfrac{1+3\frac{y}{x}}{\frac{y}{x}-3}$ shows that it is *homogeneous*. Substituting $v = \dfrac{y}{x}$ then gives $v + x\dfrac{dv}{dx} = \dfrac{1+3v}{v-3}$, or $x\dfrac{dv}{dx} = \dfrac{-v^2+6v+1}{v-3}$. Separating variables leads to $\int \dfrac{v-3}{-v^2+6v+1}dv = \int \dfrac{1}{x}dx$, or $\ln\left(-v^2+6v+1\right) = -2\ln x + C$, or $x^2\left(-v^2+6v+1\right) = C$. Back-substituting $\dfrac{y}{x}$ for v then gives the solution $-y^2+6xy+x^2 = C$ found above.

35. Rewriting the differential equation as $\dfrac{dy}{dx} = \dfrac{2x}{x^2+1}(y+1)$ shows that it is *separable*. For $y > -1$ separating variables gives $\int \dfrac{1}{y+1}dy = \int \dfrac{2x}{x^2+1}dx$, or $\ln(y+1) = \ln\left(x^2+1\right)+C$, leading to the general solution $y = C\left(x^2+1\right)-1$.

Alternatively, writing the differential equation as $y' - \dfrac{2x}{x^2+1}y = \dfrac{2x}{x^2+1}$ shows that it is *linear*. An integrating factor is given by $\rho = \exp\left(\int -\dfrac{2x}{x^2+1}dx\right) = \dfrac{1}{x^2+1}$, and multiplying the equation by ρ gives $\dfrac{1}{x^2+1}y' - \dfrac{2x}{\left(x^2+1\right)^2}y = \dfrac{2x}{\left(x^2+1\right)^2}$, or $D_x\left(\dfrac{1}{x^2+1}y\right) = \dfrac{2x}{\left(x^2+1\right)^2}$. Integrating then leads to $\dfrac{1}{x^2+1}y = -\dfrac{1}{x^2+1}+C$, or thus to the general solution $y = -1 + C\left(x^2+1\right)$ found above.

36. Rewriting the differential equation for $0 < x < \pi$, $0 < y < 1$ as $\dfrac{dy}{\sqrt{y}-y} = \cot x\, dx$ shows that it is *separable*. The substitution $y = u^2$ gives

$$\int \frac{1}{\sqrt{y}-y}dy = \int \frac{2}{1-u}du = -\ln(1-u) = -\ln\left(1-\sqrt{y}\right),$$

leading to the general solution $-\ln\left(1-\sqrt{y}\right) = \ln\sin x + C$, or $\sin x\left(1-\sqrt{y}\right) = C$, or finally $y = \left(C\csc x + 1\right)^2$.

Alternatively, writing the differential equation for $0 < x < \pi$, $0 < y < 1$ as $\frac{dy}{dx} + (\cot x)\, y = (\cot x)\sqrt{y}$ shows that it is *Bernoulli* with $n = \frac{1}{2}$. The substitution $v = y^{1/2}$ implies that $y = v^2$ and thus that $y' = 2v \cdot v'$. Substituting gives $2v \cdot v' + (\cot x)\, v^2 = (\cot x)\, v$, or $v' + \frac{1}{2}(\cot x)\, v = \frac{1}{2}\cot x$, a linear equation for v as a function of x. An integrating factor is given by $\rho = \exp\left(\frac{1}{2}\int \cot x\, dx\right) = \sqrt{\sin x}$, and multiplying the differential equation by ρ gives

$$\sqrt{\sin x} \cdot v' + \frac{1}{2}\left(\sqrt{\sin x}\,\cot x\right) v = \frac{1}{2}\sqrt{\sin x}\,\cot x,$$

or $D_x\left[\sqrt{\sin x} \cdot v\right] = \frac{1}{2}\sqrt{\sin x} \cdot \cot x$. Integrating then leads to $\sqrt{\sin x} \cdot v = \sqrt{\sin x} + C$, or $v = 1 + C\sqrt{\csc x}$. Finally, back-substituting $y^{1/2}$ for v leads to the general solution $y = \left(1 + C\sqrt{\csc x}\right)^2$ found above.